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SELECTED MATHEMATICAL REVIEWS

related to the paper in the previous section by
VAUGHN F. R. JONES, SCOTT MORRISON, AND NOAH SNYDER

MR0000898 (1,146g) 46.3X

v. Neumann, J.

On rings of operators. III.

Annals of Mathematics. Second Series **41** (1940), 94–161.

A factor is a ring of operators $\mathfrak{M}$ having the property that no element of it, except constant multiples of the identity, commute with every element of $\mathfrak{M}$. By means of a relative dimension function it has been established that five classes of factors I_n, I_∞, II_1, II_∞, III_∞ can be distinguished and that the first four classes are non-empty. [Cf. F. J. Murray and J. v. Neumann: On rings of operators, Ann. of Math. **37**, 116–229 (1936).] In the present work, examples of class III_∞ are given. If A is in $\mathfrak{M}$, the rank of $\mathfrak{M}$ is the relative dimension of its range. If A is of finite rank, it is possible to define the trace of A. Those A's of $\mathfrak{M}$ which are of finite rank form a linear multiplicative set for which $T_n(AB^*)$ is an inner product. If we complete this linear set with respect to the corresponding norm, we obtain what is called the Schmidt class. In case I_n, this set is $\mathfrak{M}$; in case I_∞, it corresponds to the operators of finite norm; in case II_1, it includes $\mathfrak{M}$, while in case III_∞, it consists simply of 0. Now if $E_1, E_2, \cdots, E_n$ are mutually orthogonal projections with $1 = E_1 + \cdots + E_n$, then $A_1 = \sum_{i=1}^n E_i A E_i$ is of finite norm if A is. Each E_i in turn can be expressed as a sum of mutually orthogonal projections $E_i = E_{i,\,1} + \cdots + E_{i,\,r}$, and for this finer subdivision there is an $A_2 = \sum E_{i,\,j} A E_{i,\,j}$. Continuing this process yields a sequence $A_1, A_2, \cdots$ with a limit (A); (A) is of finite norm if A is. Suppose that the E's are chosen so that, if F is a projection and not zero, (F) is not zero. Then if the Schmidt class is not simply 0, the set of (A)'s will contain non-zero elements of finite norm. If however no (A) is of finite norm, then $\mathfrak{M}$ must be a III_∞.

Next the examples of factors given in the previously cited joint paper are reexamined from another point of view. These examples are based on three elements, a set S, a measure function μ defined for subsets of S, and a group $\mathfrak{G}$ of transformations a of S, with certain properties one of which is that, if $S' \subset S$, then $\mu(aS') = \mu(S')$. In the present work, this condition is lightened to: $\mu(S') \neq 0$ implies $\mu(aS') \neq 0$. It is shown that certain of the resulting factors are such that there is no equivalent invariant measure and that these factors satisfy the condition given above for a III_∞. Two factors $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are said to form a factorization if the ring generated by them is the full set of operators on the space and if their intersection is the set of multiples of the identity. It is shown here that factorizations occur involving a case II and a case III, thus completing the proof of the statement that, in a factorization, any combination of cases may occur except those in which only one factor is in a case I.

From MathSciNet, December 2013

F. J. Murray

MR0094722 (20 #1234) 46.00

Dixmier, Jacques

Les algèbres d'opérateurs dans l'espace hilbertien (Algèbres de von Neumann). (French)

Cahiers scientifiques, Fascicule XXV.
Gauthier-Villars, *Paris*, 1957, vi+367 pp.

This book presents a thorough reworking and systematic development of the theory of weakly-closed *-algebras of bounded operators on a Hilbert space, which the author renames von Neumann algebras. Although only twenty years old, this has been one of the most active and fruitful areas of modern functional analysis, as the author's bibliography of nearly two hundred items attests. The present exposition divides into three large chapters: the first on the global theory, the second on the direct integral reduction theory, and a final one presenting those standard but more technical topics which do not fit into the pattern of development chosen by the author. A valuable part of the book are the numerous exercises, which present illuminating examples and sketch additional bits of theory.

