

1. Introduction

One of the signs of the maturity of a subject, and oftentimes one of the causes of it, is the appearance of technical textbooks that are accessible to strong first-year graduate students. Such textbooks would have to make a careful selection of topics that are scattered in papers and research monographs and present them in a coherent fashion, occasionally replacing the natural historical order of the development of the subject by another order that is pedagogically sensible; this textbook would also have to unify the varying notations and definitions of various sources. For a field, like the theory of automorphic forms and representations, which despite its long history, is still in many ways in its infancy, writing a textbook that satisfies the demands listed above is a daunting task, and if done successfully its completion is a real achievement that will affect the development of the subject in the coming years and decades. The two-volume book under review, along with Bump’s book [13], are precisely of this nature. The other books in this subject [42], [41], [40], [9], [7], [29], [30], [34], [28], [27], [20], [2], [47], etc., are either research monographs, conference proceedings, or long survey articles—but never graduate textbooks. Many of these books were written when the field was still shaping up, and, as a result, notations were not fixed and definitions varied from source to source. Until fairly recently, as late as 1997 when Bump’s book was published,
these, and primary research papers, were the only available sources to learn the
theory of automorphic forms beyond the case of GL(2).

$L$-functions of various sorts have played a fundamental role in arithmetic, per-
haps starting with Euler’s proof of the existence of infinitely many prime numbers.
Riemann realized the relevance of the analytic theory of zeta functions to the dis-
tribution of prime numbers in a quantitative fashion, and this realization led to
the formulation of Riemann’s Hypothesis. Dirichlet used the $L$-functions that now
bear his name to prove the Arithmetic Progression Theorem. Hadamard and de
la Vallée-Poussin used zeta functions to prove the Prime Number Theorem. Artin
observed that knowing the holomorphy properties of his $L$-functions has deep arith-
etic consequences; in fact, Artin’s Law of Reciprocity can be formulated as an
equality of two different types of $L$-functions, Abelian Artin $L$-functions on the
one hand, and Hecke $L$-functions on the other. In a similar vein, the Modularity
Conjecture, proved not too long ago \[54, 53, 10\] is also a statement about the
equality of two $L$-functions, this time the Hasse–Weil zeta function of an elliptic
curve and the $L$-function of a modular form of weight two. Hecke $L$-functions and
$L$-functions of modular forms are $L$-functions of automorphic representations in the
sense of Langlands.

The theory of automorphic representations has its genesis in the theory of ellip-
tic functions in early nineteenth century. By the late 1940s via the works of Hecke
and Maaß, in the holomorphic and real analytic cases respectively, the theory was
put in a shape that is now recognizable by anyone working in the field. These,
as well as works by Shimura, Taniyama, and others in the Japanese school in the
1960s, established the arithmetic significance of the theory. Around the same time
Harish-Chandra’s theory of infinite-dimensional representations of semi-simple and
reductive groups over local fields, especially his fundamental results on characters
and discrete series representations, had put representation theory in a usable form.
These developments along with Class Field Theory, according to the German and
Japanese schools and Artin and Tate \[3\], formed the conceptual background of the
modern theory of automorphic forms. Ultimately, it was the books by Gelfand,
Graev, and Piatetski-Shapiro \[29\] and by Jacquet and Langlands \[34\] that recast
the theory of automorphic forms in the language of infinite-dimensional representa-
tions of the reductive group $GL(n)$ over local fields and adèle rings. The book
of Godement and Jacquet \[30\] gave a definition of the principal $L$-function of an
automorphic representation of the group $GL(n)$ and established its basic analytic
properties. The Godement–Jacquet theory uses the theory of matrix coefficients.
The book under review is the first book since the appearance of \[30\] in 1972 that
gives a thorough treatment of the principal $L$-function using matrix coefficients.

2. AUTOMORPHIC $L$-FUNCTIONS

Let $G$ be a reductive group defined over a number field $F$. In very loose terms
an automorphic representation $\pi$ of $G(\mathbb{A}_F)$, $\mathbb{A}_F$ being the ring of adèles of $F$, is a
constituent of the representation obtained by the right regular action of $G(\mathbb{A}_F)$ on
$L^2(Z(\mathbb{A}_F)G(F)\backslash G(\mathbb{A}_F))$. It is a theorem of Flath that for every place $v$ of $F$, there
is an admissible representation $\pi_v$ of $G(F_v)$ such that $\pi$ is in some sense the tensor
product of the $\pi_v$’s. For almost all places $v$, the representations $\pi_v$ are unramified,
and as such have a particularly simple description. In the most important case
where $G = GL(n)$ over a non-archimedean local field $F_v$, unramified representations
of $G(F_v)$ can be described as follows: there is an unramified character of the group of
diagonal matrices $T(F_v)$ of $G(F_v)$ such that $\pi_v$ is isomorphic to a unique irreducible
subquotient of $\text{Ind}_{B(F_v)}^{G(F_v)}(\chi)$. Let us explain the notation. In this expression $B$ is the group of upper triangular matrices which can be written as a semi-direct product $T(F_v) \rtimes N(F_v)$. There is a natural map $B(F_v) \rightarrow T(F_v)$, and pulling back $\chi$ via this map gives a character of $B(F_v)$, which we again denote by $\chi$. Then $\text{Ind}(\chi)$ is the right regular action of $G(F_v)$ on the space functions $f : G(F_v) \rightarrow \mathbb{C}$ satisfying the following conditions:

