BOOK REVIEWS

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Homogeneous spaces and equivariant embeddings, by D. A. Timashev, Encyclopaedia of Mathematical Sciences, Vol. 138, Invariant Theory and Transformation Groups, 8, Springer, Heidelberg, 2011, xxii+253 pp., ISBN 978-3-642-18398-0

1. Basic concepts

Group actions on manifolds, algebraic varieties and other sets, and geometric objects have played an important role in geometry, analysis, representation theory, and physics for a long time. The book by D. A. Timashev is a welcome survey of old and new results on actions of algebraic groups on algebraic varieties. Basic facts on algebraic groups, homogeneous spaces, and equivariant embeddings are presented. Then the discussion concentrates more on a detailed description of special and interesting classes where deeper results can be obtained. Those cases include symmetric spaces, weakly symmetric spaces, spherical varieties, spaces of lower rank and complexity, and so-called wonderful varieties. Classification of several categories of homogeneous spaces are given. The book is about the algebraic side of group actions, but to complement the book, we take a more analytic viewpoint.

Let us start by recalling some basic concepts. Let G be a group, and let $\mathbf{X}$ be a set. A G-action on $\mathbf{X}$ is a map $G \times \mathbf{X} \to \mathbf{X}$, often written as $(a, x) \mapsto a \cdot x = \ell_a(x)$, such that $a \mapsto \ell_a$ is a group homomorphism from G into the group of bijections on $\mathbf{X}$. If a G-action on $\mathbf{X}$ is given, then $\mathbf{X}$ is said to be a *G-set*. For a fixed $x \in \mathbf{X}$ the map $a \mapsto a \cdot x$ is the *orbit map* and $G \cdot x$ is a *G-orbit*. The subgroup $G^x = \{a \in G \mid a \cdot x = x\}$ is the *stabilizer* of x in G. We say that $\mathbf{X}$ is *homogeneous* if $\mathbf{X} = G \cdot x$ for some point, $x \in \mathbf{X}$. In that case $\mathbf{X} = G \cdot x$ for all points x in $\mathbf{X}$. If $\mathbf{X}$ and $\mathbf{Y}$ are two G-sets, then a map $\varphi : \mathbf{X} \to \mathbf{Y}$ is G-equivariant, or a G-map, if $\varphi(a \cdot x) = a \cdot \varphi(x)$ for all $a \in G$ and all $x \in \mathbf{X}$. If $\mathbf{X}$ is homogeneous, then $gG^x \mapsto g \cdot x$ defines a G-equivariant bijection $G/G^x \simeq \mathbf{X}$. If G is a Lie group acting smoothly on a manifold $\mathbf{X}$ (always assumed separable), then G^x is closed for all $x \in \mathbf{X}$ and G/G^x is a manifold. The group G acts smoothly on G/G^x and G/G^x is isomorphic to $\mathbf{X}$ as a manifold and as a G-space. In the algebraic category, the variety $G/G^x \simeq \mathbf{X}$ if the map $aG^x \mapsto a \cdot x_o$ is separable. To explain the title of the book we can now say that $G/H \overset{\varphi}{\hookrightarrow} \mathbf{X}$ is an equivariant embedding if $\mathbf{X}$ is a normal variety with a G-action and φ is G-equivariant map with an open image in $\mathbf{X}$. In that case it is common to identify G/H with $\varphi(G/H) \subset \mathbf{X}$. Finally, if $\tau : G \to G$

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is a group homomorphism, then we set $G^\tau = \{a \in G \mid \tau(a) = a\}$ and if H is a subgroup of G, then $\mathbf{X}^H = \{x \in \mathbf{X} \mid (\forall h \in H)\ h \cdot x = x\}$.

2. Early history

The connection between geometry and groups was the focus point of Felix Klein's famous Erlanger Program in 1872. Loosely interpreted, the program states that a geometric structure is determined by its symmetry group. The Euclidean (metric) geometry corresponds to the group of rigid motions $G = \mathrm{O}(n) \ltimes \mathbb{R}^n$ where (a, b) acts on $\mathbb{R}^n$ by $v \mapsto a(v) + b$. It is also the invariance group of classical Newtonian physics. The invariance group for the Lorentzian geometry and Einstein's theory of special relativity is the Poincaré group $\mathrm{O}(1, d)^+ \ltimes \mathbb{R}^{1,d}$, where in general $\mathrm{O}(p, q)$ stands for the invariance group of the symmetric form $\beta_{p,q}((x_1, \ldots, x_{p+q})) = x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2$. The $^+$ in $\mathrm{O}(1, d)^+$ indicates the subgroup mapping the forward light cone into itself. We refer to [BH07] for detailed exposition of the role of symmetries in geometry.

Felix Klein left Leipzig in 1886 and was replaced by his friend Sophus Lie. who, in collaboration F. Engel, was creating what later became known as the theory of Lie groups. The three volumes on transformation groups [L88] were published in the years 1888 to 1893. This work was mostly local with the exception of some results related to classical groups. The global theory was developed later, in particular by H. Weyl [PW27, W24, W25, W26] and É. Cartan [C13, C23, C26, C27a, C29, C30, C32, C35]. Particularly noteworthy is Cartan's classification of Riemannian symmetric spaces, first local in [C27a] and then global in [C27b]. The modern geometric presentation of Cartan's work was first published in 1962 as the original version of [H78]. An excellent presentation of this history from the point of view of representation theory can be found in [B98] and [M92]. Other good references to learn about the subject include [CC52, B01, W11].

Cartan's work was analytic in nature, and even if some work was done earlier, at this point the algebraic viewpoint was less the focus of the development than it was later. As turning points one can mention the work of Chevalley [CT46, C47], A. Borel [B56], and A. Weil [W55a, W55b], just to name a few. Today this is a blooming field of active research.

3. Symmetric spaces

We will always assume that G is a real Lie group, in most cases contained in a complexification $G_\mathbb{C}$. One of the interesting aspects that opens up as we consider real homogeneous spaces is that one can study different real forms of the same complex homogeneous space or open G-orbits in a complex homogeneous space $G_\mathbb{C}/P_\mathbb{C}$. We concentrate our discussion on the important class of symmetric spaces and spherical, or multiplicity-free, spaces.

