RECENT ADVANCES IN SYMPLECTIC FLEXIBILITY

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ABSTRACT. Flexible and rigid methods coexisted in symplectic topology from its inception. While the rigid methods dominated the development of the subject during the last three decades, the balance has somewhat shifted to the flexible side in the last three years. In the talk we survey the recent advances in symplectic flexibility in the work of S. Borman, K. Cieliebak, T. Ekholm, E. Murphy, I. Smith, and the author.

This survey article is an expanded version of author's article [27] and his talk at the Current Events Bulletin at the AMS meeting in Baltimore in January 2014. It also uses materials from the papers [6, 9, 20, 32] and the book [8].

1. The h-principle

Many problems in mathematics and its applications deal with partial differential equations, partial differential inequalities or, more generally, with *partial differential relations* (see [48] and also [31]), i.e., any conditions imposed on partial derivatives of an unknown function. A solution of such a partial differential relation $\mathcal{R}$ is any function which satisfies this relation.

With any differential relation one can associate the underlying algebraic relation by substituting all the derivatives entering the relation with new independent functions. The existence of a solution of the corresponding algebraic relation, called a *formal* solution of the original differential relation $\mathcal{R}$, is a necessary condition for the solvability of $\mathcal{R}$. Though it seems that this necessary condition should be very far from being sufficient, it was a surprising discovery in the 1950s of geometrically interesting problems where existence of a formal solution is the only obstruction for the genuine solvability. Two of the first such non-trivial examples were the C^1-isometric embedding theorem of J. Nash and N. Kuiper [57, 68] and the immersion theory of S. Smale and M. Hirsch [55, 77]. After Gromov's remarkable series of papers, beginning with his paper [47] and culminating in his book [48], the area crystallized as an independent subject, called the h-principle.

Rigid and flexible results coexist in many areas of geometry, but nowhere else do they come so close to each other, as in symplectic topology, which serves as a rich source of examples on both sides of the *flexible-rigid spectrum.* Flexible and rigid problems and the development of each side toward the other shaped and continue to shape the subject of symplectic topology from its inception.

2. Symplectic preliminaries

To set the stage, we recall some basic notions of symplectic and contact geometry. Symplectic geometry was born as a geometric language of classical mechanics and, similarly, contact geometry emerged as a natural set-up for geometric optics and mechanics with non-holonomic constraints.

The cotangent bundle T^*M of any smooth n-dimensional manifold M carries a canonical *Liouville* 1-form λ, usually denoted pdq, which in any local coordinates $(q_1, \ldots, q_n)$ on M and dual coordinates $(p_1, \ldots, p_n)$ on cotangent fibers can be written as $\lambda = \sum_1^n p_i dq_i$. The differential $\omega := d\lambda = \sum_1^n dp_i \wedge dq_i$ is called the *canonical symplectic structure* on the cotangent bundle T^*M. In the Hamiltonian formalism of classical mechanics the cotangent bundle T^*M is viewed as the phase space of a mechanical system with the configuration space M, where the p-coordinates correspond to momenta. The full energy of the system, expressed through coordinates and momenta, i.e., viewed as a function $H \colon T^*M \to \mathbb{R}$ on the cotangent bundle (or a time-dependent family of functions $H_t \colon T^*M \to \mathbb{R}$ if the system is not conservative) is called the *Hamiltonian* of the system. The dynamics is then defined by the Hamiltonian equations $\dot{z} = X_{H_t}(z), z \in T^*M$, where the Hamiltonian vector field X_{H_t} is determined by the equation $i(X_{H_t})\omega = -dH_t$, which in the canonical (p, q)-coordinates has the form

$$X_{H_t} = \sum_1^n -\frac{\partial H_t}{\partial q_i}\frac{\partial}{\partial p_i} + \frac{\partial H_t}{\partial p_i}\frac{\partial}{\partial q_i}.$$

The flow of the vector field X_{H_t} preserves ω, i.e., $X_{H_t}^* \omega = \omega$. The isotopy generated by the vector field X_{H_t} is called *Hamiltonian*.

More generally, the Hamiltonian dynamics can be defined on any $2n$-dimensional manifold endowed with a *symplectic*, i.e., a closed and non-degenerate differential 2-form ω. According to a theorem of Darboux, any such form admits local *canonical coordinates* $p_1, \ldots, p_n, q_1, \ldots, q_n$ in which it can be written as $\omega = \sum_1^n dp_i \wedge dq_i$. Diffeomorphisms preserving ω are called *symplectomorphisms* or, in the mechanical context, *canonical transformations*. Symplectomorphisms which can be included in a time-dependent Hamiltonian flow are called *Hamiltonian*. When $n = 1$, a symplectic form is just an area form, and symplectomorphisms are area-preserving transformations. In higher dimensions symplectomorphisms are also volume preserving, but the subgroup of symplectomorphisms represents only a small part of the group of volume-preserving diffeomorphisms.

Contact geometry is an odd-dimensional counterpart of symplectic geometry. The projectivized cotangent bundle PT^*M serves as the phase space in *geometric optics*, because even in the anisotropic media the speed of light is determined by the direction of the ray. It can be interpreted as the *space of contact elements* of the manifold M, i.e., the space of all tangent hyperplanes to M. The form pdq does not descend to PT^*M but its kernel does, and hence the space of contact elements carries a canonical field of tangent hyperplanes. This field turns out to be completely non-integrable. It is called a *contact structure*. More generally, a *contact structure* on a $(2n + 1)$-dimensional manifold is a completely non-integrable field of tangent hyperplanes ξ, where the complete non-integrability can be expressed by the Frobenius condition $\alpha \wedge (d\alpha)^{\wedge n} \neq 0$ for a 1-form α (locally) defining ξ by the Pfaffian equation $\alpha = 0$. By Darboux's theorem the contact form α can be locally always written in the form $\alpha = dz + \sum_1^n p_i dq_i$ in appropriate local coordinates.

Though at first glance symplectic and contact geometries are quite different, they are in fact tightly interlinked and it is useful to study them in parallel.

An important property of symplectic and contact structures is the following stability theorem, which is due to Moser [65] in the symplectic case and to Gray [44] in the contact one: *Given a 1-parametric family of symplectic structures ω_t, or contact structures ξ_t on a manifold X, which coincide outside of a compact set and such that in the symplectic case $\omega_t - \omega_0$ belong to the same cohomology class with compact support, there exists an isotopy $h_t : X \to X$ with compact support which starts at the identity $h_0 = \mathrm{Id}$ and such that $h_t^*\omega_t = \omega_0$ or $h_t^*\xi_t = \xi_t$.*

Maximal integral (i.e., tangent to ξ) submanifolds of a $(2n+1)$-dimensional contact manifold (V, ξ) have dimension n and are called *Legendrian*. Their symplectic counterparts are n-dimensional submanifolds L of a $2n$-dimensional symplectic manifold (W, ω) which are isotropic for ω, i.e., $\omega|_L = 0$. They are called *Lagrangian* submanifolds. Here are two important examples of Lagrangian submanifolds: A diffeomorphism $f : W \to W$ of a symplectic manifold (W, ω) is symplectic if and only if its graph $\Gamma_f = \{(x, f(x)); \ x \in W\} \subset (W \times W, \omega \times (-\omega))$ is Lagrangian. A 1-form θ on a manifold M, viewed as a section of the cotangent bundle T^*M, is Lagrangian if and only if it is closed. In particular, if $H_1(M) = 0$, then Lagrangian sections are graphs of differentials of functions, and hence the intersection points of a Lagrangian with the 0-section are critical points of the corresponding *generating* function. A general Lagrangian submanifold corresponds to a *multivalued function*, called the *front* of the Lagrangian manifold. Given a submanifold $N \subset M$ (of any codimension), the set of all tangent to N hyperplanes in TM is a Legendrian submanifold of the space of contact elements PT^*M.

