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Zeta functions of graphs: a stroll through the garden, by Audrey Terras, Cambridge Studies in Advanced Mathematics, Vol. 128, Cambridge University Press, Cambridge, 2011, xii+239 pp., hardcover, US\$ 65.00, ISBN 978-0-521-11367-0

The book by Audrey Terras, under review, has the lovely subtitle *A stroll through the garden*. I will follow its style in strolling through the world of zeta functions, and begin, like her, with their ancestors from geometry and number theory. Only later will we arrive at zeta functions of graphs, the main focus of the book.

1. What is a zeta function?

There has been, throughout mathematics, a wealth of functions masquerading as "zeta functions"—to the point that Atle Selberg (more on him later) is said to have called for a Comprehensive Test Ban treaty to halt their further proliferation. We will attempt, in this section, to determine wherefrom in its pedigree a mathematical object may honestly claim to be a zeta function.

1.1. **Euler.** Zeta functions are particular kinds of convergent Dirichlet series (see below), but they nevertheless predate them. Leonhard Euler (1707–1783) was probably the first to realize their power. He computed the sum

$$1 + 1/4 + 1/9 + \cdots + 1/n^2 + \cdots = \frac{\pi^2}{6},$$

by doing formal manipulations on the series

$$\zeta(s) = 1 + 2^{-s} + \cdots + n^{-s} + \cdots$$

and deriving the value $\zeta(2)$. (He actually computed the $\zeta(2n)$ for all $n \in \mathbb{N}$, using Bernoulli numbers; the values at odd numbers are still a mystery.) Using the uniqueness of prime factorization, he obtained the formula

$$\zeta(s) = \frac{1}{(1-2^{-s})(1-3^{-s})(1-5^{-s})\cdots(1-p^{-s})\cdots}.$$

We arrive at the first two properties of zeta functions: 1. *They must be a series which converges at interesting points, with interesting special values*; and 2. *They must also admit interesting factorizations as products.*

1.2. **Dirichlet.** A sequence of numbers $a_1, a_2, \dots$ gives rise to a *Dirichlet series* $f(s) = \sum_{n\ge1} a_n n^{-s}$. This is but a method of encoding the infinitely many numbers (a_n) into a single mathematical object f. If the numbers a_n grow at most polynomially, then f will actually be an analytic function on a half-plane $\{\Re(s) > \lambda\}$; and the abscissa of convergence λ is related to the growth of the sequence in that $a_1 + \cdots + a_n \precsim n^\lambda$. Much finer asymptotic information on the sequence (e.g., in the form of an approximation $a_n \propto n^\alpha(\log n)^\beta$) can be obtained by finer analytic properties of f. This goes under the headline of *Abelian* and *Tauberian* theorems.

Gustav Lejeune Dirichlet (1805–1859) did much more than get his name attributed to a general form of formal series; he considered expressions of the form $\sum_{n\ge1} \chi(n)n^{-s}$ for a multiplicative character χ of $\mathbb{Z}/N\mathbb{Z}$. Such expressions are

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called *L-functions*. They are used, e.g., to deduce the infinity of primes in arithmetic progressions.

1.3. **Riemann.** The analytical aspects of $\zeta(s)$ were further studied by Bernhard Riemann (1826–1866). He showed in 1859 that Euler's function $\zeta(s)$ fulfills the functional equation

$$\zeta(s) = 2^s \pi^{s-1} \sin(\pi s/2)\Gamma(1-s)\zeta(1-s),$$

which allows $\zeta(s)$ to be analytically continued beyond its domain of convergence $\{\Re(s) > 1\}$. Furthermore, he noted that if the zeroes of $\zeta(s)$ were located on the line $\{\Re(s) = 1/2\}$, this would explain the apparent randomness of primes: the fact that they behave very much as if the probability that n is prime were of the order $1/\log n$, and they were independent of the probabilities that other numbers were prime. We see here two more features of zeta functions: 3. *They should admit a functional equation*; and 4. *They should admit an analytic continuation which tells us something about the original sequence they come from.*

The so-called *Riemann hypothesis*, namely that all zeroes of $\zeta(s)$ are on the line $\{\Re(s) = 1/2\}$, is one of the outstanding open problems in mathematics.