The first chapter is probably the most important, since the global theory is the principal aspect of the general theory to be developed since the original von Neumann and von Neumann-Murray papers. Its goal is the global decomposition of a von Neumann algebra according to type and finiteness, and its essential tool is the theory of traces, upon which the key definitions are based. This procedure avoids the more difficult global dimension theory of projections (which is here relegated to the third chapter), but it does not yield the complete classification analysis.

The first two sections of the first chapter give the elementary definitions of the subject and the standard ways of forming new algebras from old: the reduced algebras generated by projections, direct sums of algebras and tensor products of algebras, in addition to the basic process of forming the commutator algebra $\mathcal{A}'$. The third section introduces the various topologies found to be useful in studying operator algebras—uniform, strong, weak, ultra strong and ultra weak—and investigates the ideas and improvements stemming from the original von Neumann theorem that the strong closure of a *-algebra $\mathcal{A}$ of operators is its double commutator algebra $\mathcal{A}''$. The author next treats positive linear functionals, the homomorphisms they induce, and the special functionals (normal, ultra weakly continuous) generated by elements of the underlying Hilbert space. The central theorem of this section is the complete answer to the question of to what extent a normal homomorphism between von Neumann algebras implies a spatial isomorphism of the underlying Hilbert spaces. Section five develops the elementary theory of Hilbert algebras, which can be regarded as the abstract presentation of the way in which the left and right regular representations are treated on locally compact groups. The group situation is discussed in the exercises. These algebras are of crucial importance to the present scheme of development because they provide their generated von Neumann algebras with plenty of traces, upon which, as mentioned above, the classification theory is made to depend. In section six, probably the major section of the book, the theory of traces is developed. In particular, the basic connection with Hilbert algebras is established, and the classification and decomposition according to finiteness accomplished. The next two sections effect the global analysis according to type, and the last section contains examples of the various kinds of factors.

The course of the second chapter of necessity runs closer to the original von Neumann treatment in its general outline, although there is much change in detail and improvement in exposition. After a preliminary development of the theory of the direct integral of a measurable field of Hilbert spaces and the notion of a decomposable operator, we have the notion of a diagonalizable operator (a direct integral of constant operators), and the theorem that a bounded operator is decomposable if and only if it commutes with the commutative algebra $\mathcal{Z}$ of all diagonalizable operators. The heart of the chapter is the material on decomposable algebras. Basic are the theorem (§ 3, Theorem 3) that if $\mathcal{A}$ and $\mathcal{A}'$ are both decomposable, into the families $\mathcal{A}(\zeta)$ and $\mathcal{A}'(\zeta)$ respectively, then $\mathcal{A}(\zeta)$ and $\mathcal{A}'(\zeta)$ commute for almost all ζ, and are complementary factors if $\mathcal{Z}$ is the center of $\mathcal{A}$; the theorem (§ 3, Lemma 1) that if the base space Z has a countable basis then the measurability of the field $\mathcal{A}'(\zeta)$ follows from that of $\mathcal{A}(\zeta)$—this is the point in the theory where von Neumann had to resort to the theory of analytic sets, and the present author still finds this necessary; and the theorem (§ 6, Theorem 2) that if $\mathcal{Z}$ is a commutative von Neumann algebra such that $\mathcal{Z}'$ is countably generated over $\mathcal{Z}$, then the underlying Hilbert space can be decomposed as a direct integral in such a way that $\mathcal{Z}$ becomes the algebra of all diagonalizable operators. These three theorems together imply the original von Neumann decomposition theorem, to the effect that a von Neumann algebra over a separable Hilbert space is (essentially) uniquely expressible with respect to its center as a direct integral of factors, and it is curious that the author does not explicitly state this corollary. The newer material contained in this chapter is mainly to be found in sections four and five, on the reduction theory of Hilbert algebras and traces, respectively.