1. For all $b \in B(F_v)$ and $g \in G(F_v)$, we have
   $$f(bg) = \delta(b)^{1/2} \chi(b)f(g);$$
2. for all $g \in G(F_v)$ and $k$ in some compact open subgroup $K$ of $G(F_v)$, we have
   $$f(gk) = f(g).$$

In the above definition $\delta(b)$ is the modulus character. Generically the representations $\text{Ind}_{B}^{G}(\chi)$ thus defined are irreducible and infinite dimensional. We define the local $L$-function of $\pi_v$ in the following manner. The character $\chi$ is determined by a choice of $n$ quasi-characters of $F_v^{	imes}$ in the following fashion:

$$\chi \left( \begin{pmatrix} t_1 & & \cdots & \cr \cr t_2 & & & \cr \cr & \cdots & & \cr \cr & & \cdots & \cr \cr t_n & & & \end{pmatrix} \right) = \chi_1(t_1) \cdots \chi_n(t_n).$$

We set

$$L(\pi_v, s) = \prod_{i=1}^{n} (1 - \chi_i(\varpi_v)q_v^{-s})^{-1}.$$

Alternatively, we may define $C_v(\pi_v)$ to be the conjugacy class of the diagonal matrix

$$\begin{pmatrix} \chi_1(\varpi_v) & & \cr & \cdots & \cr & \cr & & \chi_n(\varpi_v) \end{pmatrix}$$

and set

$$L(\pi_v, s) = \det(1 - q_v^{-s}C_v(\pi_v))^{-1}.$$

This description should remind the reader of the definition of the Artin $L$-function.

If $\pi = \bigotimes'_v \pi_v$, and for $v \notin S$, $S$ finite set containing all the non-archimedean places, $\pi_v$ is unramified, we set

$$L^S(\pi, s) = \prod_{v \notin S} L(\pi_v, s).$$

The main result of [30] is that there is a way to complete the definition of the $L$-function for $v \in S$, and give a definition for $L(\pi_v, s)$ such that if we set

$$(2.1) \quad L(\pi, s) = \prod_{v} L(\pi_v, s),$$

then the resulting $L(\pi, s)$ has properties similar to those satisfied by the Riemann zeta function (functional equation, analytic continuation to a meromorphic function with finitely many poles, boundedness in vertical strips). In the special case where the representation $\pi$ is irreducible cuspidal, the $L$-function has an analytic continuation to an entire function. Let us briefly explain what it means for a representation to be cuspidal. A parabolic subgroup of $\text{GL}(n)$ is an algebraic subgroup that contains a Borel subgroup. For our purposes it would be sufficient to consider
only standard parabolic subgroups, that is those containing the standard Borel subgroup of upper triangular matrices. For any such parabolic subgroup $P$ there is a partition of the number $n = n_1 + \cdots + n_r$ such that $P$ is the collection of block-upper triangular matrices of the form

$$
\begin{pmatrix}
g_1 & * \\
. & . \\
. & . \\
g_r
\end{pmatrix}
$$

with $g_i \in \text{GL}(n_i)$. The unipotent radical of $P$ is defined to be the subgroup consisting of those matrices for which for every $i$, $1 \leq i \leq r$, $g_i$ is the identity element of $\text{GL}(n_i)$. A function $\phi \in L^2(G(F)Z(A)\backslash G(\mathbb{A}))$ is called cuspidal if for the unipotent radical of every standard parabolic subgroup and for almost all $g \in G(\mathbb{A})$, we have

$$
\int_{N(F)\backslash N(\mathbb{A})} \phi(ng) \, dn = 0.
$$

The collection of cuspidal functions is an invariant subspace of $L^2(Z(\mathbb{A})G(F)\backslash G(\mathbb{A}))$ under the right action of $G(\mathbb{A})$. Furthermore, it can be shown that it is the Hilbert direct sum of irreducible subrepresentations, each of which is called an irreducible cuspidal representation.