1.4. **Dedekind.** Richard Dedekind (1831–1916) generalized the now-called Riemann zeta function $\zeta(s)$ to a wider range of algebraic structures. If $\mathbb{K}$ is a number field (a finite-dimensional field extension of $\mathbb{Q}$) and $\mathcal{O}$ is its ring of integers (elements of $\mathbb{K}$ that are roots of monic polynomials over $\mathbb{Z}$), one defines

$$\zeta_{\mathbb{K}}(s) = \sum_{\mathfrak{a} \lhd \mathcal{O}} \#(\mathcal{O}/\mathfrak{a})^{-s}; \tag{1}$$

the sum ranges over all nonzero ideals in $\mathcal{O}$. Thus $\zeta_{\mathbb{Q}}$ is the original zeta function. The quantity $\#(\mathcal{O}/\mathfrak{a})$ is known as the *norm* of the ideal $\mathfrak{a}$.

Extending previous considerations by Dirichlet, Dedekind noted that the nature of $\zeta_{\mathbb{K}}(s)$ about its pole at $s = 1$ encodes important algebraic properties of $\mathbb{K}$; for example, the class number formula expresses the residue of $\zeta_{\mathbb{K}}(s)$ at $s = 1$ as $2^{r_1}(2\pi)^{r_2} h_{\mathbb{K}} R_{\mathbb{K}} / w_{\mathbb{K}} \sqrt{|D_{\mathbb{K}}|}$ in terms of numbers of real and complex imbeddings, class number, regulator, number of roots of unity, and discriminant, respectively.

1.5. **Artin, Hasse, Weil.** In 1924, Emil Artin (1898–1962) extended Dedekind's definition to fields $\mathbb{K}$ of positive characteristic such as $\mathbb{F}_p(x)$ and its finite extensions. Such a field admits as "integers" the subring $\mathcal{O}$ of zeroes of monic polynomials over $\mathbb{F}_p[x]$; ideals again have finite index, so one may form the zeta function of $\mathbb{K}$ as in (1).

As an example, Artin considered $\mathbb{K} = \mathbb{F}_p(x)[y]/(y^2 - x^3 + x)$. He noted that $\zeta_{\mathbb{K}}$ encodes the number of points on the elliptic curve $\mathscr{E} : y^2 = x^3 - x$ in characteristic p; more precisely, for a power $q = p^k$ of p, we write

$$\begin{aligned} N_q = \#\mathscr{E}(\mathbb{F}_q) &= \#\{(x:y:z) \in \mathbf{P}^2(\mathbb{F}_q) \colon y^2 z = x^3 - xz^2\} \\ &= 1 + \#\{(x,y) \in \mathbb{F}_q^2 \colon y^2 = x^3 - x\} \end{aligned}$$

and

$$\zeta_{\mathscr{E}}(s) = \exp\left(\sum_{k \geq 1} N_{p^k} \frac{p^{-ks}}{k}\right), \tag{2}$$

then

$$\zeta_{\mathbb{K}}(s) = (1 - p^{-s})\zeta_{\mathscr{E}}(s).$$

Based on Artin's computations, Helmut Hasse (1898–1979) viewed the zeta function of the curve as the fundamental object of study, by means of which the curve could be grasped. He proved that $\zeta_{\mathscr{E}}(s)$ is a degree-2 rational function of p^{-s} and that, in analogue to the Riemann hypothesis, its zeroes lie on the axis $\{\Re(s) = 1/2\}$:

$$\zeta_{\mathscr{E}}(s) = \frac{1 - (p + 1 - N_p)p^{-s} + p^{1-2s}}{(1 - p^{-s})(1 - p^{1-s})}.$$

André Weil (1906–1998) deduced then, during the Second World War, a general expression for the zeta function counting points on a curve of higher genus; it is still a rational function of p^{-s}, and its degree is twice the genus of the curve. This gives the concrete following estimate: on a projective curve of genus g over the field $\mathbb{F}_q$, there are between $q + 1 - 2g\sqrt{q}$ and $q + 1 + 2g\sqrt{q}$ points.