The best understood multiplicity-free spaces, both with respect to structure theory as well as representation theory and analysis, are the Riemannian symmetric spaces. Let H be a closed subgroup of G. Then (G, H) is a *symmetric pair*, and H is a symmetric subgroup, if there is an involution $\tau : G \to G$ such that $(G^\tau)_o \subseteq H \subseteq G^\tau$ where the index $_o$ indicates the connected component containing the identity element. If $H \subset G$ is symmetric, then we say that $\mathbf{X} = G/H$ is a symmetric space. If G is semisimple without compact factors and G^τ is a maximal compact subgroup of G, then τ is called a Cartan involution of G and we usually write θ instead of τ and $K = G^\theta$. In this case K is always connected if G is connected and G/K is simply connected. It is a *Riemannian symmetric space of the noncompact type*.

A simple example of a symmetric space is the group itself. Let H be a Lie group, and let $G = H \times H$. Identify H with the diagonal subgroup $\{(a,a) \mid a \in H\} = G^\tau \subset G$, where the involution τ is given by $\tau(a,b) = (b,a)$. The map $(a,b)H \mapsto ab^{-1}$ defines a G isomorphism $G/H \simeq H$ intertwining the action of G on G/H with the right-left action $(a,b)\cdot x = axb^{-1}$ on H.

Another well-known example is the upper half-plane $\mathbb{C}^+ = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$. In this case $G = \mathrm{SL}(2,\mathbb{R})$ acts by fractional linear transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}\,.$$

The stabilizer of $i = \sqrt{-1}$ is the subgroup $\mathrm{SO}(2)$ and $\mathbb{C}^+ \simeq \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$. The Cayley transform $z \mapsto (z-i)/(z+i)$ identifies $\mathbb{C}^+$ with the unit disc $D = \{z \in \mathbb{C} \mid |z| < 1\}$ giving us a simple example of a *bounded symmetric domain* or a *hermitian symmetric space*. Finally the open cone of positive-definite $n\times n$-matrices is symmetric and isomorphic to $\mathrm{GL}(n,\mathbb{R})/\mathrm{O}(n)$ under the map $a\mathrm{O}(n) \mapsto aa^T$.

It is often convenient to realize G/G^τ as a closed submanifold of G using the smooth map $\eta(aG^\tau) = a\tau(a)^{-1}$. The image of η is contained in the set $\{a \in G \mid \tau(a) = a^{-1}\}$ and if G/G^τ is connected, then the image is exactly the connected component containing the unit element e. The G-action on G/G^τ is transformed into the action $a \cdot b = ab\tau(a)^{-1}$. As an example, if $\tau : \mathrm{GL}(n,\mathbb{R}) \to \mathrm{GL}(n,\mathbb{R})$ is the involution $a \mapsto (a^{-1})^T$, then $\eta(\mathrm{GL}(d,\mathbb{R})/\mathrm{O}(d))$ is the above realization of the positive definite matrices with the $\mathrm{GL}(n,\mathbb{R})$-action $a\cdot x = axa^T$.

On the Lie algebra level, we say that $(\mathfrak{g},\mathfrak{h})$ is a symmetric pair if there exists a nontrivial involutive Lie algebra homomorphism $d\tau : \mathfrak{g} \to \mathfrak{g}$ such that $\mathfrak{h} = \mathfrak{g}^{d\tau}$. If (G,H) is a symmetric pair corresponding to the involution τ, then the derived involution $d\tau : \mathfrak{g} \to \mathfrak{g}$ is an involution on $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{s}$ where $\mathfrak{h} = \mathfrak{g}^{d\tau}$ and $\mathfrak{s}$ is the (-1)-eigenspace of $d\tau$. Furthermore, $\mathfrak{h}$ is the Lie algebra of H, $T(\mathbf{X}) \simeq G \times_H \mathfrak{s}$, and $T^*(\mathbf{X}) \simeq G \times_H \mathfrak{s}^*$ as homogeneous vector bundles. If G is semisimple, then the Killing form restricted to $\mathfrak{s}$ defines an H-invariant nondegenerate bilinear form on $\mathfrak{s}$ and hence a pseudo-Riemannian structure on $\mathbf{X}$. It is Riemannian if and only if $H/Z(G)$, $Z(G)$ the center of G, is compact.

Let $x_o = eH$, and let $\mathbf{X} = G/H$. Then $\mathbf{X}$ is a globally symmetric space with symmetry given by $s_{b\cdot x_o}(a\cdot x_o) = \eta(b)\cdot(\tau(a)\cdot x_o)$. In the realization of G/G^τ inside G we have the simpler expression $s_x(y) = x\tau(y)x$. As $d\tau|_{\mathfrak{s}} = -\mathrm{id}$ we have $(ds_x)_x = -\mathrm{id}_{T_x\mathbf{X}}$ for all $x \in \mathbf{X}$.