More generally, if $r : \text{GL}(n, \mathbb{C}) \to \text{GL}(N, \mathbb{C})$ is an algebraic representation, we may define

$$
L(\pi_v, s, r) := \det(1 - q_v^{-s}r(C_v(\pi_v)))^{-1},
$$

and set

$$
L^S(\pi, s, r) = \prod_{v \not\in S} L(\pi_v, s, r).
$$

Langlands [39] conjectures that it is possible to complete the definition of the $L$-function for the remaining places in such a way that the completed $L$-function has desirable analytic properties as above. In this generality, the conjecture is widely open.

One of the methods of proving analytic continuation and functional equations for $\text{GL}(n)$ automorphic representations is the method of zeta integrals. Let $\pi = \bigotimes_v \pi_v$ be an automorphic cuspidal representation of $\text{GL}(n)$, and let $V_\pi$ be its associated space of cuspidal functions. In very loose terms, the idea is to cook up an integral $Z(s, \phi)$ for $s \in \mathbb{C}$ and $\phi \in V_\pi$ with a number of properties. First, we expect $Z(s, \phi)$ to define an entire function of $s$ (provided that we expect the target $L$-function to be entire, but this is not always the case), and satisfy a functional equation with $s \mapsto 1 - s$. Next, we expect that for appropriate $\phi$ and $\Re s$ large enough, we would have a factorization

$$
Z(s, \phi) = \prod_v Z_v(s),
$$

where the $Z_v$’s would depend on $\phi$ and are rational functions of $q_v^{-s}$. Further, for almost all places $v$, we would have

$$
Z_v(s) = L(s, \pi_v, r).
$$
It is in general rather easy to write down zeta integrals that have a functional equation—a number of available zeta integrals use Eisenstein series for this reason, but factorizability and the expression in terms of local Langlands $L$-functions are typically difficult to achieve. Factorizability of the zeta integral usually follows from local uniqueness results, e.g., matrix coefficients, Whittaker models, Bessel models, etc. Recent work by Sakellaridis \[48\] puts a number of zeta integrals in a conceptual framework involving spherical subgroups of reductive groups.

Tate and Iwasawa, independently, give the construction of the $L$-function of a Hecke character $\chi : F^x A^x \to \mathbb{C}^x$ using a zeta integral. The construction is as follows. Let $\Phi$ be a Schwartz–Bruhat function on $\mathbb{A}$, and consider the integral

$$Z(s, \chi, \Phi) = \int_{\mathbb{A}^x} \Phi(x) \chi(x) |x|^s d^x x,$$

with $d^x x$ an appropriately normalized measure on $\mathbb{A}^x$. One can then use this integral to prove all the basic properties of the $L$-function \[51\]. A Hecke character is an automorphic cuspidal representation of $GL(1)$, and the $L$-function considered by Tate and Iwasawa is its principal $L$-function. It is truly remarkable that this method also works for $GL(n)$. The Godement–Jacquet construction for $GL(n)$ is as follows. Let $(\pi, V_\pi)$ be an automorphic cuspidal representation of $GL(n)$ over a number field $F$ which for simplicity we assume to have unitary central character. Let $\phi_1, \phi_2 \in V_\pi$. Define a function $\beta(g)$ for $g \in GL_n(\mathbb{A})$ as follows:

$$\beta(g) = \int_{Z(\mathbb{A})GL_n(F)\backslash GL_n(\mathbb{A})} \phi_1(hg)\overline{\phi_2(h)} dh.$$

This function is called a matrix coefficient of $\pi$. Here, too, $dg$ comes from an appropriately normalized Haar measure on $GL_n(\mathbb{A})$. Now let $\Phi$ be a Schwartz–Bruhat function on $M_{n \times n}(\mathbb{A})$. For $s \in \mathbb{C}$ with $\Re s$ large, we set

$$Z(s, \Phi, \beta) = \int_{GL_n(\mathbb{A})} \Phi(g) |\beta(g)| |\det(g)|^{s + \frac{n-1}{2}} dg.$$

By using this integral representation, one can prove the basic properties of the $L$-function \[2.1\]. In the case where $n = 1$ and $\pi$ is a Hecke character, this recovers the Tate integral.

The integral of Godement–Jacquet is not the only available integral representation in the market. The integral of \[34\] uses the theory of Whittaker functions. For a survey of such integral representations, see \[12\], \[14\], and \[22\].