The *Weil conjectures* make analogous claims for all projective algebraic varieties: let $\mathscr{X}$ be an n-dimensional, nonsingular algebraic variety over $\mathbb{F}_q$. Define N_q and $\zeta_{\mathscr{X}}$ as above, then the conjectures claim in particular a factorization

$$\zeta_{\mathscr{X}}(s) = \frac{P_1(q^{-s})P_{2n-1}(q^{-s})}{P_0(q^{-s})P_{2n}(q^{-s})}$$

with $P_k(T)$ a polynomial over $\mathbb{Z}$ whose zeroes have absolute value $q^{k/2}$, so that the zeroes $\zeta_{\mathscr{X}}(s)$ lie on vertical lines $\{\Re(s) \in \{1/2, \dots, n - 1/2\}\}$. There is a functional equation

$$\zeta_{\mathscr{X}}(n - s) = \pm q^{(\frac{n}{2} - s)\chi(\mathscr{X})}\zeta_{\mathscr{X}}(s).$$

Moreover, the polynomial P_k has a cohomological interpretation as the characteristic polynomial of the action of the Frobenius endomorphism $\Phi : (x_1, \dots, x_N) \mapsto (x_1^q, \dots, x_N^q)$ on the [ℓ-adic] cohomology of $\mathscr{X}$; and if $\mathscr{X}$ is a "good reduction mod p" of a nonsingular projective variety $\mathscr{Y}$ defined over a number field, then the degree of P_k is also the dimension of $H^*(\mathscr{Y}(\mathbb{C}), \mathbb{Z})$. In particular, $P_0 = 1 - T$ and $P_{2n} = 1 - q^nT$.

The conjectures were eventually proven by Deligne in 1973, following work of Grothendieck.

1.6. **Borevich, Shafarevich, Igusa.** The field $\mathbb{F}_p$ is the base of the tower of fields $\mathbb{F}_{p^k}$. It is also the base of the tower of rings $\mathbb{Z}/p^k\mathbb{Z}$. The first tower leads to the ring of power series in $\mathbb{F}_p$ and to rational functions $\mathbb{F}_p(x)$, while the second tower leads to the p-adic ring $\mathbb{Z}_p$ and to $\mathbb{Q}$.

Let $f(x_1, \dots, x_n)$ be a polynomial with integral coefficients, and define

$$N_q(f) = \#\{(x_1, \dots, x_n) \in (\mathbb{Z}/q\mathbb{Z})^n \colon f(x_1, \dots, x_n) \equiv 0 \pmod{q}\}.$$

Zenon Borevich (1922–1995) and Igor Shafarevich (1923–) proposed to encode this information in a local zeta function; using the variable $T = p^{-s}$, they wrote

$$\zeta_{f,p}(T) = \sum_{k \geq 1} N_{p^k}(f)T^k$$

and conjectured that it is a rational function of T. This was proven by Jun-Ichi Igusa (1924–), using Hironaka's resolution of singularities. He conjectured, additionally, a striking relationship between the hypersurface $\{f = 0\}$ over $\mathbb{Z}_p$ and $\mathbb{C}$: roughly said, if α is a pole of $\zeta_{f,p}$, then $\exp(2\pi i\Re(\alpha))$ is an eigenvalue of the monodromy around a singularity of the complex hypersurface $\{f = 0\}$.

1.7. **Lefschetz, Artin (2), Mazur, Ruelle.** The zeta function (2) and its cohomological interpretation may be viewed as dynamical statements. Consider now a topological space $\mathscr{X}$, and a continuous self-map $f : \mathscr{X} \to \mathscr{X}$. Solomon Lefschetz (1884–1972) considered in 1926 the invariant

$$\Lambda(f) = \sum_{k\geq 0}(-1)^k \operatorname{trace}(f_*|H_k(\mathscr{X},\mathbb{Q}))$$

as an algebraic refinement to the set of fixed points of f: it may be expressed as a sum, over the fixed points of f, of *local indices*. It gives rise to the Lefschetz zeta function

$$\zeta_f^L(T) = \exp\left(\sum_{k\geq 1}\Lambda(f^k)\frac{T^k}{k}\right). \tag{3}$$

Michael Artin (1934– , son of Emil Artin) and Barry Mazur (1937–), on the other hand, considered the zeta function counting fixed points

$$\zeta_f^{AM}(T) = \exp\left(\sum_{k\geq 1}\#\{x\in\mathscr{X} : f^k(x)=x\}\frac{T^k}{k}\right).$$

Both are special cases of a zeta function due to David Ruelle (1935–), which takes as ingredient a matrix-valued function ϕ on $\mathscr{X}$, and is defined as

$$\zeta_{f,\phi}(T) = \exp\left(\sum_{k\geq 1}\sum_{x\in\mathscr{X} :\ f^k(x)=x}\operatorname{trace}\left(\prod_{j=0}^{k-1}\phi(f^k(x))\right)\frac{T^k}{k}\right).$$

The number-theoretical zeta functions of the previous subsection are related to the dynamical ones by the following simple observation: if $\mathscr{X}$ be an algebraic variety defined over $\mathbb{F}_q$, then the points over $\mathbb{F}_{q^k}$ are the fixed points of Φ^k, with Φ the Frobenius endomorphism of $\mathscr{X}$. Thus, the Weil conjectures translate to a statement about the singularities of $\mathscr{X}$ and the local indices of the Frobenius endomorphism.