The third, and most technical, chapter begins with the comparison theory of projections and the associated relative dimension theory, continuing the global classification theory which was not quite completed in the first chapter. It then takes up the trace mapping of a von Neumann algebra $\mathcal{A}$ into its center $\mathcal{Z}$, and discusses the factorization of a normal trace functional into one on $\mathcal{Z}$ composed with a trace mapping of $\mathcal{A}$ into $\mathcal{Z}$, as well as the existence and uniqueness of the trace mapping for semi-finite and finite algebras. There follow sections treating the "approximation theorem", the coupling operator between a von Neumann algebra and its commutator algebra, and hyperfinite factors. The chapter concludes with a demonstration of the equivalence of the trace definition of finiteness with the classical definition in terms of partial isometries. This proof requires the construction of a finite trace under the latter hypothesis, and here is found the analogue of the von Neumann-Murray theorem that the natural trace on a finite factor defined for self-adjoint T by integrating λ with respect to the dimension of the spectral projections of T, is additive. The above mentioned material on central mappings and the approximation theorem are needed for this final step in recovering the classical classification theory.

The book concludes with five short appendices presenting technical material needed in the first and second chapters.

From MathSciNet, December 2013

L. H. Loomis

MR1091619 (92b:57024) 57N10; 17B37, 57M25, 81R50

Reshetikhin, N; Turaev, V. G.

Invariants of 3-manifolds via link polynomials and quantum groups.

Inventiones Mathematicae **103** (1991), *no.* 3, 547–597.

The authors construct new topological invariants of compact oriented 3-manifolds and of framed links in such manifolds. The invariant of (a link in) a closed oriented 3-manifold is a sequence of complex numbers parametrized by complex roots of 1. For a framed link in the three-sphere the terms in the sequence are equal to the values of a Jones polynomial of the link evaluated in the corresponding roots of 1. Thus, for links in the three-sphere, the invariants in this paper are essentially equivalent to the Jones polynomial.

In this context the Jones polynomial refers to the original one-variable Jones polynomial and its relatives obtained from the quantum group $\mathrm{SL}(2)_q$. The original Jones polynomial corresponds to the fundamental representation of the quantum group. The invariants in the paper are constructed by labelling each component of the link with a given representation of the quantum group. This gives a particular Jones polynomial corresponding to the labelling. The authors show that, by using values of q that are roots of unity, and by summing (with appropriate coefficients) the Jones polynomials of a link corresponding to such a coloring, an invariant of framed links is obtained that is also invariant under the Kirby moves. The Kirby moves are modifications of framed links that give homeomorphic 3-manifolds in the class of 3-manifolds obtained by surgery on framed links.

In this way, invariants of 3-manifolds are obtained via invariants of knots and links. Each three-manifold is presented as surgery on a framed link, and the invariant of that link, being invariant under Kirby moves, is an invariant of the 3-manifold.

The paper uses a number of techniques and formulations. First of all there is the notion of a ribbon Hopf algebra (of which the universal enveloping algebra for the quantum lie algebra for $\mathrm{SL}(2)_q$ is an example). In another paper [Comm. Math. Phys. **127** (1990), no. 1, 1–26; MR1036112 (91c:57016)] the authors showed that ribbon Hopf algebras are an appropriate category of Hopf algebras for formulating invariants of framed links. In this method the invariant is formulated as a functor from a category of diagrams (with tangles as morphisms) to a corresponding module category. The functor takes a closed link diagram to a morphism from the complex numbers to itself, hence to a number. The core of the algebraic part of the paper is a careful treatment of the quantum group for $\mathrm{SL}(2)_q$ at roots of unity, showing that the representation theory is appropriate for the solution to the problem of obtaining invariants of the Kirby moves.

This paper is important as the first construction of a nontrivial 3-manifold invariant via invariants of framed links and the Kirby moves. It is also important as an instantiation of the program of invariants of 3-manifolds initiated by E. Witten [ibid. **121** (1989), no. 3, 351–399; MR0990772 (90h:57009)]. Witten's program uses ideas from quantum field theory and conformal field theory. The present paper is more elementary, but presumably produces the same invariants as the Witten program. The relationships between the Reshetikhin-Turaev approach and the Witten program will become clear as soon as the relationships between quantum groups and conformal field theory are more fully understood.