Symmetric spaces always come—at least locally—in pairs. To see this, we note that $(\mathfrak{g},\mathfrak{h})$ is a symmetric pair if and only if there exists an $\mathfrak{h}$-invariant complement $\mathfrak{s}$ of $\mathfrak{h}$ in $\mathfrak{g}$ such that $[\mathfrak{s},\mathfrak{s}] \subseteq \mathfrak{h}$. Define $\mathfrak{g}^c = \mathfrak{h} \oplus i\mathfrak{s}$. Then $\mathfrak{g}^c$ is a Lie algebra and $(\mathfrak{g}^c,\mathfrak{h})$ is a symmetric pair. Denote by G^c the simply connected Lie group with Lie algebra $\mathfrak{g}^c$ and note that τ defines an involution on G^c. As G^c is simply connected it follows that $(G^c)^\tau = \tilde{H}$ is connected ([L69, Thm. IV.3.4]) and $\mathbf{X}^c = G^c/\tilde{H}$ is simply connected. If $\mathbf{X}$ is a Riemannian symmetric spaces of the noncompact type, then the dual space $\mathbf{X}^c$ is a simply connected symmetric space of the *compact type*. In the case where G is contained in a complex Lie group $G_{\mathbb{C}}$ with Lie algebra $\mathfrak{g}_{\mathbb{C}}$ such that τ defines an involution on $G_{\mathbb{C}}$, there is a more "natural" way to define $\mathbf{X}^c$. Assume that $H = G^\tau$, and let $H_{\mathbb{C}} = G_{\mathbb{C}}^\tau$. Let G^c be the analytic subgroup of $G_{\mathbb{C}}$ corresponding to $\mathfrak{g}^c$, and let $\tilde{H} = (G^c)^\tau = G^c \cap H_{\mathbb{C}} = G^c \cap G$. Then $\mathbf{X}$ and $\mathbf{X}^c = G^c/\tilde{H}$ are two *transversal* real forms of the complex symmetric space $\mathbf{X}_{\mathbb{C}} = G_{\mathbb{C}}/H_{\mathbb{C}}$.

As mentioned earlier, the Riemannian symmetric spaces were classified by E. Cartan in 1927. As a step toward this classification, one proves the following theorem:

Theorem. *Assume that $\mathbf{X}$ is a Riemannian symmetric space. Denote by $\mathrm{I}(\mathbf{X})$ the group of isometries of $\mathbf{X}$, $G = \mathrm{I}(\mathbf{X})_o$ and K a stabilizer of a point x in $\mathbf{X}$. Then, in the compact-open topology, G is a finite-dimensional Lie group, K is a compact subgroup and (G, K) is a symmetric pair.*

- *If $\mathbf{X}$ is of noncompact type, then K is a connected maximal compact subgroup of G, $\mathbf{X}$ is simply connected, and $\mathfrak{s} \to \mathbf{X}$, $X \mapsto \mathrm{Exp}(X) = \exp(X) \cdot x$ is an analytic diffeomorphism. All maximal compact subgroups of G are conjugated by elements in G.*
- *If $\mathbf{X}$ is of the compact type, then K is not necessarily connected and hence there might be several locally isomorphic spaces corresponding to the symmetric pair $(\mathfrak{g}, \mathfrak{k})$.*
- *The space $\mathbf{X}$ is of noncompact type if and only if $\mathbf{X}^c$ is of compact type.*

An important class of irreducible Riemannian symmetric spaces are bounded symmetric domains or symmetric spaces of hermitian type classified by É. Cartan in [C35]. They are characterized by $\mathfrak{k}$ not being semisimple or, equivalently, $\mathfrak{s}_{\mathbb{C}}$ not irreducible as a K-representation. In fact the center $\mathfrak{z}$ of $\mathfrak{k}$ is one dimensional and there exists an element $z \in \mathfrak{z}$ such that $\mathrm{ad}(z)|_{\mathfrak{s}}$ defines a G-invariant complex structure on $\mathbf{X}$. Let $\mathfrak{s}_+$ denote the $+i$-eigenspace of $\mathrm{ad}(z)$ in $\mathfrak{s}_{\mathbb{C}}$. Then $\mathfrak{s}_+$ is abelian, $K_{\mathbb{C}}S_+$ ($S_+ = \exp \mathfrak{s}_+$) is a maximal parabolic subgroup in $G_{\mathbb{C}}$, and G/K is an open G-orbit in $G_{\mathbb{C}}/K_{\mathbb{C}}S_+$. It can also be realized as a bounded symmetric domain in $\mathfrak{s}_+$.

Those spaces play important role in both geometry and representation theory. They were studied by A. Borel [B52] and A. Borel and Lichnerowicz [BL52], but the realization as bounded domain was later given by Harish-Chandra [HC55]; see also [H78], Chapter VI. For more information, in particular on the orbit structure of the closure, see [KW65a, KW65b, S80, W69, W72].

The semisimple symmetric spaces were classified by M. Berger [B57] who also described the natural representation of H on $\mathfrak{s}$.

4. Examples

For $n \in \mathbb{N}$ and $1 \leqslant p \leqslant d/2$, let $U = \mathrm{SO}(n)$ and $q = n - p$. Define

$$\tau(a) = \mathrm{I}_{p,q} a \mathrm{I}_{p,q}\,, \text{ where } \mathrm{I}_{p,q} = \begin{pmatrix} \mathrm{I}_p & 0 \\ 0 & -\mathrm{I}_q \end{pmatrix}.$$

The derived involution $d\tau$ is given by the same formula. We have

$$U^\tau = \mathrm{S}(\mathrm{O}(p) \times \mathrm{O}(q)) = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \,\middle|\, a \in \mathrm{O}(p), b \in \mathrm{O}(q) \text{ and } \det a \det b = 1 \right\} \tag{4.1}$$

and $U/U^\tau = \mathrm{Gr}_p(\mathbb{R}^n)$, the space of p-dimensional subspaces of $\mathbb{R}^n$. We note that $K_o = \mathrm{SO}(p) \times \mathrm{SO}(q) \neq K$ so U/K_o and $\mathrm{Gr}_p(\mathbb{R}^n)$ are two locally isomorphic, but not diffeomorphic, symmetric spaces both corresponding to the same symmetric pair $(\mathfrak{so}(n), \mathfrak{so}(p) \times \mathfrak{so}(q))$. For $p = 1$ this corresponds to the double covering $\mathrm{S}^{n-1} \to \mathbf{P}(\mathbb{R}^n)$.

The noncompact dual of U is locally isomorphic to $G = \mathrm{SO}_o(p,q) = \mathrm{O}_o(p,q)$. Let $\mathbf{X}$ be the set of x in $\mathrm{Gr}_p(\mathbb{R}^n)$ such that $\beta_{p,q}$ is positive definite on x. The group G acts transitively on $\mathbf{X}$, the stabilizer of the point $x_o = \{(x_1, \ldots, x_p, 0, \ldots, 0)^T \mid x_1, \ldots, x_p \in \mathbb{R}\}$ is K_o, and $\mathbf{X} = G/K_o$. The closure $\overline{\mathbf{X}}$ of $\mathbf{X}$ in $\mathrm{Gr}_p(\mathbb{R}^n)$ is a compactification of $\mathbf{X}$ and $\mathbf{X} \hookrightarrow \overline{\mathbf{X}}$ is a G-equivariant embedding.