1.8. **Selberg, Ihara.** Consider now $\mathscr{X}$ a Riemannian manifold. Call a closed geodesic on $\mathscr{X}$ *primitive* if it is not a proper power; parameterizing the geodesic with unit speed as $\overline{\gamma} : [0,L] \to \mathscr{X}$, this means that there is no $k > 1$ with $\overline{\gamma}|_{[0,\frac{L}{k}]} = \cdots = \overline{\gamma}|_{[\frac{(k-1)L}{k},L]}$. Atle Selberg (1917–2007) considered the zeta function

$$\zeta_{\mathscr{X}}(s) = \prod_{\overline{\gamma}\text{ primitive}}\prod_{k\geq 0}\big(1-\exp(-(s+k)\operatorname{length}(\overline{\gamma}))\big),$$

and expressed it in terms of the Laplace operator on $\mathscr{X}$—viewing closed geodesics as periodic points of the geodesic flow on $\mathscr{X}$.

He studied it in particular in the case when $\mathscr{X}$ is a Riemann orbisurface $\mathbb{H}^2/\Gamma$, for a discrete subgroup Γ of $\mathrm{PSL}_2(\mathbb{R})$, such as $\mathrm{PSL}_2(\mathbb{Z})$. A primitive geodesic on $\mathscr{X}$ yields a hyperbolic transformation of $\mathbb{H}^2$ of which it is the axis; the length of the geodesic $\overline{\gamma}$ is the translation length of the corresponding $\gamma \in \Gamma$; if $\tau^{\pm 1}$ be the eigenvalues of γ, then $\exp(\operatorname{length}(\overline{\gamma})) = |\tau|^2 =: N(\gamma)$. The element γ is only defined up to conjugation and is primitive in the sense that it generates its own centralizer; thus

$$\zeta_{\mathscr{X}}(s) = \prod_{\gamma^\Gamma \text{ primitive conjugacy class}} \prod_{k\geq 0} \big(1 - N(\gamma)^{-s-k}\big).$$

Selberg showed, in the case $\mathscr{X} = \mathbb{H}^2/\operatorname{PSL}_2(\mathbb{Z})$, that whenever the Riemann zeta function has a zero at s, there is a corresponding zero of $\zeta_{\mathscr{X}}$ at $s/2$.

Yasutaka Ihara (1938–) studied the p-adic analogue of these Riemann surfaces. Consider a finite extension $\mathbb{K}$ of either $\mathbb{Q}_p$ or $\mathbb{F}_q((x))$, its ring of integers $\mathcal{O}$, the group $G = \operatorname{PGL}_2(\mathbb{K})$, the *upper half-plane*

$$\mathcal{H} = G/\operatorname{PGL}_2(\mathcal{O}), \tag{4}$$

and a torsion-free discrete subgroup Γ of G. For $\gamma \in \Gamma$ with eigenvalues τ_1, τ_2 put $N(\gamma) = |\operatorname{ord}(\tau_1/\tau_2)|$, the valuation of the quotient of the eigenvalues. Ihara defined then

$$\zeta_\Gamma(T) = \prod_{\gamma^\Gamma \text{ primitive conjugacy class}} \big(1 - T^{N(\gamma)}\big)^{-1}.$$

He noted that it was, in idea, an adaptation of Selberg's zeta function to the p-adic and positive-characteristic cases and, in form, a parent of Dedekind's congruence zeta functions, in which primitive conjugacy classes play the role of primes.

