BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 51, Number 2, April 2014, Pages 367–372
S 0273-0979(2013)01429-0
Article electronically published on September 23, 2013

Collected papers of C. S. Seshadri. Volume 1. Vector bundles and invariant theory, Vikraman Balaji, V. Lakshmibai, M. Pavaman Murthy, and Madhav V. Nori (Editors), Hindustan Book Agency, New Delhi, India, 2012, xxiv+1008 pp., ISBN 978-93-80250-17-5 (2 volume set)

Collected papers of C. S. Seshadri. Volume 2. Schubert geometry and representation Theory, Vikraman Balaji, V. Lakshmibai, M. Pavaman Murthy, and Madhav V. Nori (Editors), Hindustan Book Agency, New Delhi, India, 2012, xxvi+633 pp., ISBN 978-93-80250-17-5 (2 volume set)

1. Introduction

It would probably take several volumes to describe the enormous impact of Seshadri's work contained in these two volumes of his collected papers. His impressive achievements in the last several decades span a large variety of diverse topics, including moduli of vector bundles on curves, geometric invariant theory, representation theory, and Schubert calculus. We briefly address here the main results of his work presented in the volumes under review.

2. The early work

In the nineteenth century the theory of divisors on a compact Riemann surface was developed, mainly by Abel and Riemann. In modern algebraic geometry, divisors modulo linear equivalence are identified with line bundles. So, this theory is the same as the theory of line bundles. It can also be generalized to higher dimensions. Seshadri's early contributions are a sequence of important exposés on divisors in algebraic geometry written in the Chevalley seminar. This was followed by his construction of the Picard variety of a complete variety X, i.e., giving a canonical structure of a group variety to the group of Cartier divisors on X algebraically equivalent to 0 [Ses62]. The construction uses, together with the ideas of Chevalley, descent theory of Cartier for purely inseparable coverings and existence of a moduli for a rational map of a smooth curve into a commutative group variety (in the sense of Rosenlicht). This work paved way for further results on the representability of Picard functors by J. P. Murre and others.

3. Serre's problem

The most well-known contribution of Seshadri from the late 1950s is probably his ingenious short solution of Serre's problem in two variables. In his paper on algebraic coherent sheaves [Ser55, p.243], Serre posed the following question:

> *Serre's problem: Let P be a projective module over the polynomial ring $k[X_1, \dots, X_n]$ in n variables over a field k. Is then P a free module?*

Being locally free, a projective module over the polynomial ring $k[X_1, \dots, X_n]$ can be identified with a vector bundle on the affine space k^n. Hence Serre's problem

2010 *Mathematics Subject Classification.* Primary 01A75 (14-06).

can be rephrased as follows:

Are all (algebraic) vector bundles on k^n trivial?

Certainly, Serre's problem has an affirmative answer if $n = 1$, since $k[X]$ is a principal ideal domain. This was the only case for which the answer was known when Serre raised this question. Seshadri found an affirmative answer in case $n = 2$.

Theorem 3.1 ([Ses58]). *Vector bundles over the affine plane k^2 are trivial. In fact, if A is a principal ideal domain, then finitely generated projective modules over $A[X]$ are free.*

More generally, he later proved that if k is an algebraically closed field, then every algebraic vector bundle over $C \times k$, the product of a smooth curve with an affine line, is obtained by pullback of an algebraic vector bundle on C by the projection $C \times k \to C$ [Ses59]. This means that if A is the ring of functions on a smooth curve, then every finitely generated projective module over $A[X]$ is isomorphic to $Q \otimes_A A[X]$ for some finitely generated projective module Q over A.

This first step taken by Seshadri drew much attention. It attracted several mathematicians to work on this and related problems and to generate a tremendous amount of work in the next few decades. The next big leaps forward came fifteen years later. M. P. Murthy and J. Towber gave an affirmative answer to Serre's problem for $n = 3$ [MT74] and k algebraically closed. Finally, D. Quillen [Qui76] and A. A. Suslin [Suz76] independently showed that Serre's problem had an affirmative solution for all n, settling the problem completely.