The spaces $\mathrm{Gr}_p(\mathbb{R}^n)$ and $\mathrm{SO}_o(p,q)/\mathrm{SO}(p)\times\mathrm{SO}(q)$ are hermitian symmetric spaces if and only if $p = 2$. On the other hand the complex analogues $\mathrm{Gr}_p(\mathbb{C}^n) = \mathrm{SU}(n)/\mathrm{S}(\mathrm{U}(p) \times \mathrm{U}(q))$ and

$$\mathbf{Y} = \mathrm{SU}(p,q)/\mathrm{S}(\mathrm{U}(p) \times \mathrm{U}(q)) \simeq \{z \in \mathrm{M}(p \times q, \mathbb{C}) \mid I_q - z^*z > 0\}$$

are always hermitian symmetric spaces.

Denote by $\sigma : \mathbf{Y} \to \mathbf{Y}$ the complex conjugation. Then $\mathbf{Y}^\sigma = \mathbf{X}$ realizing $\mathbf{X}$ as the bounded real domain $\{x \in \mathrm{M}(p \times q, \mathbb{R}) \mid I_q - x^T x > 0\}$. The closure in $M_{p,q}(\mathbb{R})$ is a compactification of $\mathbf{X}$. It is a fact that all classical Riemannian symmetric spaces, sometimes extended by $\mathbb{R}^+$, and some exceptional symmetric spaces can be compactified in this way [Ó00]. There are several other natural compactifications of symmetric spaces. We will give examples of wonderful compactifications in Section 6. For good references on compactifications of symmetric spaces, see [BJ06, O78, Sa60] or Chapter 4 in [S84].

The above mentioned Grassmanians $\mathrm{Gr}_p(\mathbb{K}^d)$, $\mathbb{K} = \mathbb{R}, \mathbb{C}$, fits into other interesting classes of homogeneous spaces. The group $\mathrm{SL}(n, \mathbb{K})$ acts transitively on $\mathrm{Gr}_p(\mathbb{K}^d)$ and the stabilizer of the basepoint x_o is the maximal parabolic subgroup

$$\begin{aligned} P &= \left\{ \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \,\middle|\, a \in \mathrm{GL}_p, b \in \mathrm{GL}_q, \det a \det b = 1 \text{ and } x \in \mathrm{M}_{p\times q} \right\} \\ &\simeq \underbrace{\mathrm{S}(\mathrm{GL}(p,\mathbb{K}) \times \mathrm{GL}(q,\mathbb{K}))}_{L} \ltimes \underbrace{\mathbb{K}^{pq}}_{N} . \end{aligned}$$

Thus, $\mathrm{Gr}_p(\mathbb{K}^d)$ is an example of a *symmetric R-space* [KN64]. All the compact symmetric R-spaces U/K have the property that the noncompact dual G/K_o can be realized as open orbits in U/K.

The R-spaces are closely related to other classes of symmetric spaces. There exists an involution τ on G such that $L = G^\tau$. The symmetric space G/L is an example of a para-hermitian symmetric space studied in detail by S. Kaneyuki [K85, K03]. The para-hermitian structure is given by an element $x \in \mathfrak{s}$ such that $\mathrm{ad}(x)$ has the eigenvalues 0, ± 1, the group L is the centralizer of x in G, and $d\tau = e^{\pi i \mathrm{ad}(x)}$ and $G/L \simeq \mathrm{Ad}\,(G)x \subset \mathfrak{g} \simeq \mathfrak{g}^*$ is a coadjoint orbit. The symmetric space $G/G^{\theta\tau}$ has a G-invariant partial ordering and is sometimes called a *noncompactly causal* symmetric space [HÓ97, Ó91]. Finally, taking the c-dual with respect to the involution $\tau\theta$ gives a symmetric space of hermitian type G^c/K^c such that $K^c_{\mathbb{C}} S_+ \cap G = P$ and $K^c_{\mathbb{C}} = L_{\mathbb{C}}$.

5. Commutative spaces

In the category of Riemannian homogeneous spaces, multiplicity-free spaces go under several different names in the literature, such as weakly symmetric spaces (introduced in 1956 by A. Selberg [S56]) or commutative spaces. A good source of information is [W07].

A Riemannian space $\mathbf{X}$ is said to be weakly symmetric if for every $x \in \mathbf{X}$ and every nonzero $\xi \in T_x\mathbf{X}$ there exists an isometry $s_{x,\xi} : \mathbf{X} \to \mathbf{X}$, which might depend on ξ and not only the point x, such that

$$s_{x,\xi}(x) = x \text{ and } (ds_{x,\xi})_x(\xi) = -\xi\,. \tag{5.1}$$

In particular, every symmetric space is weakly symmetric. Denote again by $\mathrm{I}(\mathbf{X})$ the group of isometries of $\mathbf{X}$, $G = \mathrm{I}(\mathbf{X})_o$ and $K = G^x$ the stabilizer of $x \in \mathbf{X}$. Then K is compact and $\mathbf{X} \simeq G/K$.

If the group G acts on a set $\mathbf{Y}$, then G acts on functions on $\mathbf{Y}$ by $a \cdot f(y) = f(a^{-1} \cdot y)$. For an algebraist, the important space of functions would be the space of regular functions or algebraically defined sheaves of functions on $\mathbf{Y}$. Invariant measures and hence L^2-spaces, are not algebraic objects because algebraic groups are in general not even Hausdorff in the Zariski topology. In analysis the topology starts to play a role, and the representations are realized in locally convex topological vector spaces often constructed using invariant or quasi-invariant measures. It could be a long section by itself to discuss some of the aspects of this interesting field, but the only thing we need to know here is that if $H \subset G$ is reductive, then there always exists a G-invariant measure on $\mathbf{X}$. Hence the left translation defines a unitary representation on $L^2(\mathbf{X})$. Harmonic analysis on general weakly symmetric spaces is not as well developed as in the case of Riemannian symmetric spaces, and several natural questions, including the explicit form of the Plancherel density, are open in general.