He also showed how ζ_Γ can effectively be computed. Let q denote the order of the residue field $\mathcal{O}/\mathfrak{M}$ for the maximal ideal $\mathfrak{M}$ of $\mathcal{O}$, and set $h = \#(\Gamma\backslash\mathcal{H})$. Let $x_1, \dots, x_h \in G$ be a set of double coset representatives, let G_1 be the subgroup of G consisting of scalar multiples of matrices $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $a, b, c, d \in \mathcal{O}$, $(a, b, c, d) = \mathcal{O}$, and let $\operatorname{ord}(ad - bc) = 1$. Construct then the $h \times h$ matrix $A = (a_{ij})$ by

$$a_{ij} = \#(x_i^{-1} G_1 x_j \cap \Gamma),$$

and note that $A = (q + 1) \oplus A_0$ for a $(h - 1) \times (h - 1)$ matrix A_0.

Set $g = (q - 1)h/2 + 1$. Then Γ is isomorphic to the free group of rank g, and

$$\zeta_\Gamma(T) = (1 - T)(1 - qT)(1 - T^2)^{g-1} \det(1 - A_0 T + qT^2)^{-1}.$$

1.9. **Grunewald, Segal, Witten.** An altogether different class of zeta functions were considered by Fritz Grunewald (1949–2010) and Dan Segal (1947–). Let Γ be a group, and define

$$\zeta_\Gamma(s) = \sum_{H \leq \Gamma} [\Gamma : H]^{-s},$$

with the sum ranging over finite-index subgroups of Γ. In particular, if $\Gamma = \mathbb{Z}$, one recovers the Riemann zeta function $\zeta(s)$. For example, a simple calculation using partitions gives $\zeta_{\mathbb{Z}^d}(s) = \zeta(s) \cdots \zeta(s + d - 1)$. These functions have been studied more deeply for Γ nilpotent; indeed, every finite-index nilpotent subgroup is in a unique manner an intersection of prime-power-index subgroups, from which one gets $\zeta_\Gamma(s) = \prod_{p \text{ prime}} \zeta_{\Gamma,p}(s)$ for "local factors" $\zeta_{\Gamma,p}(s)$ counting subgroups of index a power of p.

One may then hope, in analogy with the number-theoretical results, that $\zeta_{\Gamma,p}(s)$ be a rational function of p^{-s}. This is true, e.g., for free nilpotent groups of arbitrary rank and class. Better, Grunewald and du Sautoy conjecture that the $\zeta_{\Gamma,p}$ behave "uniformly" with respect to p, in that there is a rational function $F(X, Y) \in \mathbb{Q}(X, Y)$ such that $\zeta_{\Gamma,p}(s) = F(p, p^{-s})$ holds for almost all p. This is still open in general, but is known to hold if either the rank or the class is at most 2.

These local factors are also known, in some cases, to satisfy functional equations. For example, let Γ be the free 3-generated class-2 nilpotent group, and write $\zeta_{\Gamma,p}(s) = F(p, p^{-s})$ as above. Then $F(p^{-1}, p^{-s}) = -p^{15-6s}F(p, p^{-s})$.

It seems, however, that the counting problems on nilpotent groups are in general not quite as well behaved as those in algebraic varieties over finite fields. Here is a hint to a link: consider the "normal subgroup" zeta function $\zeta^{\triangleleft}_{\Gamma,p}(s)$ counting normal subgroups of p-power index in Γ, for Γ the class-2 nilpotent group

$$\Gamma = \left\langle \begin{matrix} x_1, x_2, x_3 \\ y_1, y_2, y_3 \\ z_1, z_2, z_3 \end{matrix} \middle| \begin{matrix} [x_1, y_1] = z_3 & [x_2, y_1] = z_2 & [x_3, y_1] = z_1 \\ [x_1, y_2] = z_1 & & [x_3, y_2] = z_3 \\ [x_1, y_3] = z_2 & [x_2, y_3] = z_1 & \end{matrix} \right\rangle,$$

all other commutators being trivial. Then the elliptic curve $\mathscr{E} : y^2 = x^3 - x$ makes a surprising reappearance: there exist polynomials $P_1, P_2 \in \mathbb{Q}(X, Y)$ such that, for almost all primes p,

$$\zeta^{\triangleleft}_{\Gamma,p}(s) = P_1(p, p^{-s}) + \#\mathscr{E}(\mathbb{F}_p)P_2(p, p^{-s}).$$

The zeta function $\zeta_\Gamma(s)$ counts finite-index subgroups or, equivalently, transitive actions on finite sets. Edward Witten (1951–) considered the corresponding counting problem for linear representations. For a group G, he set

$$\zeta_G(s) = \sum_{\rho: G \to \mathrm{GL}(V) \text{ irreducible}} (\dim V)^{-s}.$$

For the classical group $G = \mathrm{SL}_2(\mathbb{C})$, one recovers the Riemann zeta function. Alex Lubotzky (1956–) considered ζ_G for lattices G in semisimple Lie groups, and derived functional equations as well as a description of the abscissa of convergence in terms of fundamental parameters of the ambient Lie group.

