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The Sherrington-Kirkpatrick model, by Dmitry Panchenko, Springer Monographs in Mathematics, Springer, New York, 2013, xii+156 pp., ISBN 978-1-4614-6288-0

The celebrated Sherrington-Kirkpatrick model for a mean field spin glass was introduced in 1975 in [6]. It is considered a prototype for the statistical mechanics of complex systems (see, for example, [5]). From the point of view of theoretical physics, this model is extremely interesting, and thousand of papers have been dedicated to the study of its subtle properties. Moreover, since its introduction it has attracted also the interest of mathematicians, due to the arising deep interplay of probabilistic and analytic aspects. As asserted by Michel Talagrand in [7], this model provides a "great challenge for mathematicians".

The description of the model is rather simple. One considers a large number N of sites, and introduces "configurations" as mappings

$$\sigma : i \in \{1, 2, \ldots, N\} \to \sigma_i = \pm 1.$$

Thus, there are 2^N possible configurations, which can be visualized as the vertices of an N-dimensional hypercube. We introduce the "overlap" q among configurations as follows:

$$q_{\sigma\sigma'} = \sum_i^N \sigma_1 \sigma'_i / N.$$

To the configurations we attach a family of Gaussian random variables $\mathcal{K} : \sigma \to \mathcal{K}(\sigma)$, all centered (i.e., $\mathbb{E}(\mathcal{K}(\sigma)) = 0$) and with joint variance given by

$$\mathbb{E}(\mathcal{K}(\sigma)\mathcal{K}(\sigma')) = q^2_{\sigma\sigma'}.$$

The typical problems we are interested in are suggested by statistical mechanics, and thus involve an infinite volume limit $N \to \infty$. In particular, one is interested in proving the existence of the "ground state energy density"

$$e_0 = \lim_{N\to\infty} (2N)^{-1/2} \min_\sigma \mathcal{K}(\sigma)$$

and to giving an explicit expression for the limit in terms of a suitable variational principle. The exponent $-1/2$ for N is here chosen in order to get a good large N behavior.

At first, this might look rather easy. For example, in the simple case, when $\mathcal{K}(\sigma)$ are independent centered unit Gaussian random variables, elementary probability considerations enable one to verify that the limit e_0 exists $\mathcal{K}$-almost surely and is given by a simple variational principle,

$$-e_0 = \min_{x>0}\left(\frac{\ln 2}{x} + \frac{x}{4}\right),$$

so that $-e_0 = \sqrt{\ln 2}$.

However, in the case where the joint variance is given in terms of the square of the overlap, one discovers that even the task of proving the existence of the limit e_0 is extremely difficult. Since the introduction of the model, almost four decades had elapsed before the way of getting the proof was found in [3].

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In order to explain the content of the book under review, we need to introduce some additional definitions.

For a given parameter $\beta \geq 0$, which plays the role of the inverse temperature, let us define the "partition function" Z_N and the "free energy" F_N as the random variables

$$Z_N(\beta; \mathcal{K}) = \sum_{\sigma} e^{\beta \sqrt{\frac{N}{2}} \mathcal{K}(\sigma)},$$

$$-\beta F_N(\beta; \mathcal{K}) = \ln Z_N(\beta; \mathcal{K}).$$

Let us also introduce a (random) discrete probability measure p_N on the configurations σ,

$$p_N(\sigma; \mathcal{K}) = Z_N^{-1}(\beta; \mathcal{K}) e^{\beta \sqrt{\frac{N}{2}} \mathcal{K}(\sigma)},$$

so that for any function A of the configuration we can define the average

$$\omega(A(\sigma)) = \sum_{\sigma} A(\sigma) p_N(\sigma; \mathcal{K}).$$

Notice that these averages are themselves random variables in the probability space of the variables $\mathcal{K}$.

We introduce also s replicas of the configurations,

$$\sigma_i^a, a = 1, \ldots, s, \ i = 1, \ldots, N,$$

and define for them the product measure, so that we have the averages

$$\Omega(A(\sigma_i^a)) = \sum_{\sigma^1} \cdots \sum_{\sigma^s} A(\sigma_i^a) p_N(\sigma^1; \mathcal{K}) \cdots p_N(\sigma^s; \mathcal{K}).$$

Finally, for a generic function F of the replicated variables σ_i^a and of $\mathcal{K}$, we introduce the global averages

$$\langle F \rangle = \mathbb{E}\Omega(F),$$

where we perform the product measure average Ω and then the average $\mathbb{E}$ over the variables $\mathcal{K}$, called "quenched" average.

We summarize in the next two theorems some main results concerning the infinite volume limit of the free energy density.

Theorem 1 (Existence)**.** *The infinite volume limit of the free energy density exists $\mathcal{K}$-almost surely and is connected to the limit for the quenched average in the form*

$$\lim_{N \to \infty} N^{-1} \ln Z_N(\beta; \mathcal{K}) = A(\beta), \ \mathcal{K}\text{-almost surely},$$

$$A(\beta) = \lim_{N \to \infty} N^{-1} \mathbb{E}(\ln Z_N(\beta; \mathcal{K})) = \sup_N N^{-1} \mathbb{E}(\ln Z_N(\beta; \mathcal{K})).$$

The essence of the proof relies on the basic super-additivity,

$$\mathbb{E} \ln Z_N \geq \mathbb{E} \ln Z_{N_1} + \mathbb{E} \ln Z_{N_2}, \ N = N_1 + N_2,$$

which was established in [3] through a simple interpolation argument.