If K is compact, then (G, K) is said to be a Gelfand pair, or $\mathbf{X} = G/K$ is a commutative space, if the Banach convolution algebra $L^1(\mathbf{X})^K$ is commutative (or equivalently the algebra $C_c^\infty(\mathbf{X})^K$ is commutative). There are other equivalent definitions; one of the more algebraic definitions is that the action of G on $\mathbb{C}[\mathbf{X}]$ is multiplicity-free. The corresponding analytic definition is that the unitary representation of G in $L^2(\mathbf{X})$ is multiplicity-free.

Assume that $G \subset G_{\mathbb{C}}$, $H \subset G$ a compact subgroup, $B \subset G_{\mathbb{C}}$ is a Borel subgroup, and (π, V) is an irreducible finite-dimensional representation of $G_{\mathbb{C}}$. We have the following equivalent conditions for $\mathbf{X}$ to be commutative (see [T11] for the algebraic side and [W07] for the analytic part):

Theorem. *Let the notation be as above. Then the following are equivalent:*

(1) $\mathbf{X}$ *is a commutative space.*
(2) $\mathbf{X}$ *is weakly symmetric.*
(3) *The representation of* G *in* $C_c(\mathbf{X})$ *is multiplicity-free.*
(4) *The representation of* G *in* $\mathbb{C}[\mathbf{X}_{\mathbb{C}}]$ *is multiplicity-free.*
(5) $\mathbf{X}_{\mathbb{C}}$ *is spherical.*
(6) $\dim V^{H_{\mathbb{C}}} \leqslant 1$.
(7) *If* $\chi : H_{\mathbb{C}} \to \mathbb{C}^*$ *a homomorphism, then*

$$\dim\{v \in V \mid (\forall h \in H_{\mathbb{C}})\ \pi(h)v = \chi(h)v\} \leqslant 1\,.$$

(8) $\mathbb{C}[\mathbf{X}]^B = \mathbb{C}$.
(9) $H_{\mathbb{C}}$ *has an open orbit in* $G_{\mathbb{C}}/B$.
(10) *There exists a* $g \in G_{\mathbb{C}}$ *such that* $\mathfrak{g}_{\mathbb{C}} = \mathrm{Ad}\,(g)\mathfrak{h}_{\mathbb{C}} \oplus \mathfrak{b}$.
(11) *There exists a Borel subalgebra* $\widetilde{\mathfrak{b}}$ *in* $\mathfrak{g}_{\mathbb{C}}$ *such that* $\mathfrak{g}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} \oplus \widetilde{\mathfrak{b}}$.
(12) *For any* $G_{\mathbb{C}}$ *variety* $\mathbf{Y}$ *and* $y \in \mathbf{Y}^{H_{\mathbb{C}}}$, *the closure of the* $G_{\mathbb{C}}$ *orbit* $G_{\mathbb{C}} \cdot y$ *has finitely many* $G_{\mathbb{C}}$*-orbits.*

(13) *For any $G_{\mathbb{C}}$ variety $\mathbf{Y}$ and $y \in \mathbf{Y}^{H_{\mathbb{C}}}$, the closure of the $G_{\mathbb{C}}$ orbit $G_{\mathbb{C}} \cdot y$ has finitely many B-orbits.*

Examples of weakly symmetric spaces that are not symmetric include some circle bundles over hermitian symmetric space; see [M71, N97] and [W07, Section 12.3]. The classification of reductive weakly symmetric spaces was completed by Yakimova [Y02]; see also [Y05]. This history and individual contributions are well documented in both [T11] and [W07].

6. The Book

In the following we assume for simplicity that G is a complex semisimple or reductive algebraic group or a Lie group, although the book treats algebraic groups over more general fields. Let $B \subset G$ be a Borel subgroup (a maximal solvable subgroup), and assume that $\mathbf{X}$ is an algebraic G-variety.

Basic definitions and results on algebraic homogeneous spaces can be found in Chapter 1. In the second chapter the *rank* and the *complexity* are introduced. The rank of a homogenous space G/H is zero if and only if H is a parabolic subgroup or, equivalently, G/H is projective. The complexity is the minimal codimension of B-orbits in $\mathbf{X}$. In particular, $\mathbf{X}$ is of complexity zero, or a spherical variety, if and only if $\mathbf{X}$ contains an open B orbit or, equivalently, such that G/B contains an open H-orbit. *Horospherical* varieties, symmetric varieties, and *toroidal* varieties are examples of spherical varieties.

Let (π, V) be a finite-dimensional representation of G. Denote by $\mathbf{P}(V)$ the projective space of lines in V. Then G acts on $\mathbf{P}(V)$ by $g \cdot [v] = [\pi(g)v]$, where $v \mapsto [v]$ denotes the canonical projection $V \to P(V)$. For example, $\mathrm{SL}(2, \mathbb{C})$ acts transitively on $\mathbf{P}(\mathbb{C}^2)$ and the the stabilizer of $[1, 0] \in \mathbf{P}(\mathbb{C})$ is the Borel subgroup B of upper-triangular matrices. Thus the rank of $\mathbf{P}(\mathbb{C}^2)$ is zero. The B-orbit through the point $[0, 1]$ is $\{[z, w] \mid z, w \in \mathbb{C} \ w \neq 0\}$, which is open and dense. Thus the complexity is zero. Similar arguments hold for $\mathrm{SL}(n, \mathbb{C})$ acting on $\mathbf{P}(\mathbb{C}^n)$.

