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Cellular automata and groups, by Tullio Ceccherini-Silberstein and Michel Coornaert, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2010, xx+439 pp., ISBN 978-3-642-14033-4

1. Cellular automata on $\mathbb{Z}$

The theory of cellular automata is, at its very root, a simplification gone wonderfully wrong.

Let us consider a function $f : \mathbb{R} \to A$. We think of the domain as a space (we even call it, grandiosely, the *universe*) and the function f as a configuration on the space described by the values in A at every point of the space. The configuration values in A are considered somewhat abstractly, their nature is not important to us, they may be quantitative (temperature, electrical conductivity, etc.) or qualitative (name of a geographic region, prevalent voting preference in the region, etc.) The set of all configurations $A^{\mathbb{R}} = \{f \mid \mathbb{R} \to A\}$ is called the *space of configurations* (on $\mathbb{R}$ over A). Of course, the same configuration looks different from a different point of view. Thus, for each point $a \in \mathbb{R}$ in the space, we introduce a shift operator $\sigma_a : A^{\mathbb{R}} \to A^{\mathbb{R}}$, to account for the "change of coordinates". More precisely, for a configuration $f \in A^{\mathbb{R}}$, the shifted configuration $\sigma_a(f)$ is defined by

$$(\sigma_a(f))(x) = f(a + x), \tag{1}$$

for $x \in \mathbb{R}$. In addition to the shifts σ_a, $a \in \mathbb{R}$, which relate configurations over the space, one may consider relations between configurations over time (say, under ideal conditions, each temperature configuration on the line at the present moment uniquely determines the temperature configurations at any future moment of time). Thus, for each future point $t \in \mathbb{R}^+$ in time, we introduce a time operator $\tau_t : A^{\mathbb{R}} \to A^{\mathbb{R}}$ relating the configurations at time 0 to the configurations at time t (in particular, we are assuming that, for all $t, s \in \mathbb{R}^+$, $\tau_t \circ \tau_s = \tau_{t+s}$ and $\tau_0 = id$).

We attempt to simplify and model the above setup by making everything discrete, homogeneous, and local. Thus, we replace the space $\mathbb{R}$ by its discrete approximation $\mathbb{Z}$, and we consider only finite sets A (for instance, we group all possible values into finitely many levels). The elements of the finite set A are called *states* (or *symbols*) and A is called the *state set* (or the *alphabet*). It is common to require $|A| \geq 2$ in order to avoid the irrelevant, yet pesky, exceptions that accompany the trivial case. The space shift operator σ_1 uniquely determines all other shifts ($\sigma_n = (\sigma_1)^n$, for $n \in \mathbb{Z}$) and we denote it by σ. Thus, the configuration space is $A^{\mathbb{Z}}$, the space of bi-infinite sequences over some finite set A, with $|A| \geq 2$, and the shift operator $\sigma : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is given by $(\sigma(f))(x) = f(1+x)$, for $f \in A^{\mathbb{Z}}$ and $x \in \mathbb{Z}$.

We also replace the nonnegative time semiaxis $\mathbb{R}^+$ by its discrete approximation $\mathbb{N}$. The time operator τ_1 uniquely determines all other time operators ($\tau_n = (\tau_1)^n$ for $n \in \mathbb{N}$) and we denote it by τ. Moreover, we assume a certain homogeneity and localness (finite propagation) property of the time operator, which says that there exists a nonnegative integer m such that, for every point x, only the states of the configuration f within distance m from x affect the state $(\tau(f))(x)$ of the updated

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configuration $\tau(f)$ at x. The homogeneity (independence of x) and localness (independence of states far from x) constitute the defining property of cellular automata. More precisely, a *cellular automaton* is a map $\tau : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ for which there exists a nonnegative integer m such that, for all points $x, y \in \mathbb{Z}$ and all configurations $f, g \in A^{\mathbb{Z}}$, if $f(x+a) = g(y+a)$, for $a \in [-m, m] = \{-m, -m+1, \ldots, m-1, m\}$, then $(\tau(f))(x) = (\tau(g))(y)$.