From MathSciNet, December 2013

Louis H. Kauffman

MR1303779 (95j:46063) 46Lxx; 19K56, 22D25, 58B30, 58G12, 81T13, 81V22, 81V70

Connes, Alain

Noncommutative geometry. (English)

Academic Press, Inc., *San Diego, CA*, 1994, xiv+661 pp., $59.95,
ISBN 0-12-185860-X

Alain Connes received the Fields Medal at the Warsaw Congress in 1983. The citation, which concentrates on his work on von Neumann algebras, mentions noncommutative geometry in passing, and states confidently that “the subject will rapidly develop much further”. This large (over 600 pages), dense, fascinating, and beautiful book is evidence of the accuracy of this prognosis.

To see what is meant by the phrase “noncommutative geometry”, consider ordinary geometry: for example, the geometry of a surface S in three-dimensional Euclidean space $\mathbf{R}^3$. Following Descartes, we study the geometry of S using coordinates. These are just three functions x, y, z on $\mathbf{R}^3$, and their restrictions to S generate, in an appropriate sense, the algebra $C(S)$ of all continuous functions on S. All the geometry of S is encoded in this algebra $C(S)$; in fact, the points of S can be recovered simply as the algebra homomorphisms from $C(S)$ to $\mathbf{C}$. In the language of physics, one might say that the transition from S to $C(S)$ is a transition from a “particle picture” to a “field picture” of the same physical situation.

One notes that the algebra $C(S)$ appearing here is commutative. It is therefore natural to suggest that it might be valuable to think of noncommutative algebras in geometric terms, as the algebras of functions on “noncommutative spaces”. Some attempts in this direction have been made from algebraic geometry, but Connes's program rests on differential geometry and incorporates deep techniques from functional analysis. Motivation for this program comes from a number of sources. One is physics: quantum mechanics asserts that physical observables such as position and momentum should be modeled by elements of a noncommutative algebra. Another is a series of examples showing that objects such as the space of (unitary) representations of a group, or the space of leaves of a foliation, may be most appropriately modeled as noncommutative spaces. A third comes from the application of K-theory to operator algebras and the link (in the case of group C^*-algebras) to the Novikov higher-signature conjecture and surgery theory. All these together with other motivations and applications are discussed in detail in the book.

Connes's book guides the reader through the appropriate noncommutative versions of four successively more refined kinds of geometric structure: measure theory, topology (including algebraic topology), differential topology (manifold theory), and differential (Riemannian) geometry. To begin with the coarsest of these, noncommutative measure theory means the theory of von Neumann algebras. Let H be a Hilbert space, $B(H)$ the set of bounded linear operators on it. A von Neumann algebra is an involutive subalgebra A of $B(H)$ which is closed in the weak topology on $B(H)$. The commutative example to keep in mind is $H = L^2(X, \mu)$, where

(X, μ) is a measure space, and $A = L^\infty(X, \mu)$ acting by multiplication; the weak topology coincides with the topology of pointwise almost everywhere convergence on bounded sets. This algebra is in a natural sense a "direct integral" of copies of $\mathbf{C}$, which is a factor—that is, a von Neumann algebra with trivial centre. Already in the original papers of Murray and von Neumann one finds the classification of factors into three types: type I, matrix algebras; type II, admitting a "continuous dimension" function with real values; type III, admitting no dimension function. Of these, the factors of type III are the most mysterious. Much of Connes's early work was devoted to elucidating their structure, and this is described in Chapter V of the book. This chapter also contains a discussion of the work of Vaughan Jones on subfactors of factors of type II, which had such remarkable applications to knot theory.

An example which is significant for many aspects of the book comes from the theory of foliations. Let $(V, \mathcal{F})$ be a compact foliated manifold. We want to study the "space of leaves" $V/\mathcal{F}$—initially only as a measure space, but then successively in the more refined categories mentioned above. If the foliation is ergodic—think of the irrational-slope flow on a torus—then the quotient measure space in the usual sense is trivial. But we can form a noncommutative von Neumann algebra whose elements are measurable families of operators on the L^2 spaces of the leaves of the foliations, and it turns out that this von Neumann algebra is equivalent in the non-singular case to the one arising from the usual measure-theoretic quotient, and that it is interesting even in the singular case; for instance, the von Neumann algebra of the irrational-slope flow is of type II, and its "continuous dimension" function is given by the unique invariant transverse measure on the foliation. There are remarkable links between this von Neumann algebra and the more classical invariants of foliations, for example the result (originally due to Steve Hurder) that if the Godbillon-Vey class of a foliation is nonzero, then the associated von Neumann algebra is of type III.