The two invariants, the rank and the complexity of $\mathbf{X}$, are described in terms of the geometry of the cotangential bundle $T^*\mathbf{X}$. General methods for calculating complexity and ranks, developed by Knop [K90] and Panyushev [P99], are discussed. The classification of homogeneous spaces of complexity $\leqslant 1$ is given in the case where $\mathbf{X} = G/H$ with H reductive. A different approach to the classification is given later in the book.

Two important problems related to equivariant embeddings $\varphi : G/H \to \mathbf{X}$ are (a) the classification of possible embeddings and (b) the description of the orbit structure of the boundary $\overline{\varphi(G/H)} \backslash \varphi(G/H)$. The most important embeddings are those where the boundary has finitely many orbits. In analysis on homogeneous spaces, one prefers embeddings where $\overline{\varphi(G/H)}$ is compact. One then studies boundary value maps defined on natural function spaces on G/H and Poission-type transforms from function spaces on the boundary to function spaces on G/H, [KMOT78, O78, O83, OS80] and [S84, Ch. 6].

In Chapter 3 the important Luna–Vust theory of embeddings [LV83] is discussed. The chapter includes an extension of this theory by D. A. Timashev [T97]. A very readable introduction to the theory based on embeddings of spherical varieties can be found in [K91]. The chapter ends with an overview on intersection theory. In the next chapter the theory of G-invariant valuations of the function field of a G-variety $\mathbf{X}$ is developed.

Chapter 5 is devoted to homogeneous spaces and spherical varieties. Several special cases like weakly symmetric spaces, symmetric spaces, and Gelfand pairs are discussed. Various characterizations of spherical homogeneous spaces are proven, and the importance of those space in representation theory is well presented. This chapter contains a detailed discussion on *toroidal embeddings*, *wonderful varieties*, the *Demazure embedding* (or wonderful compactification), and combinatorial classification of spherical subgroups and wonderful varieties.

As an example of a wonderful compactification, let H be a semisimple Lie group and let $G = H \times H$ as in Section 3. Let (π, V) be a representation of H. Then G acts on $\mathrm{End}(V)$ by $(a, b) \cdot T = \pi(a)T\pi(b)^{-1}$. The operator T is H-invariant if and only if T is an H-intertwining operator. Furthermore, an irreducible representation of G has an H-fixed vector if and only if it is of the form $\mathrm{End}(V)$ for some irreducible representation of H. In that case, Schur's Lemma implies that $\mathrm{End}(V)^H = \mathbb{C}\,\mathrm{id}$, where id is the identity operator. Thus we have an embedding $G/H \hookrightarrow \mathbf{P}(\mathrm{End}(V))$ given by $gH \mapsto [g \cdot \mathrm{id}]$. The variety $\mathbf{X} = \overline{G \cdot [\mathrm{id}]}$ is compact and contains H as an open dense orbit. It is the wonderful compactification of H; see [EJ08].

For another example of a wonderful compactification, assume that $\mathbf{X} = G/H$ is spherical and that $H = N_G(H)$. Let $k = \dim \mathfrak{h}$. Then we have a G-equivariant map $\varphi : \mathbf{X} \to \mathrm{Gr}_k(\mathfrak{g})$ given by $aH \mapsto \mathrm{Ad}\,(a)\mathfrak{h}$. The closure of $\varphi(\mathbf{X})$ is the Demazure embedding ([T11, Prop. 30.7]).

The book is well and clearly written. Sometimes the explanation is quite short for a nonexpert, but then the reader can always find exact references to the original work. The book includes several important classifications of special classes of homogeneous spaces and equivariant embeddings. The material is often clarified by well-chosen examples.

Representation theory of reductive groups plays an important role throughout the book, which makes it a good source for spherical representations, decompositions of tensor products, and restrictions. This is a book on algebraic groups, so the representations are usually finite dimensional or on spaces of regular functions on G-varieties. Unitarity and L^2-spaces, important in analysis, play no role here.

The choice of topics is motivated by the author's interest and work, but the book discusses extensively the work of others, including Brion, Knop, Luna, Vinberg, Vust, and Yakimova. The book contains an extensive list of references, both to original research articles as well as to books and monographs. It is surely a welcome guide for nonexperts who want to enter the subject and a good reference for the specialist.

References

[B57] Marcel Berger, *Les espaces symétriques noncompacts*, Ann. Sci. École Norm. Sup. (3) **74** (1957), 85–177 (French). MR0104763 (21 #3516)

[BH07] William Barker and Roger Howe, *Continuous symmetry: from Euclid to Klein*, American Mathematical Society, Providence, RI, 2007. MR2362745 (2008k:51001)

[B52] Armand Borel, *Les espaces hermitiens symétriques* (French). Séminaire Bourbaki 1951/52.

[B56] Armand Borel, *Groupes linéaires algébriques*, Ann. of Math. (2) **64** (1956), 20–82 (French). MR0093006 (19,1195h)

[B98] Armand Borel, *Semisimple groups and Riemannian symmetric spaces*, Texts and Readings in Mathematics, vol. 16, Hindustan Book Agency, New Delhi, 1998. MR1661166 (2000e:53063)

[B01] Armand Borel, *Essays in the history of Lie groups and algebraic groups*, History of Mathematics, vol. 21, American Mathematical Society, Providence, RI, 2001. MR1847105 (2002g:01010)

[BJ06] Armand Borel and Lizhen Ji, *Compactifications of symmetric and locally symmetric spaces*, Mathematics: Theory & Applications, Birkhäuser Boston Inc., Boston, MA, 2006. MR2189882 (2007d:22030)

[BL52] Armand Borel and André Lichnerowicz, *Espaces riemanniens et hermitiens symétriques*, C. R. Acad. Sci. Paris **234** (1952), 2332–2334 (French). MR0048134 (13,986c)

[C13] Élie Cartan, *Les groupes projectifs qui ne laissent invariante aucune multiplicité plane*, Bull. Soc. Math. France **41** (1913), 53–96 (French). MR1504700

[C23] Élie Cartan , *Sur un théorème fundamental de M. H. Weyl.* J. Math. Pures Appl. (9) **2** (1923), 167–192.