A consequence of the Langlands’ Functoriality Conjecture is that in some sense the Godement–Jacquet principal $L$-function is the only type of $L$-function that occurs in nature. An extremely special case of this conjecture, in very loose terms, asserts that if $r : GL_n(\mathbb{C}) \to GL_N(\mathbb{C})$ is an algebraic representation, then there is a transfer of automorphic representations on $GL(n)$ to those on $GL(N)$ in such a way that if $\pi$ transfers to $\Pi$, then

$$L(s, \pi, r) = L(s, \Pi).$$

The correct formulation of the Functoriality Conjecture involves notions of $L$-groups and maps between them; cf. \[6\], \[27\], and \[17\] for an elementary presentation. In general, the Functoriality Conjecture remains beyond reach, but some special cases are known; cf. \[34\], \[25\], \[50\], \[37\], \[36\], \[19\], \[18\], \[5\], \[4\], just to name a few examples. The conjecture in many key cases, the so-called endoscopic cases, is...
expected to follow from the trace formula in its various forms. The best sources to get a feel for the developments of the trace formula are the two websites

http://publications.ias.edu/rpl/

and

http://www.claymath.org/cw/arthur/

where the works of Langlands and Arthur are collected. A missing piece of the puzzle for many years was the Fundamental Lemma which was recently proved by Ngo [44, 1]; this, as well as the imminent appearance of Arthur’s monograph on endoscopic transfer—a draft of which is available on Arthur’s website—have opened doors to new possibilities in this subject.

We would be amiss if we did not at least mention the proof of the Local Langlands conjecture for GL($n$) by Harris and Taylor [32] and Henniart [33], as well as the Global Langlands Conjecture in the function field case by Lafforgue [38]. For reasons of space we have not discussed many of the exciting developments in the subject, e.g., the relative trace formula, the theory of the oscillator representation and its relevance to functoriality ([45, 24]), or the proof of the Sato–Tate Conjecture for non-CM elliptic curves [31, 52, 16]; neither have we discussed Backward Lifting, nor Converse Theorems [20, 21]. In the generality of a quasi-split reductive group $G$ over a number field, and in the case where the representation $\pi$ is generic, there is another approach to Langlands conjectures which is known as the Langlands–Shahidi method [50]. The method has produced some truly incredible results.

3. This book

The book under review gives an exposition of the basic theory of automorphic forms and representations on GL($n$), the associated local theory, and the Godement–Jacquet theory of principal $L$-functions. The book consists of two volumes. The first volume gives a careful treatment of the GL(1) and GL(2) theories, the GL(1) theory being Tate’s thesis. The GL(2) theory presented in the first volume is the $n = 2$ case of Godement and Jacquet’s theory for GL($n$) using matrix coefficients, and not the Jacquet–Langlands theory. The presentation here is extremely well motivated and elegant. The book starts with a review of $p$-adic numbers, adégles, and $p$-adic and adelic integration, and quickly moves to Tate’s theory of $L$-functions in the second chapter. The third chapter reviews classical automorphic forms on the upper half-space. The core material starts with chapter four where automorphic forms on the adelic GL(2) are introduced. The remaining chapters deal with automorphic representations, local representation theory, contragredient representations and matrix coefficients, unitary representations, and Godement–Jacquet integral representations for GL(2). In this, as well as the second volume, there are many exercises of varying degrees of difficulty, a lot of them with hints and solutions at the end of the book. The inclusion of these exercises makes this a particularly useful textbook for a first graduate course on automorphic representations. The second volume extends the results of the first volume to GL($n$) for arbitrary $n$. In this volume the theory is outlined parallel to chapters three through eleven of the first volume but the proofs of the more difficult theorems are only referenced or broadly sketched. I find this style of presentation of the material instructive, in that the novice who is trying to find her way around the subject does not get bogged down in the technicalities of the details of proofs of
the theorems and gets a feel for the landscape; in order for this approach to be successful, however, the reader should read the first volume carefully and thoroughly familiarize herself with the GL(2) theory. If the proofs of all the theorems were to be included, the length of the second volume would be a few times longer, and the book would be orders of magnitude harder to read. As presented, the logic and the development of the subject is natural and clear.

Due to all the necessary background, the general theory of automorphic forms is a famously difficult subject to enter. The fact that this book develops the theory in the special case of GL(n) makes it a particularly appealing textbook. After this book the natural next step for the interested reader would be to learn the theory of admissible representations of p-adic groups from [15] and the general theory of automorphic representations and Eisenstein series from the wonderful book [43]. The student of the subject should also consult the recent instructional books on the subject, e.g., [19], [11], [35], and [23]. Unfortunately, there are no textbooks providing an introduction to the general theory that are nearly as accessible as the book under review, but once the new student of the field is armed with what is presented in the lovely books of Goldfeld and Hundley, then the prospects of making headway in the subject are not nearly as difficult as they once were. This is a very good read.

References


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