3. Gromov's alternative and discovery of symplectic rigidity

It was an original idea of Henri Poincaré that Hamiltonian systems should satisfy special qualitative properties. In particular, his study of periodic orbits in the so-called restricted three-body problem led him to the following statement, now known as the "last geometric theorem of H. Poincaré": *any area-preserving transformation of an annulus $S^1 \times [0, 1]$ which rotates the boundary circles in opposite directions should have at least two fixed points.* Poincaré provided many convincing arguments why the statement should be true [74], but the actual proof was found by G. D. Birkhoff [5] in 1913, a few months after Poincaré's death. Birkhoff's proof was purely 2-dimensional, and further development of Poincaré's dream of what is now called *symplectic topology* had to wait until the 1960s when V. I. Arnold [3] formulated a number of conjectures formalizing this vision of Poincaré. In particular, one of Arnold's conjectures stated that the number of fixed points of a Hamiltonian diffeomorphism is bounded below by the minimal number of critical points of a function on the symplectic manifold.

At about the same time Gromov was proving his h-principle type results. He realized that symplectic problems exhibited some remarkable flexibility. This called into question whether Arnold's conjectures could be true in dimension > 2.

Among the remarkable results pointing toward symplectic flexibility which were proven by Gromov at the end of the 1960s and the beginning of the 1970s were the following:

- h-principle for symplectic and contact structures on open manifolds: In any homotopy class of non-degenerate (not necessarily closed) 2-forms on

an open manifold there is a symplectic form in any prescribed cohomology class. Moreover, any two such forms are homotopic as symplectic forms. Similarly, any almost contact structure, i.e., a pair (λ, η) of 1- and 2-forms on a $(2k + 1)$-dimensional open manifold which satisfies $\lambda \wedge \eta^k \neq 0$, is homotopic through almost contact structures to a pair $(\alpha, d\alpha)$.

- h-principle for Lagrangian immersions: This asserts that the Lagrangian regular homotopy classes of Lagrangian immersions $L \to X$ are in one-to-one correspondence with homotopy classes of injective Lagrangian homomorphisms $TL \to TX$.
- h-principle for ϵ-Lagrangian *embeddings* (i.e., embeddings whose tangent planes deviate from Lagrangian directions by an angle $< \epsilon$).
- h-principle for the *iso-symplectic* and *iso-contact* embeddings: For instance, in the symplectic case, *if (M, ω) and (N, η) are two symplectic manifolds such that $\dim N \geq \dim M + 4$, then any smooth embedding $f \colon M \to N$ which pulls back the cohomology class of the form η to the cohomology class of ω and whose differential df is homotopic to a symplectic bundle isomorphism can be C^0-approximated by an iso-symplectic embedding $\widetilde{f} \colon M \to N$, i.e. $\widetilde{f}^*\eta = \omega$.* For iso-symplectic and iso-contact *immersions* the h-principle holds even in codimension 2.

Gromov proved the following alternative: *either the group of symplectomorphisms (resp. contactomorphisms) is C^0-closed in the group of all diffeomorphisms or its C^0-closure coincides with the group of volume-preserving (resp. all) diffeomorphisms.*

One of the corollaries of Gromov's convex integration method was that there are no additional lower bounds for the number of fixed points of a volume-preserving diffeomorphism of a manifold of dimension ≥ 3 (compared to arbitrary diffeomorphisms). Clearly, the bound on the number of fixed points is a C^0-property, and hence, if the second part of the alternative were true, this would imply that Hamiltonian diffeomorphisms of symplectic manifolds of dimension > 2 have no special fixed point properties, and hence Poincaré's theorem and Arnold's conjectures reflected a pure 2-dimensional phenomenon. In fact, it was clear from this alternative, that all basic problems of symplectic topology are tightly interconnected.

Here are some of such problems, besides Gromov's alternative:

Problems 3.1.   (1) Extension of symplectic and contact structures to the ball from a neighborhood of the boundary sphere.
   (2) 1-parametric version of the previous question: *Is it true that two structures on the ball which coincide near the boundary and which are formally homotopic relative to the boundary are homotopic as genuine structures and hence isotopic?*
   (3) Fixed point problems for symplectomorphisms. More generally, Lagrangian intersection problem: *Do Lagrangian manifolds under certain conditions have more intersection points than is required by topology?*
   (4) Are there any non-formal obstructions to Legendrian isotopy?

For instance, as was already pointed out above, the resolution of Gromov's alternative in favor of flexibility would imply that symplectic fixed point Arnold conjectures are wrong in dimension > 2. Similarly, the "flexible" resolution of both problems (1) and (2) would imply the "flexible" resolution of Gromov's alternative,

as we explain below. Thus, a resolution of Gromov's alternative became a question about the existence of symplectic topology as a subject.

At the beginning of the 1980s the alternative was resolved in favor of rigidity in the series of works [21, 22], [4], [12], and it culminated in Gromov's paper [46] in which he introduced his method of (pseudo-)holomorphic curves in symplectic manifolds, which brought a genuine revolution into this subject.

As we already stated above, the rigid resolution of the symplectic and contact alternatives implied that Problems 3.1(1) and 3.1(2) cannot simultaneously have flexible solutions. Indeed, consider, for instance, the contact case. Let (M, ξ) be a $(2n + 1)$-dimensional contact manifold, and let $f : M \to M$ be isotopic to the identity diffeomorphism which is somewhere not contact. Let us first note that according to Gromov's h-principle for open contact manifolds, the analogs of Problems 3.1(1) and 3.1(2) do have positive answers for neighborhoods of discs of positive codimension. Assuming that both questions 3.1(1) and 3.1(2) have positive answers as well, we consider a small triangulation of M and inductively modify f by a C^0-small isotopy to make it contact on neighborhoods of k-skeleta of the triangulation, $k = 0, \ldots, 2n + 1$. Suppose that we already constructed a diffeomorphism $f_{k-1} : M \to M$ preserving ξ on a neighborhood of the $(k - 1)$-skeleton C_{k-1}. Take a neighborhood $G_\sigma \supset \sigma$ of a k-simplex σ. Consider a contact structure on $\mathcal{O}p\,(\partial G_\sigma \cup \sigma) \subset G_\sigma$ which coincides with ξ on $\mathcal{O}p\,\partial G_\sigma$ and with $(f_{k-1})_*\xi$ on $\mathcal{O}p\,\sigma$. Using a positive resolution of Problem 3.1(1), we can then extend it as a contact structure on G_σ which belongs to the same ξ relative to ∂G_σ homotopy class of almost contact structures. Hence, assuming a positive resolution of Problem 3.1(2) together with a Gray–Moser argument then implies existence of a compactly supported in G_σ diffeomorphism $g_\sigma : M \to M$ such that $g_\sigma^* \xi = \widetilde{\xi}$. Note that one can arrange that supports of g_σ and $g_{\widehat{\sigma}}$ are disjoint if $\sigma \cap \widehat{\sigma} = \varnothing$. Then the composition g of the diffeomorphisms g_σ for all k-simplices σ is C^0-small provided that the triangulation and the neighborhoods G_σ have small diameters. But then the diffeomorphism $f_k := g \circ f_{k-1}$ is C^0-close to f and preserves the contact structure ξ on a neighborhood of the k-skeleton C_k. Continuing by induction, we construct a contactomorphism f_{2n+1} which is C^0-close to f. But this contradicts to the C^0-closedness of the group of contact diffeomorphisms.