The noncommutative analogue of topology depends on the theory of C^*-algebras. These are involutive subalgebras of $B(H)$ which are closed for the norm topology: the standard (in fact, the only) commutative example is the algebra of continuous functions vanishing at infinity on a locally compact Hausdorff space. The noncommutative quotient construction can be applied in the C^*-algebraic setting also. An example of particular interest arises by applying the quotient construction to the trivial action of a discrete group Γ on a point; one obtains the (reduced) group C^*-algebra $C^*_r(\Gamma)$, which may be regarded as an analytical completion of the group algebra $\mathbf{C}\Gamma$. Connections with geometric topology arise when Γ is the fundamental group of a manifold.

By the theorem of Gel′fand and Naĭmark, the study of commutative C^*-algebras is exactly the same as the study of locally compact Hausdorff spaces. It is natural therefore to ask whether the tools of algebraic topology extend to the noncommutative C^*-algebraic world. It turns out that topological K-theory and its S-dual theory, K-homology, have natural extensions to the category of C^*-algebras. Moreover, there is a natural relationship with index theory in the case of C^*-algebras arising from the noncommutative quotient construction. Recall that Atiyah and Singer showed that the index of a family of elliptic operators over a base B should be thought of as an element of the K-theory group $K^*(B)$. Analogously, a family of elliptic operators along the leaves of a foliation, or on the fibres of any other quotient, has an index in the K-theory of the noncommutative quotient C^*-algebra. A very general conjecture is then that the whole C^*-algebra K-theory group is exactly generated by indices of this kind. This is known as the Baum-Connes conjecture, and it has been verified in a large number of different situations. It also has noteworthy implications: for example, the Baum-Connes conjecture for a discrete group Γ implies the Novikov conjecture for Γ, which (via surgery theory) is crucial to the topology of high-dimensional manifolds having Γ as fundamental group.

Chapter II of the book contains a detailed account of this material. The general theory is illustrated with many beautiful examples. Connes gives particular prominence to groupoids arising from geometric situations, and the K-theory maps arising from associated deformations of algebras. For example, the tangent groupoid is an algebraic encoding of the idea that the tangent bundle of a manifold M can be obtained by blowing up the diagonal in $M \times M$; the associated convolution algebra encapsulates the index-theoretic aspects of the pseudodifferential calculus.

The next stage in the development is the noncommutative analogue of differential topology. Granted that K-theories seem to be "natural" (co)homology theories in the noncommutative category, one might still search for a noncommutative analogue of ordinary (co)homology. The strategy in the search for the noncommutative analogue of some notion is first to find a way of formulating the commutative notion in "function" rather than "point" terms, and then to try replacing the functions by elements of a noncommutative algebra. Now the version of cohomology that is most naturally expressed in "function" terms is de Rham cohomology, but (since it involves differentiation) it must be formulated in terms of the dense subalgebra of smooth functions inside the C^*-algebra of continuous ones. In the absence of a general smoothing theory for C^*-algebras (as there is, say, for high-dimensional topological manifolds) the choice of such a subalgebra has to be made ad hoc. Often, however, a natural such subalgebra is available, and one refers to it as a smooth structure on the C^*-algebra in question.

Suppose a smooth structure has been chosen. How should we construct "noncommutative de Rham theory"? Connes's answer is his theory of cyclic cohomology, expounded in Chapter III of the book. Cyclic cohomology has received an extensive purely algebraic development in recent years [see, for example, J.-L. Loday, *Cyclic homology*, Springer, Berlin, 1992; MR1217970 (94a:19004)], but Connes's discussion remains close to the original geometric motivation. One of the deepest applications is to the construction of the transverse fundamental class for a foliation: this is a map from the K-theory of the "transverse space" to the real numbers, and it would have an easy construction if there were an invariant transverse Riemannian metric. In general no such metric can be found, but Connes noticed that this problem was analogous to one that he had already surmounted in his work on factors of type III. Following this insight, and combining ideas from cyclic theory (here the choice of an appropriate dense subalgebra is a very delicate matter) and sophisticated versions of the Thom isomorphism in K-theory, Connes was able to construct the desired map. Geometric results about foliations, including the theorem of Hurder mentioned above, are immediate corollaries.