2. Zeta functions of graphs

Jean-Pierre Serre (1926–) remarked that Ihara's zeta function from §1.8 may be interpreted purely in terms of graphs. Indeed, the homogeneous space $\mathcal{H}$ from (4) is a $(q+1)$-regular tree, which can be constructed as follows. Recall some notation from §1.8: we have $G = \mathrm{PGL}_2(\mathbb{K})$, $U = \mathrm{PGL}_2(\mathcal{O})$, and note $[G_1 : U] = q + 1$. Choose left coset representatives $\pi_0, \dots, \pi_q \in G_1$ such as, if $\mathfrak{M} = \pi\mathcal{O}$, the matrices $\left(\begin{smallmatrix} \pi & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & x \\ 0 & \pi \end{smallmatrix}\right)$ for representatives x of $\mathcal{O}/\mathscr{M}$. Finally, connect in $\mathcal{H}$ the vertices γU and $\gamma U \pi_i$ by an edge, for all $\gamma U \in \mathcal{H}$ and all $i \in \{0, \dots, q\}$.

The group G then acts on the tree $\mathcal{H}$ by graph isometries, with U the stabilizer of a vertex and G_1 the stabilizer of an edge. The quotient $\Gamma \backslash G$ is then a finite graph, and the matrix $A = (a_{ij})$ is its adjacency matrix.

We then arrive at a purely graph-theoretical definition of the Ihara zeta function: Let $\mathscr{G}$ be a graph, consisting of vertices and edges. A *cycle* is a cyclically ordered sequence of oriented edges such that no edge backtracks on its predecessor; a *closed path* is a cycle with a marked starting point. More formally, one has $\mathscr{G} = (V, E)$, two sets of vertices, and oriented edges, respectively, and maps $E \ni e \mapsto e^+ \in V$, $E \ni e \mapsto e^- \in V$ and $E \ni e \mapsto \overline{e} \in E$ satisfying $\overline{\overline{e}} = e$ and $\overline{e}^+ = e^-$. Then a cycle is a sequence $\gamma = (e_0, e_1, \dots, e_n = e_0)$ considered up to cyclic permutation, with $e_i^+ = e_{i+1}^-$ and $\overline{e_i} \neq e_{i+1}$. The cycle's *length* is $|\gamma| = n$. It is *primitive* if it is not a proper power of a cycle, that is, there does not exist a subcycle $(e_0, \dots, e_m = e_0)$

with $m|n$ and $e_{mj+k} = e_k$ for all $j \in \{0, \dots, m/n\}, k \in \{0, \dots, m\}$. The Ihara zeta function is

$$\zeta_{\mathscr{G}}(T) = \prod_{\gamma \text{ primitive cycle}} (1 - T^{|\gamma|})^{-1}. \tag{5}$$

Following Serre's observation, the zeta function of the discrete subgroup Γ is then nothing but the Ihara zeta function of the graph $\Gamma\backslash G/U$.

In fact, Ruelle's zeta function from Section 1.7 may also, sometimes, be expressed in terms of graphs. For instance, let Σ be a finite set, and let $F \subset \Sigma^*$ be a finite set of "forbidden" words. The *subshift of finite type* is the dynamical system

$$\Lambda = \{\sigma \in \Sigma^{\mathbb{Z}} \colon \text{ no word in } F \text{ occurs as a subword of } \sigma\},$$

on which $\mathbb{Z}$ acts naturally by the shift map. Assume without loss of generality that no words of F are proper subwords of one another. Construct the graph whose vertex set V is the set of all proper prefixes of words in F, and put for all $\sigma_i \in \Sigma$ an edge from $\sigma_1 \dots \sigma_n$ to $\sigma_2 \dots \sigma_n \sigma_{n+1}$ labeled σ_{n+1} whenever both belong to V. Then Λ is in bijection with biinfinite paths in this graph, and periodic orbits of period n under the shift map are in bijection with cycles of length n (there is no notion of backtracking). A formal manipulation with exponentials and logarithms shows that Ruelle's zeta function is then exactly as in formula (5).