In fact, an argument similar to Gromov's famous non-squeezing theorem also implies that there are additional obstructions to the extendability of symplectic structures; see Section 6.2. This still left open the possibility of a flexible resolution of Problem 3.1(2) in the symplectic case. The problem remains open today.[1] In the contact case by now both problems are answered; see the discussion in Section 5.7.

After Gromov's paper, the rigid side of symplectic topology began unravelling with an exponentially increasing speed. We just mention here the discovery of Floer homology, Hofer's metric, Gromov–Witten invariants, Symplectic Field Theory, the link with the mathematical theory of Mirror Symmetry, as well applications to lower-dimensional topology such as Taubes's "Gromov–Witten = Seiberg–Witten" theorem, the Heegaard Floer homology of Ozsváth and Szabó, and the embedded contact homology of Hutchings and Taubes.

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[1]Gromov proved that any standard at infinity symplectic structure on $\mathbb{R}^4$ is symplectomorphic to the standard one. Hence, this reduced the isotopy classification of such structures to the computation of π_0 of the group of compactly supported diffeomorphisms of $\mathbb{R}^4$.

Applications of holomorphic curves in Hamiltonian dynamics brought us closer to the realization of Poincaré's dream of establishing qualitative properties of mechanical systems (e.g., existence and the number of periodic trajectories) without actually solving the equations of motion. In particular, the *Weinstein conjecture*, which asserted the existence of periodic trajectories of Reeb vector fields, was proven in many cases (see [56, 86]) and in full generality in dimension 3 (see [80]).

4. FLEXIBLE MILESTONES AFTER THE RESOLUTION OF GROMOV'S ALTERNATIVE

Though in a shadow of successes on the rigid side, the flexible side of symplectic topology also had over the years a number of success stories. Here are examples of some interesting developments with a distinctly flexible flavor.

Overtwisted contact structures in dimension 3. It was understood in 1989 (see [23]) that in the world of 3-dimensional contact manifolds there is an important dichotomy. If a contact manifold contains the so-called *overtwisted disc*, i.e., an embedded disc which along its boundary is tangent to the contact structure, then the contact structure becomes very flexible and abides by a certain *h*-principle: *two overtwisted contact structures which are homotopic as plane fields are homotopic as contact structures*, and hence in view of Gray's theorem are isotopic. Non-overtwisted contact manifolds are called *tight*, and that is where the rigid methods of symplectic topology are applicable.

The classification of overtwisted contact structures yields similar flexibility results for Legendrian knots in overtwisted contact 3-manifolds. Namely, Legendrian knots in the complement of an overtwisted disc (called *loose* in [28]) also satisfy an *h*-principle. The high-dimensional analog of loose knots is discussed in Section 5.1 below.

It was recently discovered that an analog of classification of overtwisted contact structures holds in all dimensions; see Section 5.7 below.

Donaldson's almost holomorphic sections. We already mentioned above Gromov's *h*-principle for iso-symplectic embeddings in codimension > 2. By applying holomorphic curve technique, it is not difficult to construct counter-examples to a similar *h*-principle in codimension 2. However, Simon Donaldson used his theory of *almost holomorphic sections* of complex line bundles over almost Kähler manifolds to prove the following, among other remarkable results:

Theorem 4.1 ([13])**.** *For any closed $2n$-dimensional symplectic manifold (M, ω) with an integral cohomology class $[\omega] \in H^2(M)$ and a sufficiently large integer k, there exists a codimension 2 symplectic submanifold $\Sigma \subset M$ which represents the homology class that is Poincaré dual to $k\omega$. Moreover, the complement $M \setminus \Sigma$ has a homotopy type of an n-dimensional cell complex (which is the case for complements of hyperplane sections in complex projective manifolds).*

Furthermore, he proved the following *symplectic Lefschetz pencil theorem*:

Theorem 4.2 ([15])**.** *If (V, ω) is a symplectic manifold with an integral cohomology class $[\omega] \in H^2(M)$, then for a sufficiently large integer k there exists a topological Lefshetz pencil in which the fibers are symplectic manifolds representing a homology class dual to $k[\omega]$.*

By definition the topological Lefschetz pencil is equivalent to the complex algebraic one near all the singularities.

E. Giroux [41] adapted Donaldson's theory to the contact case. His theory of open book decompositions of contact manifolds plays an important role in contact topology.

Symplectic embeddings of polydiscs. Let us denote by $P(r_1, \ldots, r_n)$ the polydisc $\{|z_1| \leq r_1, \ldots, |z_n| \leq r_n\} \subset \mathbb{C}^n$, where we assume $r_1 \leq r_2 \leq \cdots \leq r_n$. If $P(r_1, \ldots, r_n)$ symplectically embeds into $P(R_1, \ldots, R_n)$, then the famous Gromov non-squeezing theorem implies that $r_1 \leq R_1$. We also have the volume constraint $r_1 \cdots r_n \leq R_1 \cdots R_n$.

It was a common belief that when $n > 2$ there should exist further constraints on the radii besides the Gromov width and volume constraints. However, Larry Guth proved the following remarkable result on the flexible side, which showed that room for additional constraints is very limited.

Theorem 4.3 ([49])**.** *There exists a constant $C(n)$ depending on the dimension n such that if $C(n)r_1 \leq R_1$ and $C(n)r_1 \cdots r_n \leq R_1 \cdots R_n$, then a polydisc $P(r_1, \ldots, r_n)$ symplectically embeds into $P(R_1, \ldots, R_n)$.*

4.1. **Existence of Stein complex structure.** Stein manifolds are complex manifolds which admit proper holomorphic embeddings into $\mathbb{C}^N$. According to a theorem of H. Grauert, a Stein manifold can also be characterized as a manifold which admits an exhausting strictly plurisubharmonic function. Here the word *exhausting* means *proper and bounded below*, while a real-valued function $\phi : V \to \mathbb{R}$ on a complex manifold V is called *strictly plurisubharmonic or i-convex* if the Hermitian form $-dd^{\mathbb{C}}\phi = 2i\partial\bar{\partial}\phi$, which in local holomorphic coordinates is given by a matrix $\left(\frac{\partial^2 \phi}{\partial z_i \partial \bar{z}_j}\right)$, is positive definite. For an arbitrary complex manifold with a complex structure J, we will use the term J-convex, instead of strictly plurisubharmonic, to stress the dependence on the complex structure J. Here we denote by $d^{\mathbb{C}}\phi(X) := d\phi(iX)$ the differential twisted by the operator of multiplication by $\sqrt{-1}$. It can easily be seen that critical points of a Morse strictly plurisubharmonic function on a complex n-dimensional manifold have index $\leq n$, and hence the Morse theory implies that a Stein manifold of complex dimension n has a homotopy type of a cell complex of real dimension n.