A second motivation for cyclic cohomology, equally significant, is the idea of "quantized calculus", expounded in Connes's Chapter IV. If A is an algebra of operators on some Hilbert space H, and F an operator on H with $F^2 = 1$, then the definition $da = [F, a] = Fa - aF$ obeys the basic rule $d^2 = 0$ of de Rham theory. If in addition F and A are "almost commuting" enough that any product of sufficiently many commutators $[F, a]$ is of trace class, then the trace can be used to define an "integral" of these "differential forms" and we have the ingredients for a "noncommutative de Rham current", i.e. a cyclic cocycle. A helpful example is $A = C^\infty(M)$, M a compact manifold of dimension n, and F some (zeroth order) pseudodifferential involution. The commutators are then pseudodifferential operators of order -1, and any product of more than n of them is therefore trace-class. In this example we see that the notion of dimension in quantized calculus is related to the "degree of traceability" of certain commutators.

Not all cyclic classes arise in this way. In fact, such a cyclic class belongs to the image of a "Chern character" map from K-homology. It follows, therefore, that if such a cyclic class is paired with an element of K-theory the result must be an integer, because the evaluation of pairing between K-theory and K-homology ultimately reduces to the computation of a Fredholm index. In this way one obtains integrality results somewhat analogous to those provided by the Atiyah-Singer index theorem. An application discussed in detail by Connes is the work of J. Bellissard on the existence of integer plateaux of conductivity in the quantum Hall effect. This concerns the flow of electricity in a crystal subjected to electric and magnetic fields. In general, the "natural" observables in this problem generate an algebra isomorphic to that associated to the irrational-slope foliation on a torus, and the integrality arises from an associated cyclic 2-cocycle.

The last chapter in the book is entitled "The metric aspect of noncommutative geometry", and begins an investigation of the analogue of Riemannian geometry and Yang-Mills theory in the noncommutative context. Many readers will focus with particular interest on the interpretation of the Standard Model of particle physics in terms of a noncommutative space which is the product of an ordinary manifold with a (very small) discrete space. Connes writes: "Our contribution should be regarded as an interpretation, of a geometric nature, of all the intricacies of the most accurate phenomenological model of high-energy physics. . . . It does undoubtedly confirm that high-energy physics is in fact unveiling the fine structure of space-time. Finally, it gives a status to the Higgs boson as just another gauge field, but one corresponding to a finite difference rather than a differential."

The book is magnificently produced, and contains so much new material that it completely supersedes the French edition [*Géométrie non commutative*, InterEditions, Paris, 1990; MR1079062 (92e:58016)]. The depth and variety of Connes' thought will lead the serious reader to explore the background and proofs of the results in more detail, and he or she will find a comprehensive bibliography for this purpose.

On the back cover, the publishers describe the book as "the definitive treatment", but the subject is still developing so rapidly that the sense of finality conveyed by this phrase hardly seems appropriate. Vaughan Jones' comment, also on the cover, is more apt: "A milestone for mathematics".

From MathSciNet, December 2013

John Roe

MR1642584 (99m:46148) 46L37; 19K99, 46L60, 46L80, 46N55, 82B10

Evans, David E.; Kawahigashi, Yasuyuki

Quantum symmetries on operator algebras. (English)

Oxford Mathematical Monographs.

The Clarendon Press, Oxford University Press, *New York*, 1998, xvi+829 pp., $200.00, ISBN 0-19-851175-2

This monumental book presents some of the most exciting developments of the past 15 years within operator algebra theory and its interaction with mathematical physics and algebraic topology. A main basis for these developments was established by V. F. R. Jones, who received the Fields medal in 1990 for his groundbreaking work on subfactors, link invariants and quantum statistical mechanics. Besides Jones, many others have worked since the early eighties on these and related themes, and these efforts have created a literature which it would be impossible to cover even within more than the well over 800 pages of the monograph under review. Thus, choices and hence omissions have to be made.