For now, there is just one graph zeta function that we may compute readily: that of the n-gon graph $\mathscr{C}_n$, which has two primitive cycles of length n, whence $\zeta_{\mathscr{C}_n}(T) = (1 - T^n)^{-2}$. Graphs with more than one cycle contain infinitely many primitive cycles, and it is not even clear that (5) converges anywhere—our first requirement for a function to be called a zeta function.

There are two remarkable determinantal formulas that show for any finite graph $\mathscr{G}$ that $\zeta_{\mathscr{G}}$ is the inverse of a polynomial. Consider first the two $E \times E$ matrices B and J called, respectively, *edge adjacency* and *inversion* matrices, defined by

$$B_{e,f} = \begin{cases} 1 & \text{if } e^+ = f^-, \\ 0 & \text{otherwise,} \end{cases} \qquad J_{e,f} = \begin{cases} 1 & \text{if } \overline{e} = f, \\ 0 & \text{otherwise.} \end{cases}$$

Consider next the two $V \times V$ matrices A and D called, respectively, *vertex adjacency* and *valency* matrices, defined by

$$A_{v,w} = \#\{e \in E \colon e^- = v \text{ and } e^+ = w\},$$
$$D_{v,w} = \begin{cases} \#\{e \in E \colon e^- = v\} & \text{if } \overline{v} = w, \\ 0 & \text{otherwise.} \end{cases}$$

Then one has the following identities, due to Hyman Bass (1932–) and Ki-ichiro Hashimoto (1950–):

$$\begin{aligned} \frac{1}{\zeta_{\mathscr{G}}(T)} &= \det(1 - (B - J)T) \\ &= (1 - T^2)^{\#E - \#V} \det(1 - AT + (D - 1)T^2). \end{aligned} \tag{6}$$

For example, the graph zeta function of the complete graph $\mathscr{K}_4$ is

$$\zeta_{\mathscr{K}_4}(T) = \frac{1}{(1 - T^2)^2 (1 - T)(1 - 2T)(1 + T + 2T^2)^3}.$$

Note that the exponent $\#E - \#V$ is the Euler characteristic of $\mathscr{G}$.

2.1. **Functional equation.** Let us move to our second criterion—a functional equation. Nothing should be expected from general graphs, but regular graphs do satisfy such a functional equation. Let $\mathscr{G}$ be a graph in which every vertex has valency $q+1$. Then

$$\zeta_{\mathscr{G}}(1/qT) = \left(q^2T^2\frac{1-T^2}{1-q^2T^2}\right)^{\#E-\#V}(qT^2)^{\#V}\zeta_{\mathscr{G}}(T),$$

as follows from a simple manipulation of (6). Note that there is no issue of analytic continuation, since $\zeta_{\mathscr{G}}$ is a rational function.

2.2. **The Riemann hypothesis?** There cannot be, strictly speaking, any equivalent of the Riemann hypothesis, because $\zeta_{\mathscr{G}}$ has no finite zeroes. However, the location of the poles of $\zeta_{\mathscr{G}}$ has a particular meaning.

The *spectrum* of a graph is the multiset of eigenvalues of its adjacency matrix A. If the graph $\mathscr{G}$ is $(q+1)$-regular, then $q+1$ is an eigenvalue of A, corresponding to the constant eigenvector on V. If $\mathscr{G}$ is bipartite, then $-(q+1)$ is also an eigenvalue. If $\mathscr{G}$ is connected, then these are the unique largest eigenvalues. The Alon–Boppana theorem asserts that, as $\#V$ increases, all the other eigenvalues tend to be constrained in $[-2\sqrt{q}, 2\sqrt{q}]$. A $(q+1)$-regular graph $\mathscr{G}$ is called *Ramanujan* if all its eigenvalues lie in this interval, except possibly $\pm(q+1)$. It is so-called because the first infinite family of examples (due to Lubotzky, Phillips, and Sarnak) was constructed using the Ramanujan Conjecture on coefficients of the Dedekind η-function—which by the way follows from the Weil conjectures. Their graphs are Cayley graphs of $\mathrm{PSL}_2(\mathbb{F}_q)$ for appropriate generating sets.

Now, again using (6), we see that the spectrum of A is intimately connected to the poles of $\zeta_{\mathscr{G}}$, and in fact $\mathscr{G}$ is Ramanujan if and only if the poles of $\zeta_{\mathscr{G}}(q^{-s})$ occur only on $\{\Re(s) = 1/2\}$ and at $\{0,1\}$.