[C26] Élie Cartan, *Sur une classe remarquable d'espaces de Riemann*, Bull. Soc. Math. France **54** (1926), 214–264 (French). MR1504900

[C27a] Élie Cartan, *Sur une classe remarquable d'espaces de Riemann. II*, Bull. Soc. Math. France **55** (1927), 114–134 (French). MR1504909

[C27b] Élie Cartan, *Sur certaines formes Riemanniennes remarquables des géométries à groupe fondamental simple*, Ann. Sci. École Norm. Sup. (3) **44** (1927), 345–467 (French). MR1509283

[C29] Élie Cartan, *Sur la détermination d'un système orthogonal complet dans un espace de Riemann symétrique clos.* Rend. Circ. Mat. Palermo **53** (1929), 217–252.

[C30] Élie Cartan, *La théorie finis et continus et l'Analysis Situs.* Mém. Sci. Math. **XLII**, Gauthier-Villars, Paris, 1930.

[C32] Élie Cartan, *Les espaces riemanniens symétriques.* Verh. Internat. Congr. Math. (1932), 152–161.

[C35] Élie Cartan, *Sur les domaines bornés homogeènes de l'espace de n variables complexes.* Abb. Math. Sem. Univ. Hamburg **11** (1935), 116–162.

[CC52] S. S. Chern and C. Chevalley, *Élie Cartan and his mathematical work*, Bull. Amer. Math. Soc. **58** (1852), 217–250.

[C47] Claude Chevalley, *Algebraic Lie algebras*, Ann. of Math. (2) **48** (1947), 91–100. MR0019603 (8,435d)

[CT46] Claude Chevalley and Hsio-Fu Tuan, *On algebraic Lie algebras*, Proc. Nat. Acad. Sci. U.S.A. **31** (1945), 195–196. MR0012279 (7,4a)

[EJ08] S. Evens and B. F. Jones, *On the wonderful compactification.* Lecture note, arXiv:0801.0456v1.

[HC55] Harish-Chandra, *Representations of semisimple Lie groups. IV*, Amer. J. Math. **77** (1955), 743–777. MR0072427 (17,282c)

[H78] Sigurdur Helgason, *Differential geometry, Lie groups, and symmetric spaces*, Pure and Applied Mathematics, vol. 80, Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1978. MR514561 (80k:53081)

[HÓ97] Joachim Hilgert and Gestur Ólafsson, *Causal symmetric spaces: geometry and harmonic analysis*, Perspectives in Mathematics, vol. 18, Academic Press Inc., San Diego, CA, 1997. MR1407033 (97m:43006)

[H97] A. Hurwitz, *Über die Erzeugung der Invarianten durch Integration.* Nachr. Kgl. Gew. Wiss. Göttingen, Math.-Phys. Klasse 1897, 71–90.

[K85] Soji Kaneyuki, *On classification of para-Hermitian symmetric spaces*, Tokyo J. Math. **8** (1985), no. 2, 473–482, DOI 10.3836/tjm/1270151228. MR827002 (88c:32045a)

[K03] Soji Kaneyuki, *Compactification of parahermitian symmetric spaces and its applications. II. Stratifications and automorphism groups*, J. Lie Theory **13** (2003), no. 2, 535–563. MR2003159 (2004k:32036)

[KMOT78] M. Kashiwara, A. Kowata, K. Minemura, K. Okamoto, T. Ōshima, and M. Tanaka, *Eigenfunctions of invariant differential operators on a symmetric space*, Ann. of Math. (2) **107** (1978), no. 1, 1–39, DOI 10.2307/1971253. MR485861 (81f:43013)

[K90] Friedrich Knop, *Weylgruppe und Momentabbildung*, Invent. Math. **99** (1990), 1–23.

[K91] Friedrich Knop, *The Luna-Vust theory of spherical embeddings*, Proceedings of the Hyderabad Conference on Algebraic Groups (Hyderabad, 1989), Manoj Prakashan, Madras, 1991, pp. 225–249. MR1131314 (92m:14065)

[KW65a] Adam Korányi and Joseph A. Wolf, *Realization of hermitian symmetric spaces as generalized half-planes*, Ann. of Math. (2) **81** (1965), 265–288. MR0174787 (30 #4980)

[KW65b] Joseph A. Wolf and Adam Korányi, *Generalized Cayley transformations of bounded symmetric domains*, Amer. J. Math. **87** (1965), 899–939. MR0192002 (33 #229)

[KN64] Shoshichi Kobayashi and Tadashi Nagano, *On filtered Lie algebras and geometric structures. I*, J. Math. Mech. **13** (1964), 875–907. MR0168704 (29 #5961)

[L26] H. Levy, *Forma canonica dei ds^2 per i quali si annullano i simboli di Riemann a cinque indici.* Rend. Acc. Lincei **3** (1926), 65–69.

[L88] S. Lie, *Theorie der Transformation Gruppen I, II, III.* Unter mitwirkung von F. Engels, Leipzig 1888, 1890, 1893.

[L69] Ottmar Loos, *Symmetric spaces. I: General theory*, W. A. Benjamin, Inc., New York-Amsterdam, 1969. MR0239005 (39 #365a)

[LV83] D. Luna and Th. Vust, *Plongements d'espaces homogènes*, Comment. Math. Helv. **58** (1983), no. 2, 186–245, DOI 10.1007/BF02564633 (French). MR705534 (85a:14035)

[M92] George W. Mackey, *The scope and history of commutative and noncommutative harmonic analysis*, History of Mathematics, vol. 5, American Mathematical Society, Providence, RI, 1992. MR1171011 (93g:22006)

[M71] Hans Maaß, *Siegel's modular forms and Dirichlet series*, Lecture Notes in Mathematics, vol. 216, Springer-Verlag, Berlin, 1971. MR0344198 (49 #8938)

[N97] Hiêú Nguyêñ, *Weakly symmetric spaces and bounded symmetric domains*, Transform. Groups **2** (1997), no. 4, 351–374, DOI 10.1007/BF01234540. MR1486036 (98k:53067)

[Ó91] G. Ólafsson, *Symmetric spaces of Hermitian type*, Differential Geom. Appl. **1** (1991), no. 3, 195–233, DOI 10.1016/0926-2245(91)90001-P. MR1244444 (94g:22034)

[Ó00] Gestur Ólafsson, *Analytic continuation in representation theory and harmonic analysis*, Global analysis and harmonic analysis (Marseille-Luminy, 1999), Sémin. Congr., vol. 4, Soc. Math. France, Paris, 2000, pp. 201–233 (English, with English and French summaries). MR1822362 (2002d:22014)

[ÓP12] G. Ólafsson and A. Pasquale, *Ramanujan's Master theorem for the hypergeometric Fourier transform on root systems.* Preprint, arXiv:1211.0024.