The overall focus of the authors is on combinatorial rather than analytic aspects of the theories. The later chapters are particulary centered around the viewpoint emanating from Ocneanu's paragroup theory together with its many applications. On the other hand, Tomita-Takesaki theory is essentially avoided, and the analytic parts of the classification of subfactors (such as S. Popa's theory of amenability) are mentioned without proofs. But what should interest the reader is the surprisingly comprehensive account actually given by the book, and which we proceed to outline.

The first five chapters present what the authors conceive as "the very basics of operator algebras": basic operator theory, C^*-algebras, K-theory (in the operator algebraic set-up), one-parameter semigroups, and von Neumann algebras (in particular type II_1-factors). With due preparation in functional analysis, these together with the first sections of Chapter 9 (on subfactors) provide material for a year-long graduate course on modern aspects of operator algebra theory.

The remaining 10 chapters (6–15) treat more advanced subjects pertaining to the developments alluded to in the first paragraph of this review. These fall into three main groups: (1) Operator algebras and mathematical physics: the fermion algebra (Chapter 6), the Ising model (Chapter 7), conformal field theory (Chapter 8). These chapters provide the necessary background from mathematical physics for the newer developments treated by the subsequent chapters. (2) Subfactors and their invariants, in particular combinatorial aspects: Subfactors and bimodules (Chapter 9), axiomatization of paragroups (Chapter 10), string algebras and flat connections (Chapter 11). It is possible to read most of these chapters without, or prior to, a detailed reading of the first group. (3) Applications of subfactors and finer invariants: topological quantum field theory (Chapter 12), rational conformal field theory and paragroups (Chapter 13), commuting squares of II_1-factors (Chapter 14), automorphisms of subfactors and central sequences (Chapter 15). These chapters include many recent results of the authors.

In the preface, the authors state: "This book is aimed at graduate students and researchers in operator algebras and graduate students and researchers in the interface between operator algebras and statistical mechanics, algebraic, topological and conformal field theory and low-dimensional topological invariants." All of these will find a rich source of study and references in this extensively annotated and documented volume, and not least a wide range of new perspectives on their own subject.

From MathSciNet, December 2013

Carl Winsløw

MR2747966 (2012a:46121) 46L37; 46L10, 46L40

Popa, Sorin

Classification of actions of discrete amenable groups on amenable subfactors of type II.

International Journal of Mathematics **21** (2010), *no.* 12, 1663–1695.

This is the published version (with typos corrected and a few additional references) of an I.H.E.S. preprint dating back to 1992. Since then, the preprint was widely circulated in the mathematical community and its results have been fully accepted. Although the initial motivation of the article was applications to the classification of type III_λ subfactors, it turned out that its techniques proved to be useful in other parts of von Neumann algebras theory, such as the currently very active topic of deformation/rigidity theory. These considerations led the author to revive his paper and publish it in a refereed journal.

Let us briefly recall some background about the classification of amenable factors. In his seminal paper [Ann. of Math. (2) **104** (1976), no. 1, 73–115; MR0454659 (56 #12908)], A. Connes proved the uniqueness of the amenable type II_1 and type II_∞ factors, denoted by R and $R^\infty = R\otimes\mathcal{B}(\ell^2(\mathbb{N}))$ respectively. Moreover, for $0<\lambda<1$, using the discrete decomposition as a crossed product $M = N\rtimes\sigma$ of any type III_λ factor, which he had previously established [Ann. Sci. École Norm. Sup. (4) **6** (1973), 133–252; MR0341115 (49 #5865)], he also proved the uniqueness of the type III_λ amenable factor. In this decomposition, N is a type II_∞ factor, called the core of M, and σ is an automorphism of N of modulus λ. The core of an amenable type III_λ factor M is R^∞, and the uniqueness of M relies on the classification of the automorphisms of R^∞ [A. Connes, Ann. Sci. École Norm. Sup. (4) **8** (1975), no. 3, 383–419; MR0394228 (52 #15031)]. Further classification results for actions of finite groups and general amenable groups on amenable type II factors were proved by V. F. R. Jones and A. Ocneanu respectively.