Serre's problem had many applications, including Serre's construction and Kronecker's question. For an affine variety $X \subset \mathbb{A}^n$, let $I(X)$ be the ideal of all polynomials vanishing on X, and let $\mu(I(X))$ be the minimum number of generators of $I(X)$. Serre gave a method by which $\mu(I(X))$ can be determined in some cases provided Serre's problem has affirmative solution. Generally, $\mu(I(X))$ is bigger than the number of equations needed to define X set theoretically. Freeness of projective modules also plays an important role in settling the following question (attributed to Kronecker): Can any curve C in $\mathbb{A}^n$ be defined set theoretically by $n - 1$ equations? By Krull's principal ideal theorem, this is the least possible such number. In case of a smooth curve C, this question was settled in the affirmative by D. Ferand for $n = 3$ [Szp79, p. 75] and by N. Mohan Kumar for $n \geq 4$ [Kumar78, Corollary 5, p. 235]. In the singular case a positive answer in positive characteristic is due to Cowsik and Nori [CN78]. To my knowledge, the question is open in general.

4. Mumford's geometric invariant theory and moduli spaces

David Mumford's deep work in geometric invariant theory in the early 1960s was a revolution in classical invariant theory. Seshadri had also turned his attention to the problem of constructing a quotient of an algebraic variety X by a group variety G. Existence of such quotients is a far more difficult problem in algebraic geometry than in analytic or differential geometry. Seshadri showed that if X is normal, then the obstruction to constructing an algebraic geometric orbit space comes from a finite group action [Ses63I], and in the case when G acts freely, he gave sufficient conditions for existence of a quotient. In the case when G is an Abelian variety with a free action satisfying certain conditions, he showed that the orbit space exists as an algebraic variety [Ses63II] and $X \to X/G$ is a locally isotrivial principal G-bundle. Seshadri's most recent contribution to invariant theory (with P. Sastry) is an algebraic geometric proof of Mumford's conjecture on reductivity and geometric reductivity [SaSes2011].

Vector bundles on the projective line were classified by Grothendieck; they are direct sums of line bundles. A complete classification of vector bundles on elliptic curves was done by Atiyah [At57]. The case of vector bundles on higher genus curves turned out to be much more difficult and needed a whole new machinery. In his talk at the International Congress of Mathematicians in 1962, Mumford announced his Geometric Invariant Theory (GIT). He outlined how GIT can be used to solve various moduli problems—moduli of curves, Abelian varieties, and of vector bundles on curves. For every vector bundle E on a smooth curve, Mumford defined the slope of E to be the rational number

$$\mu(E) = \frac{\text{degree}\,(E)}{\text{rank}\,E}.$$

He defined the vector bundle to be stable (resp., semistable) if for every proper subbundle $F \subset E$, one has $\mu(F) < \mu(E)$ (resp., $\mu(F) \le \mu(E)$). He went on to construct the moduli space of stable vector bundles of fixed rank r and degree d as a quasi-projective variety.

5. Work of M. S. Narasimhan and C. S. Seshadri

The seminal work of M. S. Narasimhan and C. S. Seshadri in the late 1960s began a new era in the theory of vector bundles on a compact Riemann surface X. André Weil [W38] had shown that a vector bundle on a smooth curve X is associated to a representation of the fundamental group of the curve if and only if all its indecomposable components have degree zero. Narasimhan and Seshadri went a step further proving the following deeper result.

Theorem 5.1 ([NarSes65]).

(1) *A vector bundle of degree zero on X is stable if and only if it is associated to an irreducible unitary representation of the fundamental group of X.*

(2) *A vector bundle is associated to a unitary representation of the fundamental group if and only if it is polystable; i.e., it is a direct sum of stable vector bundles of degree zero.*

This beautiful work showed that the moduli space of stable holomorphic vector bundles of fixed rank r and degree 0 (resp., degree $d \neq 0$) can be identified, as a topological space, with the space of conjugacy classes of irreducible unitary representations of the fundamental group $\pi_1(X)$ (resp., Fuchsian groups). It has various generalizations (both in algebraic and differential geometry): to vector bundles and principal G-bundles on curves and higher dimensional varieties (Hitchin–Kobayashi correspondence), parabolic bundles, generalized parabolic bundles, and the list goes on (see [Bh2003] for a detailed account). It had tremendous impact on the theory of vector bundles, as well as on many other areas, including symplectic geometry, 4-dimensional topology, and mathematical physics.

On the physics side, Maxwell's equations governing electro-magnetic phenomena in empty space can be put into Langrangian form by introducing a one form A_μ called an electromagnetic potential. Yang and Mills had constructed non-Abelian version of the electromagnetic Langrangian by introducing matrix valued

A_μ. Atiyah and Bott showed that irreducible unitary representations realise the Morse theoretic minimum of the Yang–Mills functional, thus relating the work of Narasimhan and Seshadri to the Young–Mill theory. This pioneering work lead to several fruitful research directions. Construction of the moduli space of stable vector bundles as symplectic quotients helped to obtain many deep results on the geometry, cohomology, and topology of these moduli spaces.