[O78] Toshio Ōshima, *A realization of Riemannian symmetric spaces*, J. Math. Soc. Japan **30** (1978), no. 1, 117–132. MR0477175 (57 #16716)

[O83] Toshio Ōshima, *A definition of boundary values of solutions of partial differential equations with regular singularities*, Publ. Res. Inst. Math. Sci. **19** (1983), no. 3, 1203–1230, DOI 10.2977/prims/1195182027. MR723467 (86d:35009)

[OS80] Toshio Ōshima and Jirō Sekiguchi, *Eigenspaces of invariant differential operators on an affine symmetric space*, Invent. Math. **57** (1980), no. 1, 1–81, DOI 10.1007/BF01389818. MR564184 (81k:43014)

[P99] D. I. Panyushev, *Complexity and rank of actions in invariant theory*, J. Math. Sci. **95** (1999), 1925–1985.

[PW27] F. Peter and H. Weyl, *Die Vollständigkeit der primitiven Darstellungen einer geschlossenen kontinuierlichen Gruppe*, Math. Ann. **97** (1927), no. 1, 737–755, DOI 10.1007/BF01447892 (German). MR1512386

[Sa60] Ichirô Satake, *On representations and compactifications of symmetric Riemannian spaces*, Ann. of Math. (2) **71** (1960), 77–110. MR0118775 (22 #9546)

[S80] Ichirô Satake, *Algebraic structures of symmetric domains*, Kanô Memorial Lectures, vol. 4, Iwanami Shoten, Tokyo, 1980. MR591460 (82i:32003)

[S84] Henrik Schlichtkrull, *Hyperfunctions and harmonic analysis on symmetric spaces*, Progress in Mathematics, vol. 49, Birkhäuser Boston Inc., Boston, MA, 1984. MR757178 (86g:22021)

[S24] I. Schur, *Neue Anwendungen der Integralrechnung auf Probleme der Invariantentheorie. I, II.* Sitzungsber. Preuss. Akad. Wiss. Berlin 1924, 189–208, 297–321.

[S56] A. Selberg, *Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series*, J. Indian Math. Soc. (N.S.) **20** (1956), 47–87. MR0088511 (19,531g)

[T97] D. A. Timashëv, *Classification of G-manifolds of complexity* 1, Izv. Ross. Akad. Nauk Ser. Mat. **61** (1997), no. 2, 127–162, DOI 10.1070/im1997 v061n02ABEH000119 (Russian, with Russian summary); English transl., Izv. Math. **61** (1997), no. 2, 363–397. MR1470147 (98h:14058)

[T11] Dmitry A. Timashev, *Homogeneous spaces and equivariant embeddings*, Encyclopaedia of Mathematical Sciences, vol. 138, Springer, Heidelberg, 2011. Invariant Theory and Algebraic Transformation Groups, 8. MR2797018 (2012e:14100)

[W55a] André Weil, *On algebraic groups of transformations*, Amer. J. Math. **77** (1955), 355–391. MR0074083 (17,533e)

[W55b] André Weil, *On algebraic groups and homogeneous spaces*, Amer. J. Math. **77** (1955), 493–512. MR0074084 (17,533f)

[W24] H. Weyl, *Zur Theorie der Darstellung der einfachen kontinuierlichen Gruppen*, Sitzungsber. Preuss. Akad. Wiss. Berline, 1924, 338–345.

[W25] H. Weyl, *Theorie der Darstellung kontinuierlicher halbeinfacher Gruppen durch lineare Transformationen. I, II, II und Nachtrag*, Math. Zeitschr. **23** (1925), 271–309, **24** (1926), 328–376, 377–395. 789–791.

[W26] H. Weyl, *Zur Darstellungstheorie und Invariantenabzählung der Projektiven, der Komplex- und der Drehungsgruppe*, Acta Math. **48** (1926), no. 3-4, 255–278, DOI 10.1007/BF02565334 (German). MR1555225

[W69] Joseph A. Wolf, *The action of a real semisimple group on a complex flag manifold. I. Orbit structure and holomorphic arc components*, Bull. Amer. Math. Soc. **75** (1969), 1121–1237. MR0251246 (40 #4477)

[W72] Wolf, Joseph A., *The fine structure of hermitian symmetric spaces*, Symmetric Spaces (W. Boothby and G. Weiss, eds.), Marcel Dekker, New York, 1972. MR0404716 (53 #8516)

[W11] Joseph A. Wolf, *Spaces of constant curvature*, 6th ed., AMS Chelsea Publishing, Providence, RI, 2011. MR2742530 (2011j:53001)

[W07] Joseph A. Wolf, *Harmonic analysis on commutative spaces*, Mathematical Surveys and Monographs, vol. 142, American Mathematical Society, Providence, RI, 2007. MR2328043 (2008f:22008)

[Y02] O. S. Yakimova, *Weakly symmetric Riemannian manifolds with reductive isometry group*, Math. Sb. **195** (2004), no. 4, 143–160. MR1934062 (2004b:53076)

[Y05] Oksana Yakimova, *Gelfand pairs*, Bonner Mathematische Schriften [Bonn Mathematical Publications], 374, Universität Bonn Mathematisches Institut, Bonn, 2005. Dissertation, Rheinische Friedrich-Wilhelms-Universität Bonn, Bonn, 2004. MR2206354 (2006m:22018)

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