A subsequent challenging problem is the classification of amenable inclusions of factors $N\subset M$ with finite Jones index. The crucial tool is the standard invariant, which is roughly described as follows: Let $N\subset M$ be a finite index inclusion of type II_1 factors and let $M\supset N\supset N_1\supset\cdots\supset N_k\supset\cdots$ be a corresponding tunnel of subfactors. The sequence of commuting squares $\{N_k'\cap N\subset N_k'\cap M\}_k$ together with canonical traces is encoded in a graph type object called the standard invariant $\mathcal{G}_{N,M}$. We set $N^{\mathrm{st}} = \overline{\bigcup_k N_k'\cap N}$ and $M^{\mathrm{st}} = \overline{\bigcup_k N_k'\cap M}$. Then $N^{\mathrm{st}}\subset M^{\mathrm{st}}$ is called the standard model of $N\subset M$. Under a suitable assumption of amenability for the inclusion, called strong amenability, the author proved [Acta Math. **172** (1994), no. 2, 163–255; MR1278111 (95f:46105)] that $N\subset M$ is isomorphic to its standard model, and thus completely classified by its standard invariant, a fact analogous to the uniqueness of the amenable type II_1 factor. A similar result applies for strongly amenable inclusions of type II_∞ factors.

Concerning the study of some inclusions of type III_λ factors, a next step to be achieved is the classification of properly outer automorphisms of strongly amenable inclusions of type II factors.

Given an automorphism θ of a finite index inclusion $N \subset M$ of type II factors, a basic ingredient is the standard part θ^{st} of θ introduced by P. Loi. It is an automorphism of the standard model $N^{\mathrm{st}} \subset M^{\mathrm{st}}$ and it implements an action γ_θ on $\mathcal{G}_{N,M}$, thus defining the standard invariant $(\mathcal{G}_{N,M}, \gamma_\theta)$ of the equivariant inclusion.

We now state the first main result of the paper. Let θ be a properly outer trace-scaling automorphism of a strongly amenable inclusion $N^\infty = N \otimes \mathcal{B}(\ell^2(\mathbb{N})) \subset M^\infty = M \otimes \mathcal{B}(\ell^2(\mathbb{N}))$ of type II_∞ factors (i.e. $N \subset M$ is a strongly amenable inclusion of type II_1 factors). Then there is an isomorphism α from $N^\infty \subset M^\infty$ onto $N^{\mathrm{st}} \otimes R^\infty \subset M^{\mathrm{st}} \otimes R^\infty$ such that $\alpha \circ \theta \circ \alpha^{-1} = \theta^{\mathrm{st}} \otimes \sigma_0$, where σ_0 is a model automorphism of R^∞ having the same module as θ. The uniqueness of σ_0, up to conjugacy, is provided by Connes' classification of automorphisms of R^∞. So, the action θ is completely classified by the action it implements on the standard invariant of the inclusion.

The application to the classification of some inclusions of type III_λ factors $(0 < \lambda < 1)$ follows immediately. Precisely, let $\mathcal{N} = N^\infty \rtimes \sigma \subset \mathcal{M} = M^\infty \rtimes \sigma$ where $N^\infty \subset M^\infty$ is a strongly amenable inclusion of type II_∞ factors and σ is a λ-scaling automorphism of M^∞ leaving N^∞ globally invariant. Then $\mathcal{N} \subset \mathcal{M}$ is isomorphic to $((N^{\mathrm{st}} \otimes R^\infty) \rtimes (\sigma^{\mathrm{st}} \otimes \sigma_0) \subset (M^{\mathrm{st}} \otimes R^\infty) \rtimes (\sigma^{\mathrm{st}} \otimes \sigma_0))$, where σ_0 is a model automorphism of R^∞ of modulus λ.

The case of a properly outer cocycle action θ of a discrete amenable group G on a strongly amenable inclusion $N \subset M$ of type II_1 factors is considered in the last section. It is shown that such an action is cocycle conjugate to the tensor product of θ^{st} and a properly outer model cocycle action of G on the amenable type II_1 factor R.

The results rely in part on tools developed by the author in several previous papers, and in particular in his influential above-mentioned paper. Moreover, non-commutative ergodic theory techniques such as the construction of local Rokhlin towers are of fundamental importance in the subtle proofs.

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