Theorem 4.4 ([24])**.** *Let (V, J) be any manifold of dimension $2n > 4$, and let $\phi : V \to \mathbb{R}$ be an exhausting Morse function without critical points of index $> n$. Then there exists an integrable complex structure $\widetilde{J}$ on V homotopic to J for which the function ϕ is target equivalent to a $\widetilde{J}$-convex function. In particular, $(V, \widetilde{J})$ is Stein.*

Interestingly, complex dimension 2 belongs to the domain of symplectic rigidity. An analog of Theorem 4.4 is wrong in this dimension. For instance, $S^2 \times \mathbb{R}^2$ does not admit any Stein complex structure; see [59].

What is transpired from the proof of Theorem 4.4 is that it is useful to define a symplectic analog of a Stein manifold. The corresponding notion of *Weinstein* manifold, crucial for an understanding of Morse theoretic properties of Stein structures, was introduced in [30]. It formalized the Stein handlebody construction from [24] and symplectic handlebody construction from A. Weinstein's paper [88]. We discuss this notion and related results in Section 5.5 below.

5. Renaissance of the h-principle in symplectic topology

The last three years witnessed a number of quite unexpected advances on the flexible side of symplectic topology.

5.1. **Loose Legendrian knots.** It turns out that in contact manifolds of dimension > 3 there is a remarkable class of Legendrian embeddings—or as we also refer to them *Legendrian knots*, discovered by Emmy Murphy in [67]—which satisfies a certain form of an h-principle. These knots are called *loose*, in analogy with loose knots in overtwisted contact manifolds; see [28]. A remarkable fact about Murphy's loose knots is that, in contrast with the 3-dimensional case, they exist in *all* contact manifolds of dimension > 3.

Stabilization. The *stabilization construction* for Legendrian submanifolds, first introduced in [24] and also described in [8, 67], can be informally defined as follows. In an appropriate Darboux coordinate system near a point on a Legendrian submanifold Λ, its front projection has a cuspidal edge; see Figure 5.1. Then given a domain U near a cusp edge on the lower branch of the front, we push it up so over U the lower branch crosses over the upper one. The Legendrian submanifold corresponding to the modified front is called the U-stabilization of Λ.

More precisely, in the standard contact

$$\mathbb{R}^{2n-1}_{\mathrm{st}} = \left(\mathbb{R}^{2n-1} \ , \ \xi_{\mathrm{st}} = \ker\left(dz + \sum_1^{n-1} p_i dq_i \right) \right),$$

where $(p_1, q_1, \ldots, p_{n-1}, q_{n-1}, z)$ are coordinates in $\mathbb{R}^{2n-1}$, consider a diffeomorphic to $\mathbb{R}^{n-1}$ Legendrian submanifold

$$\Lambda_{\mathrm{cu}} = \left\{ (p_1, q_1, \ldots, p_{n-1}, q_{n-1}, z) \colon q_1 = \frac{1}{2} p_1^2, \ q_2 = \cdots = q_{n-1} = 0, \ z = -\frac{1}{3} p_1^3 \right\}.$$

One can check that given any Legendrian $(n-1)$-submanifold $\Lambda \subset Y$ in a contact $(2n-1)$-manifold Y, any point $u \in \Lambda$ has an arbitrarily small neighborhood $\Omega \subset Y$ that admits a contactomorphism

$$\Phi \colon (\Omega, \Lambda \cap \Omega) \to (\mathbb{R}^{2n-1}_{\mathrm{st}}, \Lambda_{\mathrm{cu}}), \quad \Phi(u) = 0.$$

The stabilization construction is a local modification of a Legendrian knot in a neighborhood of a point. It replaces the preimage of Λ_{cu} by a preimage of another Legendrian Λ_{cu}^U, which coincides with Λ_{cu} at infinity. We describe this modification below.

The two branches of the front Γ_{cu} of the Legendrian Λ_{cu}, i.e., the projection to the $(x_1, \ldots, x_{n-1}, z)$-coordinate subspace, are graphs of the functions $\pm h$, where

$$h(q) = h(q_1, \ldots, q_{n-1}) = \tfrac{2\sqrt{2}}{3} \, q_1^{\frac{3}{2}},$$

defined on the half-space $\mathbb{R}^{n-1}_+ := \{ q = (q_1, \ldots, q_{n-1}) \colon q_1 \geq 0 \}$. Let U be a domain with smooth boundary contained in the interior of $\mathbb{R}^{n-1}_+$, $U \subset \mathrm{Int}\left(\mathbb{R}^{n-1}_+\right)$. Pick a non-negative function $\phi \colon \mathbb{R}^{n+1}_+ \to \mathbb{R}$ with the following properties:

- ϕ has compact support in $\mathrm{Int}\left(\mathbb{R}^{n-1}_+\right)$;
- the function $\widetilde{\phi}(q) := \phi(q) - 2h(q)$ is Morse, $U = \widetilde{\phi}^{-1}([0, \infty))$, and 0 is a regular value of $\widetilde{\phi}$.
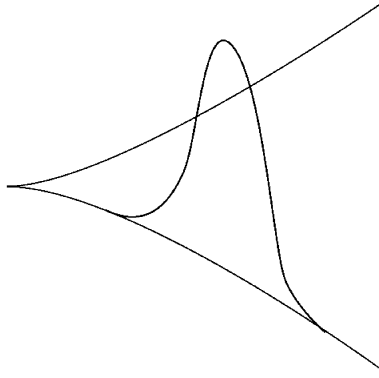
FIGURE 5.1. Stabilization

Consider the front Γ_{cu}^{U} in $\mathbb{R}^{n-1} \times \mathbb{R}$ obtained from Γ_{cu} by replacing the lower branch of Γ_{cu}, i.e., the graph $z = -h(q)$, by the graph $z = \phi(q) - h(q)$. Since ϕ has compact support, the front Γ_{cu}^{U} coincides with Γ_{cu} outside a compact set. Consequently, the Legendrian embedding $\Lambda_{\mathrm{cu}}^{U} \colon \mathbb{R}^{n-1} \to \mathbb{R}^{2n-1}$ defined by the front Γ_{cu}^{U} coincides with Λ_{cu} outside a compact set.

It turns out that if (and only if) the Euler characteristic of the domain U is equal to 0, then the Legendrian submanifolds Λ_{cu} and $\Lambda_{\mathrm{cu}}^{U}$ are *formally* Legendrian isotopic via a compactly supported Legendrian isotopy (however, they are never Legendrian isotopic if $U \neq \varnothing$). We recall that a formal Legendrian isotopy connecting Legendrian embeddings $f_0, f_1 \colon \Lambda \to (Y, \xi)$ is a pair $f_t, \Phi_t, t \in [0,1]$, where f_t is a smooth isotopy and $\Phi_t \colon T\Lambda \to \xi$ is a family of Lagrangian homomorphisms connecting $\phi_0 = df_0$ and $\Phi_1 = df_1$, and such that the paths of homomorphisms df_t, Φ_t are homotopic with fixed endpoints as paths of injective homomorphisms $T\Lambda \to TY$. We also note that when $n = 1$, the Euler characteristic $\chi(U)$ is always positive, and hence the stabilization construction never preserves the formal isotopy class of a 1-dimensional Legendrian knot. This is the main point where the theory in high dimensions deviates from the 1-dimensional case.