The variational principle for $A(\beta)$ is given in

Theorem 2 (The variational principle)**.**

$$A(\beta) = \inf_x \left(\ln 2 + f(0, 0; \beta, x) - \frac{1}{2} \beta^2 \int_0^1 q \ x(q) \ dq \right),$$

*where x is the **Parisi functional order parameter**, which can be taken as a piecewise constant nondecreasing function*

$$x : [0,1] \ni q \to x(q) \in [0,1],$$

and $f(0,0;\beta,x)$ is the value at $q=0, y=0$ of the function

$$f : [0,1] \times (-\infty,+\infty) \ni (q,y) \to f(q,y;\beta,x),$$

defined as solution of the partial differential equation

$$\partial_q f + \frac{1}{2}\partial^2_{yy} f + \frac{1}{2}x(q)(\partial_y f)^2 = 0,$$

with final value at $q = 1$

$$f(1,y;\beta,x) = \ln\cosh(\beta y).$$

The proof relies on the following sum rule, established in [4] through interpolation arguments

$$N^{-1}\mathbb{E}(\ln Z_N(\beta;\mathcal{K})) = \ln 2 + f(0,0;\beta,x) - \frac{1}{2}\beta^2 \int_0^1 q\ x(q)\ dq + R_N,$$

holding for any functional order parameter x, where the error term R_N is nonnegative. By taking the limit when $N \to \infty$ and neglecting the error term, we obtain the upper bound in the statement of the theorem. It took a dramatic *tour de force* to prove that the error term vanishes in the infinite volume limit, provided that the order parameter is chosen in the optimal way. This fundamental breakthrough result was due to Michel Talagrand [8].

The strategy developed by Dmitry Panchenko in a series of papers and concisely and brilliantly expounded in the book under review, amounts to a true revolution in order to establish the missing lower bound. It has extremely relevant consequences for the rigorous analysis of complex systems.

Let me mention that the treatment of the model given in theoretical physics (see, for example, [5]) provides some important properties for the overlap distribution under the averages $\langle\ \rangle$ introduced above. It turns out that the overlaps have the **ultrametric property**, *i.e.*, the joint distribution for any three overlaps among replicas q_{12},q_{13},q_{23} has support on the region where the inequality $q_{12} \geq \min(q_{13}, q_{23})$ holds. Moreover, the optimal functional order parameter is $x(q) = P(q_{12} \leq q)$ and the joint distribution under $\langle\ \rangle$ of overlaps $q_{ab}, 1 \leq a \leq b \leq s$, can be uniquely described in terms of the single overlap distribution.

Some years ago, a surprising constraint on the overlap distribution was found by Ghirlanda and Guerra in [2]. Consider the overlap distribution for s replicas $q_{a,b}, 1 \leq a \leq b \leq s$, then increase the number of replicas to $s+1$. It turns out that, assuming for instance that $i = 1$, the distribution of $q_{1,s+1}$, conditioned to the values of the overlaps among the first s replicas, is such that $q_{1,s+1}$ takes exactly either one of the values $q_{1b}, b = 2,\ldots,s$, or is independent from the first overlaps, with the respective probabilities all equal to s^{-1}. The Ghirlanda-Guerra identities were proven to be valid for a large class of interactions, with the possible exclusion of some values for the parameters. They have a very simple physical interpretation. Namely, if the interaction depends on some additional random term of the type $e^{A(\sigma;\chi)}$ in the partition function, where χ is some external noise, then in the infinite volume limit the quantity $N^{-1}A(\sigma;\chi)$ behaves as a constant under

$\langle\ \rangle$. This is a manifestation of some kind of generalized equivalence of ensembles in statistical mechanics.

Ultrametricity implies the Ghirlanda-Guerra identities. However, it was not clear whether the converse statement would also hold. That is why in the paper [2] the cautious subtitle "Toward Parisi ultrametricity" was used.

Dmitry Panchenko has been able to prove that this is indeed the case: Ghirlanda-Guerra identities have, as a necessary consequence, ultrametricity. The essential part of his proof is a very clever exploitation of the Dovbysh-Sudakov representation for an infinite symmetric, weakly exchangeable, and positive definite random array $R = (R_{a,b})_{a,b\geq 1}$. Thus, the important property of ultrametricity is rigorously established for the overlap distribution.

Moreover, some very simple considerations, coming from the so-called cavity method [5], lead one to conclude that in the infinite volume limit the limiting free energy density can be expressed, in the notation of Theorem 2, as

$$A(\beta) = \ln 2 + f(0,0;\beta,x) - \frac{1}{2}\beta^2 \int_0^1 q\ x(q)\ dq$$

for some properly chosen functional order parameter x. With the upper bound already established, this result implies trivially the lower bound. This establishes Theorem 2 without going through the complex procedure of [8].

The book under review gives a very clear and useful description of the general frame of the Sherrington-Kirkpatrick mean field spin glass model. It also describes the original strategy exploited by the author in order to get control of the infinite volume limit of the free energy, in the frame of the Parisi variational principle, as well as some properties of the infinite volume limit of the overlap distribution, such as ultrametricity. The book is a very valuable addition to the literature in the field, and can be seen as complementary to other existing works, as for example [1, 7].

I would recommend this book especially to young researchers. They can learn from it a lot about the scientific results, which could be extended to other important cases of hard optimization problems. It also provides an insight on developing original strategies when working with the difficult mathematical problems arising in studies of complex systems.

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