An equivalent, and perhaps more intuitive, way of understanding the defining property of cellular automata is through the notion of a *local update rule*, which is a function $\mu : A^{[-m,m]} \to A$ that associates a single state to each possible pattern of states defined on the central interval $[-m, m]$ around 0. This local update rule, by extension, defines a cellular automaton $\tau : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ by

$$(\tau(f))(x) = \mu\left(\sigma_x(f)|_{[-m,m]}\right), \tag{2}$$

for $f \in A^{\mathbb{Z}}$ and $x \in \mathbb{Z}$. It is instructive to ponder the above formula and absorb what it actually says. In order to update the configuration f at x, we use the shift $\sigma_x(f)$ to look at the given configuration with x as the center of reference, we restrict our attention to the pattern of states $\sigma_x(f)|_{[-m,m]}$ within distance m from x, and we update the state at x by following the local update rule μ.

Maps $\tau : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ that commute with the shift σ are called $\mathbb{Z}$-*equivariant*. The defining property of cellular automata implies that they are $\mathbb{Z}$-equivariant. Indeed,

$$\begin{aligned}\sigma\tau(f)(x) &= \tau(f)(1+x)\\ &= \mu\left(\sigma_{1+x}(f)|_{[-m,m]}\right) = \mu\left(\sigma_x(\sigma(f))|_{[-m,m]}\right) = \tau\sigma(f)(x).\end{aligned}$$

This observation is part of one of the fundamental results in symbolic dynamics, the Curtis–Hedlund Theorem, characterizing cellular automata as exactly the maps $\tau : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ that are $\mathbb{Z}$-equivariant and are continuous with respect to the prodiscrete topology on $A^{\mathbb{Z}}$, i.e., the product topology induced by the discrete topology on A. Note that the prodiscrete topology turns the configuration space $A^{\mathbb{Z}}$ into a Cantor set (this is one of the instances where $|A| \geq 2$ is indeed a necessary assumption).

As a direct corollary of the Curtis–Hedlund Theorem, the image of every cellular automaton is (topologically) closed and it is invariant under the space shift σ. A subset of the configuration space that is closed and shift invariant is called a *subshift* (the configuration space itself is, confusingly, and perhaps regrettably, often called shift, or full shift). Subshifts can be characterized as follows. A subset S of the configuration space $A^{\mathbb{Z}}$ is a subshift if and only if there exists a set F of finite patterns of states that defines it (i.e., S consists exactly of the configurations in which none of the patterns from F appears anywhere in the configuration; this is why the patterns in the defining set F are usually called *forbidden patterns*). At its core, symbolic dynamics is the study of subshifts, their applications, representations, invariants (used to distinguish subshifts up to topological conjugacy), and surjective, continuous, and $\mathbb{Z}$-equivariant maps between them (generalizing the concept of cellular automata and called *factor maps*). Of particular interest, especially in applications, are the *shifts of finite type* (subshifts for which the defining set of finite forbidden patterns is finite) and *sofic shifts* (images of shifts of finite type under factor maps).

2. Cellular automata on $\mathbb{Z}^d$

It is natural to consider universes other than $\mathbb{Z}$ and cellular automata associated to such universes. One of the first universes that naturally comes to mind after considering the "line" $\mathbb{Z}$ is the "plane" $\mathbb{Z}^2$ and, more generally, the d-dimensional space $\mathbb{Z}^d$, for $d \geq 2$. All concepts and results mentioned above are applicable, *mutatis mutandis*, in this setting as well (local update rules; the Curtis–Hedlund Theorem; the notion of a subshift as a closed, space shift invariant subset; the characterization of subshifts as subsets defined by sets of forbidden patterns; and so on).

Nevertheless, despite the deep analogies and transfer of concepts and results, the change of scenery, once everything is considered in $\mathbb{Z}^d$, for $d \geq 2$, is so significant that Douglas Lind famously described it as a "trip through the Swamp of Undecidability" [Lin04]. This we're-not-in-Kansas-anymore declaration is due to the fact that even basic questions that one might ask about shifts of finite type on $\mathbb{Z}^2$ are undecidable, starting from the most fundamental one—Is the shift of finite type defined by a given set of finitely many finite forbidden patterns empty or not? The last question is directly related to the problem of Wang tiles.