Now, given a Legendrian $(n-1)$-submanifold Λ of a contact $(2n-1)$-manifold Y and contactomorphism

$$\Phi \colon (\Omega, \Lambda \cap \Omega) \to (\mathbb{R}_{\mathrm{st}}^{2n-1}, \Lambda_{\mathrm{cu}}),$$

$\Omega \subset Y$ is a neighborhood of a point of $u \in \Lambda$, and we replace $\Omega \cap \Lambda$ with $\Phi^{-1}(\Lambda_{\mathrm{cu}}^{U})$. The resulting Legendrian embedding Λ^{U} which coincides with Λ outside of Ω is called the *U-stabilization* of Λ in Ω.

5.2. **Murphy's theorem.** A Legendrian embedding $\Lambda \to Y$ of a connected manifold Λ is called *loose* if it is isotopic to the stabilization of another Legendrian knot. We point out that looseness depends on the ambient manifold. A loose Legendrian embedding Λ into a contact manifold Y need not be loose in a smaller neighborhood Y', $\Lambda \subset Y' \subset Y$.

The above construction shows that a Legendrian submanifold $\Lambda \subset Y$ can be made loose by stabilizing it in an arbitrarily small neighborhood of a point and even without changing its formal Legendrian isotopy class.

It was known since the early days of the h-principle that formally isotopic Legendrian knots become isotopic after sufficiently many stabilizations. In dimension 3 the corresponding proof was carried out in [50]. Moreover, it was shown by J. Etnyre and K. Honda in [34] that *no a priori given number of stabilizations of 1-dimensional Legendrian knots is sufficient.*

It was an unexpected discovery of E. Murphy that *in dimension > 1 one stabilization is always enough.* Namely, she proved the following h-principle for loose Legendrian knots in contact manifolds of dimension $2n - 1 > 3$:

Theorem 5.1 ([67])**.** *Any two loose Legendrian embeddings which coincide outside a compact set and which can be connected by a formal compactly supported Legendrian isotopy can be connected by a genuine compactly supported Legendrian isotopy.*

5.3. **Lagrangian caps.** Murphy's discovery was followed by a number of other results, which seemed to be out of reach before that. In particular, it turned out that Lagrangian embeddings with loose Legendrian boundaries also satisfy an h-principle.

The story begins with the following question: Let B be the round ball in the standard symplectic $\mathbb{R}^{2n}$. *Is there an embedded Lagrangian disc $\Delta \subset \mathbb{R}^{2n} \setminus \mathrm{Int}\, B$ with $\partial \Delta \subset \partial B$ such that $\partial \Delta$ is a Legendrian submanifold and Δ transversely intersects ∂B along its boundary?*

If $n = 2$, then such a Lagrangian disc does not exist: its existence contradicts the so-called *slice Bennequin inequality*; see [75]. Until recently no such examples were known in higher dimensions either. Surprisingly, it turns out that when $n > 2$, then Lagrangian discs with *loose* Legendrian boundaries satisfy an h-principle, which, in particular, implies that they exist in abundance:

Theorem 5.2 ([32])**.** *Let L be a smooth manifold of dimension $n > 2$ with nonempty boundary such that its complexified tangent bundle $TL \otimes \mathbb{C}$ is trivial. Then there exists an exact Lagrangian embedding $f : (L, \partial L) \to (\mathbb{R}^{2n} \setminus \mathrm{Int}\, B, \partial B)$ with $f(\partial L) \subset \partial B$ such that $f(\partial L) \subset \partial B$ is a Legendrian submanifold and f is transverse to ∂B along its boundary ∂L.*

Note that the triviality of the bundle $TL \otimes \mathbb{C}$ is a necessary (and, according to Gromov's h-principle for Lagrangian immersions [48], sufficient) condition for existence of *any Lagrangian immersion $L \to \mathbb{C}^n$.*

Given a symplectic manifold (X, ω), we say that $L \subset M$ is a *Lagrangian submanifold with an isolated conical point* if it is a Lagrangian submanifold away from a point $p \in L$, and there exists a symplectic embedding $f : B_\varepsilon \to X$ such that $f(0) = p$ and $f^{-1}(L) \subset B_\varepsilon$ is a Lagrangian cone. Here B_ε is the ball of radius ε in the standard symplectic $\mathbb{R}^{2n}$. Note that this cone is automatically a cone over a Legendrian sphere in the sphere ∂B_ε endowed with the standard contact structure given by the restriction to ∂B_ε of the Liouville form $\lambda_{\mathrm{st}} = \frac{1}{2} \sum_1^n (p_i dq_i - q_i dp_i)$. For instance, Givental's Lagrangian Whitney umbrella (see [43]) can be viewed as a Lagrangian cone over the simplest stabilization of the trivial Legendrian knot.

As a special case of Theorem 5.2 (when ∂L is a sphere) one has

Corollary 5.3 ([32])**.** *Let L be an n-dimensional, $n > 2$, closed manifold such that the complexified tangent bundle $T^*(L \setminus p) \otimes \mathbb{C}$ is trivial. Then L admits an exact Lagrangian embedding into $\mathbb{R}^{2n}$ with exactly one conical point. In particular,*

an n-sphere admits a Lagrangian embedding to $\mathbb{R}^{2n}$ with one conical point for each $n > 2$.

5.4. **Lagrangian non-intersections.** The conical singularity with appropriate loose Legendrian asymptotics in Corollary 5.3 can be resolved into an immersion with one self-intersection point. This leads to surprising constructions of Lagrangian immersions with a minimal number of self-intersection points, which at first glance are going against popular Arnold type Lagrangian intersection conjectures. In particular,

Theorem 5.4 ([20]). *Let L be an n-dimensional closed manifold with trivial bundle $TL \otimes \mathbb{C}$. We denote by $s(L)$ the minimal number of double points of a Lagrangian immersion of L into the standard symplectic $\mathbb{R}^{2n}$. Then the following hold:*

(1) *If n is odd or if L is non-orientable, then $s(L) \in \{1, 2\}$.*
(2) *If $n = 3$, then $s(L) = 1$.*
(3) *If n is even and L is orientable, then for $\chi(L) < 0$, $s(L, \sigma) = \frac{1}{2}|\chi(L)|$, and for $\chi(L) \geq 0$, either $s(L) = \frac{1}{2}\chi(L)$ or $s(L) = \frac{1}{2}\chi(L) + 2$.*

The case $n = 2$ is due to D. Sauvaget [78]. It is interesting to compare Theorem 5.4 with the results of [17, 19] which show that if n is even, then the standard n-sphere is the only homotopy n-sphere that admits a self-transverse Lagrangian immersion into Euclidean space with only one double point. This means, in particular, that when $\dim(L)$ is even and $\chi(L) > 0$, then $s(L)$ is generally *not determined* by the homotopy type of L. The following result constrains the homotopy type of a manifold for which this phenomenon may occur.