2.3. **Special values.** The pole of $\zeta_{\mathscr{G}}(T)$ at $T=1$ has a special significance. At least if $\mathscr{G}$ is connected and nonbipartite, the order of the pole is precisely $\#E-\#V$, and the corresponding coefficient

$$\kappa(\mathscr{G}) = -\left(\lim_{T\to 1^-}(\#E-\#V)2^{\#E-\#V+1}(1-T)^{\#E-\#V+1}\zeta_{\mathscr{G}}(T)\right)^{-1}, \tag{7}$$

called the *complexity* of $\mathscr{G}$, is the number of maximal subtrees of $\mathscr{G}$.

2.4. **Infinite graphs.** Simple power series manipulations let one rewrite $\zeta_{\mathscr{G}}(T)$ in a form closer to (3):

$$\zeta_{\mathscr{G}}(T) = \exp\left(\sum_{k\geq 1}\frac{c_k}{k}T^k\right), \tag{8}$$

with c_k equal to the number of closed paths without backtracking (even between the beginning and the end). If furthermore the graph $\mathscr{G}$ is admits a vertex-transitive group of automorphisms, then there are as many closed paths starting from each vertex, and so c_k is divisible by $\#V$. Fix a vertex $* \in V$. Then the perhaps more familiar power series $\sum_{k\geq 0}(c_k/\#V)T^k$, with $c_0 = \#V$, counts the number of loops without backtracking based at $*$, and is sometimes known as the *Green function* for the backtracking-free random walk on $\mathscr{G}$.

Serre considered sequences of finite graphs $(\mathscr{G}_n)$, and called such a sequence *equidistributed* if the spectrum of $\mathscr{G}_n$, viewed as an atomic measure on $[-(q+1), (q+1)]$, converges weakly. He showed that this happens if and only if the power series $\zeta_{\mathscr{G}_n}(T)^{1/\#V_n}$ converges in $\mathbb{R}[[T]]$, and equivalently if and only if the coefficients $c_k(\mathscr{G}_n)/\#V_n$ converge for every $k \geq 0$.

If $\mathscr{G}$ is an infinite graph with basepoint $*$ and with vertex-transitive automorphism group, or more generally a graph that looks "sufficiently the same" about every vertex, then the coefficients $c_k/\#V$ still make sense, as the number of loops based at $*$ of length k and without backtracking. Therefore, the expression

$$\zeta_{\mathscr{G}}(T)^{1/\#V} = \exp\left(\sum_{k\geq 1} \frac{c_k}{\#Vk} T^k\right)$$

makes sense and defines the normalized zeta function of the infinite graph $\mathscr{G}$. If there be an exhaustion of $\mathscr{G}$ by finite subgraphs $\mathscr{G}_n$ whose boundary in $\mathscr{G}$ is asymptotically vanishing with respect to their number of vertices, then the functions $\zeta_{\mathscr{G}_n}(T)^{1/\#V_n}$ converge to $\zeta_{\mathscr{G}}(T)^{1/\#V}$.

3. Graphs and algebraic objects

In this brief review, we have not covered more of the exciting topics linking graph theory with other parts of mathematics; in particular, random matrices, various notions of chaos, and deeper analogies such as, for instance, the *Picard group* of a graph; this extends the numerical connection between the complexity $\kappa(\mathscr{G})$ of a graph (see (7)) with the class number of a number field, in the form of a *Riemann–Roch* theorem for graphs.

Another strand that we have not followed is the generalization of the zeta functions to L-functions. In the case of a number field, one weighs the zeta function with respect to an additive character on the ring of integers; in the case of a Riemann surface, or of a graph, one considers a cover and a representation of its deck transformation group, and weighs closed geodesics in the base according to the trace of their monodromy action on the cover.

The book by Audrey Terras presents the reader with a delightful stroll through a wealth of diverse mathematical topics. The text's numerous illustrations and colour graphics provide the reader with welcome distractions. In contrast with this survey, the book focuses mainly on zeta functions of graphs, and almost half of the book is devoted to the study of finite graph coverings and the relations they entail on respective zeta and L-functions.

One generally thinks of algebraic number theory as living in a very pure and abstract universe, in high contrast with the more murky world of combinatorics. The various links we exhibited testify on the contrary to the unity of mathematics.

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