Cellular automata on $\mathbb{Z}^2$ were studied by von Neumann, who was interested in their self-replicating capacity. They became very popular, even among non-mathematicians, with the advent of Conway's *Game of Life*, which is a cellular automaton on $\mathbb{Z}^2$ defined by a very simple local update rule, but with very rich and interesting behavior (not an uncommon situation among cellular automata, given that some cellular automata, even over $\mathbb{Z}$, happen to be universal Turing machines).

3. A few historical remarks

The beginnings of symbolic dynamics are usually attributed to the end of the 19th century work of Hadamard. In his study of systems of differential equations, Hadamard used sequences of symbols, with each symbol representing a region in a space X, to model the way a point moves around the space under iterations of a self-map on X (in the particular case, the space was a surface of negative curvature and the self-map was a discretized version of a geodesic flow). Such sequences are called *symbolic itineraries* and were soon used (and are used productively to this day) in many other situations to model, real or complex, continuous or discrete, dynamical systems. It is worth mentioning that Hadamard already found it necessary to consider shifts of finite type (*sans* the terminology, of course). Namely, he observed that not every possible sequence of symbols corresponds to the itinerary sequence of some point, and that the admissible sequences were exactly the sequences in which certain patterns of two consecutive symbols do not appear.

The "classical period", represented by the works of Hadamard, Poincaré, Birkhoff, Artin, and their contemporaries, ends roughly by the end of 1930s with the works of Morse and Hedlund [MH38], who were the first to fully break away from the view of symbolic dynamics as a simplification/modelling tool and its origins in the continuous/differentiable world and consider it on its own.

About ten years later, the use of shifts of finite type supporting Markov probability measures in the seminal work of Shannon [Sha48] on coding and communication definitely confirmed the "grown up" status of symbolic dynamics as an area of mathematics that both finds multiple applications and uses techniques from multiple areas of mathematics for its own development.

The list of areas in which symbolic dynamics nowadays finds applications, in mathematics as well as outside of it (ranging from physics and computer science to engineering and medicine), is extremely long and varied, justifying our playful opening reference to the theory of cellular automata as a simplification gone wonderfully wrong.

4. Cellular automata on arbitrary groups

The jump from cellular automata on $\mathbb{Z}^d$ to cellular automata on an arbitrary group G as a universe is natural and, for the most part, goes smoothly. Indeed, every group G provides a homogeneous space (it "looks the same" from any point) on which we may define configurations $f : G \to A$, for some finite state set A. Further, it comes with naturally defined space shifts operators $\sigma_g : A^G \to A^G$, for $g \in G$, based on translations within the group (translations within the space), that relate configurations with respect to different points and are defined, for $f \in A^G$ and $x \in G$, by

$$(\sigma_g(f))(x) = f(g^{-1}x). \tag{3}$$

A keen-eyed reader might notice that the definitions of space shifts (1) and (3) disagree, since the latter subtracts where the former adds. We may think of this discrepancy as yet another instance of the eternal dilemma "did our train just move or it was theirs" or, closer to the context, when we shift a configuration between the identity e and the element g in G, do we want the new configuration to "look" near e as the old one near g, or to "look" near g as the old one near e? Definition (3) is compatible with the latter choice. However, this still does not explain why one would make the switch from the choice made in (1). One practical reason, dictated by the historical context of the origins of symbolic dynamics—origins that can be traced to areas of mathematics in which functions are almost exclusively written on the left and, accordingly, left group actions are preferred—is that (3) ensures that the shift action $\sigma : G \to \mathsf{Sym}(A^G)$, defined by $g \mapsto \sigma_g$ is a left action of G on the configuration space A^G, regardless of whether G is abelian or not.

We again think of cellular automata as maps $\tau : A^G \to A^G$ that update every configuration f at x in G depending only on the pattern of states on a predefined finite "neighborhood" of x. More precisely, we may define a local update map $\mu : A^M \to A$, where M is any finite subset of the universe G, called the *memory set* (this set plays the role of the central interval $[-m, m]$), and we may extend it to a cellular automaton $\tau : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ by setting

$$(\tau(f))(x) = \mu\left(\sigma_{x^{-1}}(f)|_M\right)$$

for $f \in A^G$ and $x \in G$ (compare with (2) and notice again the adjustment involving the inverse, forced by definition (3)).