Theorem 5.5 ([20]). *Let L be an even dimensional spin manifold with $\chi(L) > 0$. If $s(L) = \frac{1}{2}\chi(L)$, then $\pi_1(L) = 1$ and $H_{2k+1}(L) = 0$ for all k. In particular if $\dim L > 4$, then L has the homotopy type of a CW-complex with $\chi(L)$ even-dimensional cells and no odd-dimensional cells.*

It is interesting to note that even for the standard odd-dimensional sphere S^{2k+1} the construction in Theorem 5.4 provides an immersion with a single double point of Maslov index 1, which is different from the standard Whitney Lagrangian immersion $S^{2k+1} \to \mathbb{R}^{4k+2}$, where the intersection point has index $2k + 1$. Using Polterovich's surgery [82] we then get

Corollary 5.6. *There exists a Lagrangian embedding $S^1 \times S^{2k} \to \mathbb{R}^{4k+2}_{\mathrm{st}}$ for which the generator of the first homology of positive action has non-positive Maslov index $2 - 2k$. In particular, there exists a Lagrangian embedding $S^1 \times S^2 \to \mathbb{R}^6_{\mathrm{st}}$ with zero Maslov class.*

The problem of possible values of Maslov class for Lagrangian embeddings of $S^1 \times S^{n-1} \to \mathbb{R}^{2n}$ was first discussed by Polterovich in [81, 82], who also constructed for an even n a Lagranigian embedding $S^1 \times S^{n-1} \to \mathbb{R}^{2n}$ with Maslov classes $n - 1$, in addition to the standard Lagrangian embedding with Maslov class 2, which one gets by resolving the double point of the standard Whitney Lagrangian immersion $S^n \to \mathbb{R}^{2n}$. For an odd n the only previously known Lagrangian embedding was the standard embedding with Maslov class 2. We also note that Fukaya, Oh, Ohta, and Ono proved (see [51]) that the only possible Maslov class values for Lagranigian embeddings $S^1 \times S^{n-1} \to \mathbb{R}^{2n}$ for n odd are 2 and $2 - 2n$.

5.5. **Flexible Stein and Weinstein manifolds.**

5.5.1. *Stein and Weinstein manifolds.* Weinstein manifolds, which we define below, are symplectic counterparts of Stein complex manifolds. Their symplectic tpology is another playground where the theory of loose Legendrian knots yields interesting applications.

Definition. A *Weinstein structure* on an open manifold V is a triple (ω, X, ϕ), where

- ω is a symplectic form on V;
- $\phi : V \to \mathbb{R}$ is an exhausting Morse (or generalized Morse, i.e., having either non-degenerate or birth-death critical points) function;
- X is a complete vector field which is Liouville for ω (i.e., $L_X \omega = \omega$) and gradient-like for the function ϕ.

The quadruple (V, ω, X, ϕ) is then called a *Weinstein manifold*, and the function ϕ is called a *Lyapunov* function for the Liouville field X.

We note that though any Weinstein structure (ω, X, ϕ) can be perturbed to make the function ϕ Morse, in 1-parameter families of Weinstein structures birth-death points are generically unavoidable.

We will also consider Weinstein *cobordism* structures. Let W be a compact manifold with boundary $\partial W = \partial_+ W \amalg \partial_- W$. A Morse (or generalized Morse) function $\phi : W \to \mathbb{R}$ is called *defining* if $\partial_\pm W$ are regular level sets of ϕ with $\phi|_{\partial_- W} = \min \phi$ and $\phi|_{\partial_+ W} = \max \phi$. The notion of a *Weinstein cobordism* (W, ω, X, ϕ) differs from that of a Weinstein manifold only in replacing the condition that ϕ is exhausting by the requirement that ϕ is a defining function, and by replacing the completeness condition of X by the requirement that X is inward transverse to $\partial_- W$ and outward transverse to $\partial_+ W$. A Weinstein cobordism with $\partial_+ W = \emptyset$ is called a *Weinstein domain.*

In the complex geometric context, let us recall that a J-convex (see Section 4.1) function $\phi : V \to \mathbb{R}$ on a complex manifold (V, J) serves as a potential of a Kähler metric $H_{J,\phi}(X, Y) := g_{J,\phi}(X, Y) - i\omega_{J,\phi}(X, Y)$, where $\omega_{J,\phi} = -dd^{\mathbb{C}}\phi$ and $g_{J,\phi}(X, Y) = \omega_{J,\phi}(X, JY)$. The gradient $X_{J,\phi} := \nabla_{J,\phi}\phi$ of the function ϕ with respect to the metric $g_{J,\phi}$ is a Liouville field for $\omega_{J,\phi}$, i.e., $L_{X_{J,\phi}}\omega_{J,\phi} = \omega_{J,\phi}$. If (V, J) is Stein, then for any exhausting J-convex function $\phi : V \to \mathbb{R}$ the vector field $X_{J,\phi}$ can be made complete by composing ϕ with any function $h : \mathbb{R} \to \mathbb{R}$ with positive first and second derivatives; see [8]. Assuming that this was already done, we associate with a Stein complex manifold (V, J) together with an exhausting J-convex (generalized) Morse function $\phi : V \to \mathbb{R}$ a Weinstein structure $\mathfrak{W}(V, J, \phi) = (V, \omega_{\phi,J}, X_{J,\phi}, \phi)$.

By a *Stein cobordism* structure on a cobordism W, we understand a pair (J, ϕ) where J is an integrable complex structure on W and $\phi : W \to \mathbb{R}$ a defining J-convex function. A Stein cobordism with empty $\partial_- W$ is called a *Stein domain.* As in the manifold case, any Stein cobordism structure (J, ϕ) on W determines a Weinstein cobordism structure $\mathfrak{W}(J, \phi) = (W, \omega_{J,\phi}, X_{J,\phi}, \phi)$.

The following result is an upgrade of Theorem 4.4.

Theorem 5.7 ([8]). (1) *Let $\mathfrak{W} = (V, \omega, X, \phi)$ be a Weinstein structure. Then there exists*

- *an integrable complex structure J on V for which ϕ is J-convex and*
- *a homotopy of Weinstein structures $\mathfrak{W}_t = (\omega_t, X_t, \phi)$ connecting $\mathfrak{W}_0 = \mathfrak{W}$ and $\mathfrak{W}_1 = \mathfrak{W}(V, J, \phi)$.*

(2) *Let V be a manifold of dimension $2n \neq 4$, let $\phi : V \to \mathbb{R}$ be an exhausting Morse function without critical points of index $> n$, and let η be a non-degenerate 2-form. Then there exists a Weinstein structure (ω, X, ϕ) on V such that ω and η are homotopic as non-degenerate forms.*

A similar result holds in the cobordism case.

5.5.2. *Flexibility.* Each Weinstein manifold or cobordism can be cut along regular level sets of its Lyapunov function ϕ into Weinstein cobordisms that are *elementary* in the sense that there are no trajectories of the Liouville vector field connecting different critical points of ϕ. An elementary $2n$-dimensional Weinstein cobordism (W, ω, X, ϕ), $n > 2$, is called *flexible* if the attaching spheres of all index n handles form in $\partial_- W$ a *loose Legendrian link*, i.e., each of its components is loose in the complements of the others. A Weinstein cobordism or manifold structure (ω, X, ϕ) is called *flexible* if it can be decomposed into elementary flexible cobordisms.

A $2n$-dimensional Weinstein structure (ω, X, ϕ), $n \geq 2$, is called *subcritical* if all critical points of the function ϕ have index $< n$. Any subcritical Weinstein structure in dimension $2n > 4$ is by definition flexible.

Remark 5.8. The property of a Weinstein structure being subcritical is not preserved under Weinstein homotopies because one can always create a pair of critical points of index n and $n - 1$. It is an open problem *whether or not flexibility is preserved under Weinstein homotopies.*

The following results are proven in [8] using in a crucial way the theory of loose Legendrian knots.