6. Compactified moduli spaces of vector bundles

The moduli spaces of stable vector bundles are noncompact. Seshadri introduced the notion of a semistable vector bundle on a smooth curve and constructed projective moduli spaces of equivalence classes of semistable vector bundles (of fixed rank and degree) under certain equivalence. These spaces gave a natural compactification of the moduli spaces of stable bundles. As remarked by Mumford, Seshadri's construction is a perfect representative example and a forerunner of all GIT constuctions of compactifications in a whole range of moduli problems. Introducing different notions of semistability, Maruyama and Gieseker constructed different moduli spaces of vector bundles (and torsionfree sheaves) on smooth surfaces and later on higher dimensional smooth varieties.

Seshadri and Mehta's construction of the moduli spaces of parabolic bundles had an additional feature, namely the choice of polarization [MS80]. Parabolic bundles are vector bundles with parabolic structures at finitely many points. The parabolic structure at a point consists of a flag of vector subspaces of the fibre of the bundle at that point and real numbers called weights attached to the flag. To construct the moduli, Seshadri needed to make a suitable choice of polarization depending on the weights. This paved way to later constuctions of moduli spaces of vector bundles with various additional structures like framed bundles, generalized parabolic bundles, etc. Seshadri and Mehta associated parabolic bundles to unitary representations of the fundamental group of the open Riemann surface. Recently, Balaji and Seshadri have generalized them to parahoric bundles which are torsors under Bruhat–Tits group schemes (not necessarily semisimple).

7. Vector bundles on singular curves

When a smooth curve degenerates to a singular curve, the Jacobian (the variety of line bundles) of the smooth curve degenerates to the compactified Jacobian (the variety of torsionfree sheaves of rank 1) of the singular curve. Such degenerations have been studied classically. A lot of work on Picard schemes was done in modern times as well by Kleiman, Altman, Kleppe, Oda, Seshadri, D'Souza, and others. However, not much was known for bundles of higher ranks on singular curves. M. S. Narasimhan and P. E. Newstead constructed a projective moduli space of torsionfree sheaves on an integral singular curve [Ne78]. Seshadri generalized it to nonintegral curves introducing a notion of semistability depending on polarizations on different components of the curve [Ses82], the construction having good specialisation property. These constructions give nonnormal compactifications of the moduli spaces of vector bundles on singular curves.

Almost a decade later, this reviewer introduced generalized parabolic bundles which gave a very effective tool to study bundles on a singular curve by relating them to bundles on the normalization of the curve [Bh92]. This provided normal compactifications of the moduli spaces of vector bundles on singular curves. In a couple of interesting papers, degenerations of moduli spaces of vector bundles on curves were studied by Seshadri and Nagaraj [NagSes97], [NagSes99] generalising also results of Gieseker in rank 2. The classification of G-bundles on singular curves turned out to be much more difficult. It was done (more generally on singular varieties) by this reviewer [Bh2004].

8. Schubert geometry and representation theory

In the beginning of the the second volume of the collected works, Seshadri himself has given a beautiful overview of his Standard Monomial Theory (SMT). His exposition nicely describes the development of the theory over the years, explains the theory in detail, and touches upon its applications. An interested reader should certainly read it; I shall only say a few things about it.

The Standard Monomial Theory, initiated by Seshadri in the early 1970s and developed by Seshadri, V. Lakshmibai, and C. Musili in a series of long papers, constructs nice bases for finite dimensional representations of a semisimple algebraic group G or more generally for Demazure modules. This enabled them to formulate LS conjectures giving indexing of the SMT bases. This led to a new character formula for the representations which was quite different from the Weyl character formula. SMT can also be used to study the decomposition into irreducible components of tensor products of irreducible G-modules.

SMT also leads to a concrete determination of the ideal (or ideal sheaf) of a Schubert variety embedded in the generalized flag variety associated with the group. Hence, it provides a very effective and systematic tool for studying the geometry of Schubert varieties. Consequently, it has several applications in the study of the geometry of the Schubert variety. It is useful for the determination of Cohen–Macaulay property, types of singularities, and the singular locus of the Schubert variety. It was also used to prove vanishing of higher cohomologies of line bundles on Schubert varieties, which was in fact the initial aim of SMT. Later, some of these were also achieved by A. Ramanathan and V. Mehta using the new technique of Frobenius splitting.

A number of interesting classes of varieties can be regarded as open subsets of Schubert varieties. These include determinantal varieties, varieties of complexes, varieties associated to certain quivers, and varieties defined by rings of invariants in classical invariant theory. Hence, SMT has applications to the study of these varieties as well.

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