An analog of the Curtis–Hedlund Theorem holds in the general case and characterizes the cellular automata on a group G over the finite alphabet A as the maps $\tau : A^G \to A^G$ that are continuous with respect to the prodiscrete topology on A^G and are G-equivariant (i.e., $\tau\sigma_g = \sigma_g\tau$, for all $g \in G$). Many other classical notions in symbolic dynamics lift nicely and easily to the general case, as do many results, but there are notable exceptions, and the book under review builds aptly around a few of the more involved and interesting situations. Once again, our model goes wonderfully wrong, fails to stay in Kansas, leads to more than meets the eye, and provides a few lessons, this time in group theory.

5. The book, Garden of Eden, and a question of Gottschalk

There are at least two high quality books offering treatments of symbolic dynamics on $\mathbb{Z}$, one by Lind and Marcus [LM95] and one by Kitchens [Kit98]. The choices are more limited when one is interested in symbolic dynamics on groups other than $\mathbb{Z}$, especially if one wants to get past $\mathbb{Z}^2$, the second most popular choice. One of the exceptions is the book by Coornaert and Papadopoulos [CP93] in which symbolic dynamics on Gromov hyperbolic groups is considered.

The book under review is currently the only text of comparable size and level which devotes its attention to the general case of cellular automata on arbitrary groups from the very start, rather than as an afterthought. After the first chapter, in which the basics are established, the authors focus on their main topic of interest, surjunctivity, and take the reader on an illuminating journey that starts with basic definitions and ends with presentations of the latest research results in the area.

The notion of surjunctivity was introduced in 1973 by Gottschalk [Got73] and is the cellular automata version of the property "injective-implies-surjective", a type of property that reappears in many settings throughout mathematics. A group G is called *surjunctive* if every injective cellular automaton on G is surjective. The question of Gottschalk [Got73] asking if all groups are surjunctive is still open.

A chapter in the book on residually finite groups is followed by a chapter in which surjunctivity is proved for locally residually finite groups, then a chapter on amenable groups is followed by a chapter in which surjunctivity is proved for locally residually amenable groups, and a chapter on sofic groups, a common generalization of residually finite and amenable groups, ends with a proof that all sofic groups are surjunctive (the last result is due to Gromov [Gro99] and Weiss [Wei00]).

The parts of the material in which various classes of groups and their basic properties are introduced may be used independently from the material focusing on cellular automata. For instance, the author of this review has used the carefully written and current presentation of amenable groups, which includes most standard characterizations, along with complete proofs that these characterizations are equivalent, in a graduate course that had nothing to do with symbolic dynamics.

A recent characterization of amenability in terms of cellular automata is also provided, namely, a group G is amenable if and only if every surjective cellular automaton on G is pre-injective (a cellular automaton $\tau : A^G \to A^G$ is pre-injective if, for any two configurations f and g, the equality $\tau(f) = \tau(g)$ implies that either $f = g$ or f and g disagree on an infinite subset of G). The forward direction of this characterization is a corollary of the Garden of Eden Theorem, proved in full generality, for all amenable groups, by Ceccherini-Silberstein, Machì, and Scarabotti [CSMS99], extending the classical works of Moore and Myhill (see [Bur70]) concerning Garden of Eden patterns on $\mathbb{Z}^2$. The converse is due to Bartholdi [Bar10]. Recall that Garden of Eden pattern for a given cellular automaton is a finite pattern of states without a predecessor, i.e., a pattern that does not appear in the image of the automaton.

In the last chapter, the authors add more structure and consider configuration spaces V^G in which the alphabet V is a vector space (not necessarily finite). The definition of cellular automata $\tau : V^G \to V^G$ is adjusted to this setting and, in addition to the homogeneity and localness, their linearity is required (note that V^G is naturally a vector space). The authors confirm the Kaplansky Stable Finiteness Conjecture for group algebras of sofic groups (a result previously established by Elek and Szabó [ES04] by using embeddings in simple continuous von Neumann regular rings) in terms of linear cellular automata. Namely, one easily observes that the group algebra $\mathbb{F}[G]$ of a group is stably finite if and only if, for all $n > 0$, every injective linear cellular automaton $\tau : (\mathbb{F}^n)^G \to (\mathbb{F}^n)^G$ is surjective, and the authors prove that this is indeed the case for all sofic groups (this is a linear version of Gromov–Weiss surjunctivity theorem). The chapter ends with a reformulation of Kaplansky Zero Divisor Conjecture in terms of linear cellular automata. Namely, for a torsion free group G and a field $\mathbb{F}$, the group algebra $\mathbb{F}[G]$ has no zero divisors if and only if every nonzero linear cellular automaton $\tau : \mathbb{F}^G \to \mathbb{F}^G$ is pre-injective.