Theorem 5.9. (1) *Let $\mathfrak{W} = (V, \omega, X, \phi)$ be a flexible Weinstein manifold of dimension $2n > 4$. Then there exists a flexible Weinstein homotopy $\mathfrak{W} = (V, \omega_t, X_t, \phi_t)$, $t \in [0, 1]$, with $\mathfrak{W}_0 = \mathfrak{W}$, which is fixed outside a compact set and such that the Morse function ϕ_1 has minimal number of critical points allowed by the Morse theory. If ϕ has finitely many critical points, then the homotopy can be made fixed at infinity.*

(2) *Let $\mathfrak{W}_0 = (\omega_0, X_0, \phi_0)$ and $\mathfrak{W}_1 = (\omega_1, X_1, \phi_1)$ be two flexible Weinstein structures on a manifold V of dimension $2n$. Suppose that ω_0 and ω_1 are homotopic as non-degenerate (not necessarily closed) 2-forms. Then $\mathfrak{W}_0$ and $\mathfrak{W}_1$ can be connected by a homotopy $\mathfrak{W}_t = (\omega_t, X_t, \phi_t)$, $t \in [0, 1]$, of flexible Weinstein structures.*

An analog of Theorem 5.9 also holds for Weinstein *cobordisms*. Combined with Theorem 5.7, we also get an analog of Theorem 5.9 in the Stein case. We will formulate it here only for Stein cobordisms.

Theorem 5.10 ([8]). (1) *Let $\mathfrak{W} = (W, J, \phi)$ be a flexible Stein cobordism of dimension $2n > 4$. Then there exists a homotopy of defining J-convex functions $\phi_t : W \to \mathbb{R}$, $t \in [0, 1]$, with $\phi_0 = \phi$, such that the Morse function ϕ_1 has minimal number of critical points allowed by the Morse theory.*

(2) *Any two flexible Stein cobordism structures (W, J_0) and (W, J_1) which are homotopic as almost complex structures are homotopic as flexible Stein structures.*

In particular, we have the following Stein/Weinstein version of the h-cobordism theorem.

Corollary 5.11 (Weinstein and Stein h-cobordism theorem). *Any flexible Weinstein structure on a product cobordism $W = Y \times [0, 1]$ of dimension $2n > 4$ is homotopic to a Weinstein structure (W, ω, X, ϕ), where $\phi : W \to [0, 1]$ is a defining function without critical points. Similarly, any flexible Stein cobordism (W, J) which is diffeomorphic to $Y \times [0, 1]$ admits a defining J-convex function without critical points.* □

We note that the rigid methods allowed Seidel–Smith and McLean to show that without the flexibility assumption the above claim is wrong; see [63, 83].

5.5.3. *Symplectomorphisms of flexible Weinstein manifolds.* Theorem 5.9 has the following consequence for symplectomorphisms of flexible Weinstein manifolds.

Corollary 5.12. *Let $\mathfrak{W} = (V, \omega, X, \phi)$ be a flexible Weinstein manifold of dimension $2n > 4$, and let $f : V \to V$ be a diffeomorphism such that $f^* \omega$ is homotopic to ω through non-degenerate 2-forms. Then there exists a diffeotopy $f_t : V \to V$, $t \in [0, 1]$, such that $f_0 = f$, and f_1 is an exact symplectomorphism of (V, ω).*

Remark 5.13. Even if $\mathfrak{W}$ is of finite type, i.e., ϕ has finitely many critical points, and $f = $ id outside a compact set, the diffeotopy f_t provided by Theorem 5.12 will be in general *not equal to the identity* outside a compact set.

5.5.4. *Equidimensional symplectic embeddings of flexible Weinstein manifolds.* The following result about equidimensional symplectic embeddings of flexible Weinstein domains is proven in [32] as an application of Lagrangian caps technique.

Theorem 5.14 ([32]). *Let (W, ω, X, ϕ) be a flexible Weinstein domain with Liouville form λ. Let Λ be any other Liouville form on W such that the symplectic forms ω and $\Omega := d\Lambda$ are homotopic as non-degenerate (not necessarily closed) 2-forms. Then there exists an isotopy $h_t : W \hookrightarrow W$ such that $h_0 = $ Id and $h_1^* \Lambda = \varepsilon \lambda + dH$ for some small $\varepsilon > 0$ and some smooth function $H : W \to \mathbb{R}$. In particular, h_1 defines a symplectic embedding $(W, \varepsilon \omega) \hookrightarrow (W, \Omega)$.*

Corollary 5.15 ([32]). *Let (W, ω, Z, ϕ) be a flexible Weinstein domain, and let (X, Ω) be any symplectic manifold of the same dimension. Then any smooth embedding $f_0 : W \hookrightarrow X$, such that the form $f_0^* \Omega$ is exact and the differential $df : TW \to TX$ is homotopic to a symplectic homomorphism, is isotopic to a symplectic embedding $f_1 : (W, \varepsilon \omega) \hookrightarrow (X, \Omega)$ for some small $\varepsilon > 0$. Moreover, if $\Omega = d\Lambda$, then the embedding f_1 can be chosen in such a way that the 1-form $f_1^* \Lambda - i_Z \omega$ is exact. If, moreover, the Liouville vector field dual to Λ is complete, then the embedding f_1 exists for an arbitrarily large constant ε.* □

5.6. **Topology of polynomially and rationally convex domains.** We finish this section by discussing the implications of the above flexibility results for a problem of a high-dimensional complex analysis concerning the topology of polynomially and rationally convex domains.

Polynomial, rational, and holomorphic convexity. Recall the following complex analytic notions of convexity for domains in $\mathbb{C}^n$. For a compact set $K \subset \mathbb{C}^n$, one defines its *polynomial hull* as

$$\widehat{K}_{\mathcal{P}} := \{z \in \mathbb{C}^n \mid |P(z)| \leq \max_{u \in K} |P(u)| \text{ for all complex polynomials } P \text{ on } \mathbb{C}^n\},$$

and its *rational hull* as

$$\widehat{K}_{\mathcal{R}} := \{z \in \mathbb{C}^n \mid |R(z)| \leq \max_{u \in K} |R(u)| \text{ for all rational functions } R = \frac{P}{Q}, \ Q|_K \neq 0\}.$$

Given an open set $U \supset K$, the *holomorphic hull of K in U* is defined as

$$\widehat{K}_{\mathcal{H}}^U := \{z \in U \mid |f(z)| \leq \max_{u \in K} |f(u)| \text{ for all holomorphic functions } f \text{ on } U\}.$$

A compact set $K \subset \mathbb{C}^n$ is called *rationally* (resp. *polynomially*) *convex* if $\widehat{K}_{\mathcal{R}} = K$ (resp. $\widehat{K}_{\mathcal{P}} = K$). An open set $U \subset \mathbb{C}^n$ is called *holomorphically convex* if $\widehat{K}_{\mathcal{H}}^U$ is compact for all compact sets $K \subset U$. A compact set $K \subset \mathbb{C}^n$ is called *holomorphically convex* if it is the intersection of its holomorphically convex open neighborhoods. We have

$$\text{polynomially convex} \implies \text{rationally convex} \implies \text{holomorphically convex}.$$

According to a theorem of E. Levi [58], any holomorphically convex domain $W \subset \mathbb{C}^n$ has *weakly i-convex* (= pseudo-convex) boundary ∂W. The converse statement that the interior of any domain in $\mathbb{C}^n$ with weakly i-convex boundary is holomorphically convex is known as the *Levi problem*. It was resolved in an increasingly more general context in the series of papers begining from K. Oka's paper [73] to the paper [16] of F. Docquier and H. Grauert.