The book is a great read and a great resource. It is decidedly self-contained, modulo basic understanding of topology, analysis, and group theory, as well as readiness to peek into the ten appendices, with topics ranging from nets, filters, and ultrafilters in topological spaces, to symmetric groups, free groups, to accounts of Banach–Alaoglu Theorem, Markov–Kakutani Fixed Point Theorem, and Hall Marriage Theorem (and to its more general finite-to-one version, Hall Harem Theorem). The basic material is supported by a trove of exercises (over 300) and notes at the end of each chapter, and the book ends with a selection of 17 open problems.

References

[Bar10] Laurent Bartholdi, *Gardens of Eden and amenability on cellular automata*, J. Eur. Math. Soc. (JEMS) **12** (2010), no. 1, 241–248, DOI 10.4171/JEMS/196. MR2578610 (2011e:05282)

[Bur70] Arthur W. Burks (ed.), *Essays on cellular automata*, University of Illinois Press, Urbana, Ill., 1970. MR0299409 (45 #8457)

[CP93] Michel Coornaert and Athanase Papadopoulos, *Symbolic dynamics and hyperbolic groups*, Lecture Notes in Mathematics, vol. 1539, Springer-Verlag, Berlin, 1993. MR1222644 (94d:58054)

[CSMS99] T. G. Ceccherini-Silberstein, A. Machì, and F. Scarabotti, *Amenable groups and cellular automata*, Ann. Inst. Fourier (Grenoble) **49** (1999), no. 2, 673–685 (English, with English and French summaries). MR1697376 (2000k:43001)

[ES04] Gábor Elek and Endre Szabó, *Sofic groups and direct finiteness*, J. Algebra **280** (2004), no. 2, 426–434, DOI 10.1016/j.jalgebra.2004.06.023. MR2089244 (2005d:16041)

[Got73] Walter Gottschalk, *Some general dynamical notions*, Recent advances in topological dynamics (Proc. Conf. Topological Dynamics, Yale Univ., New Haven, Conn., 1972; in honor of Gustav Arnold Hedlund), Lecture Notes in Math., vol. 318, Springer, Berlin, 1973, pp. 120–125. MR0407821 (53 #11591)

[Gro99] M. Gromov, *Endomorphisms of symbolic algebraic varieties*, J. Eur. Math. Soc. (JEMS) **1** (1999), no. 2, 109–197, DOI 10.1007/PL00011162. MR1694588 (2000f:14003)

[Kit98] Bruce P. Kitchens, *Symbolic dynamics*, Universitext, Springer-Verlag, Berlin, 1998. One-sided, two-sided and countable state Markov shifts. MR1484730 (98k:58079)

[Lin04] Douglas Lind, *Multi-dimensional symbolic dynamics*, Symbolic dynamics and its applications, Proc. Sympos. Appl. Math., vol. 60, Amer. Math. Soc., Providence, RI, 2004, pp. 61–79. MR2078846 (2005f:37038)

[LM95] Douglas Lind and Brian Marcus, *An introduction to symbolic dynamics and coding*, Cambridge University Press, Cambridge, 1995. MR1369092 (97a:58050)

[MH38] Marston Morse and Gustav A. Hedlund, *Symbolic Dynamics*, Amer. J. Math. **60** (1938), no. 4, 815–866, DOI 10.2307/2371264. MR1507944

[Sha48] C. E. Shannon, *A mathematical theory of communication*, Bell System Tech. J. **27** (1948), 379–423, 623–656. MR0026286 (10,133e)

[Wei00] Benjamin Weiss, *Sofic groups and dynamical systems*, Sankhyā Ser. A **62** (2000), no. 3, 350–359. Ergodic theory and harmonic analysis (Mumbai, 1999). MR1803462 (2001j:37022)

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