We call a domain $W \subset \mathbb{C}^n$ *i-convex* if its boundary is i-convex. Note that any weakly i-convex domain in $\mathbb{C}^n$ can be C^∞-approximated by a slightly smaller i-convex one.

Topology of polynomially and rationally convex domains. Any i-convex domain $W \subset \mathbb{C}^n$ admits a defining i-convex function, so in particular it admits a defining Morse function without critical points of index $> n$ (see, e.g., [8]). It follows that any holomorphically, rationally, or polynomially convex domain has the same property. We already stated above, see Theorem 4.4, that for $n \geq 3$, any domain in $\mathbb{C}^n$ with such a Morse function is smoothly isotopic to an i-convex one.

It turns out, in the spirit of Theorem 4.4, that for $n \geq 3$ there are no additional constraints on the topology of rationally convex domains.

Theorem 5.16 ([9]). *A compact domain $W \subset \mathbb{C}^n$, $n \geq 3$, is smoothly isotopic to a rationally convex domain if and only if it admits a defining Morse function without critical points of index $> n$.*

The next result gives necessary and sufficient constraints on the topology of polynomially convex domains.

Theorem 5.17 ([9]). *A compact domain $W \subset \mathbb{C}^n$, $n \geq 3$, is smoothly isotopic to a polynomially convex domain if and only if it satisfies the following topological condition:*

(T) *W admits a defining Morse function without critical points of index $> n$, and $H_n(W; G) = 0$ for every abelian group G.*

The "only if" part is well known; see [2] and also [35]. Note that, in view of the universal coefficient theorem, condition (T) is equivalent to the condition

(T') W admits a defining Morse function without critical points of index $> n$, $H_n(W) = 0$, and $H_{n-1}(W)$ has no torsion.

Further analysis of condition (T) yields

Proposition 5.18 ([9])**.** (a) *If W is simply connected, then condition* (T) *is equivalent to the existence of a defining Morse function without critical points of index $\geq n$.*

(b) *For any $n \geq 3$ there exists a (non-simply connected) domain W satisfying condition* (T) *with $\pi_n(W, \partial W) \neq 0$. In particular, W does not admit a defining function without critical points of index $\geq n$.*

Theorems 5.16 and 5.17 are consequences of the following more precise result for flexible Stein domains:

Theorem 5.19 ([9])**.** *Let (W, J) be a flexible Stein domain of complex dimension $n \geq 3$, and let $f : W \hookrightarrow \mathbb{C}^n$ be a smooth embedding such that f^*i is homotopic to J through almost complex structures. Then (W, J) is deformation equivalent to a rationally convex domain in $\mathbb{C}^n$. More precisely, f is smoothly isotopic to an embedding $g : W \hookrightarrow \mathbb{C}^n$ such that $g(W) \subset \mathbb{C}^n$ is rationally convex, and g^*i is Stein homotopic to J. If in addition $H_n(W; G) = 0$ for every abelian group G, then $g(W)$ can be made polynomially convex.*

We note that the proofs of all the above results concerning polynomial and rational convexity are based, in addition to the theory of flexible Weinstein manifolds and analytic techniques from the book [8], on the following complex analytic characterization of the polynomial and rational convexity.

The following classical criterion for polynomial convexity goes back to K. Oka's paper [73] (see also [79, Theorem 1.3.8]).

Criterion 5.20. *An i-convex domain $W \subset \mathbb{C}^n$ is polynomially convex if and only if there exists an exhausting i-convex function $\phi : \mathbb{C}^n \to \mathbb{R}$ such that $W = \{\phi \leq 0\}$.*

To formulate a criterion for rational convexity, consider the following condition on a J-convex domain W in a complex manifold (X, J):

(R) There exists a J-convex function $\phi : W \to \mathbb{R}$ such that $W = \{\phi \leq 0\}$, and the form $-dd^{\mathbb{C}}\phi$ on W extends to a Kähler form ω on the whole X.

The following criterion for rational convexity was proved by S. Nemirovski [69] as a corollary of a result of J. Duval and N. Sibony [18, Theorem 1.1].

Criterion 5.21. *An i-convex domain $W \subset \mathbb{C}^n$ is rationally convex if and only if it satisfies condition* (R)*.*

The situation in complex dimension 2 is more complicated. The following theorem of S. Nemirovski and K. Siegel gives necessary and sufficient conditions for a disc bundle over a (not necessarily orientable) surface to be realizable as a rationally convex domain in $\mathbb{C}^2$. Recall that a 2-disc bundle over a non-orientable surface with the orientable total space has a well-defined integer valued Euler number e; see [89].

Let $D(\chi, e)$ denote the orientable D^2-bundle over an orientable surface of Euler characteristic χ, with Euler number e. Let $\tilde{D}(\chi, e)$ denote the non-orientable bundle over a non-orientable surface of Euler characteristic χ, with Euler number e.

Theorem 5.22 ([70])**.** *The following manifolds can be realized as rationally convex domains in $\mathbb{C}^2$:*

- *$D(\chi, 0)$ for $\chi \neq 2$;*
- *$\tilde{D}(\chi, e)$ for $(\chi, e) \neq (1, -2)$ and $e \in \{2\chi - 4, 2\chi + 4, \ldots, -2\chi - 4 + 4[\chi/4 + 1]\}$.*

Moreover, these are the only possibilities.

5.7. **Classification of overtwisted contact structures in all dimensions.** It was recently discovered that there exists a high-dimensional analog of overtwisted contact structures (see Section 4) and that the extension h-principle and the corresponding classification result holds in all dimensions, not only in dimension 3.

Let us recall that a formal homotopy counterpart of a contact structure is an *almost contact structure.* This is a defined up to a scalar factor pair (λ, ω), where λ is non-vanishing 1-form on M, possibly with local coefficients in a non-trivial 1-bundle, and ω is a non-degenerate 2-form on the hyperplane field $\xi = \{\alpha = 0\}$ with coefficients in the same local system. In the coorientable case, i.e., when TM/ξ is trivial, the existence of almost contact structure is equivalent to the existence of a *stable almost complex structure* on M, i.e., a complex structure on the bundle $TM \oplus \varepsilon^1$, where ε^1 is the trivial line bundle over M.

We already mentioned above that in 1969 M. Gromov [45] proved a parametric h-principle for contact structures on an *open* manifold M: *any almost contact structure is homotopic to a genuine one, and two contact structures are homotopic if they are homotopic as almost contact structures.*

For 3-dimensional closed manifolds J. Martinet [61] and R. Lutz [60] proved the non-parametric existence h-principle for 3-manifolds. Then D. Bennequin [4] discovered a first contact rigidity result: the 1-parametric h-principle fails for contact structures on S^3. As was already mentioned in Section 4, building on Bennequin's work, in [23] the author introduced a dichotomy of 3-dimensional contact manifolds into *tight* and *overtwisted* and established a parametric h-principle for overtwisted contact structures.

A lot of progress was achieved in the last three decades in the direction of contact rigidity. Tight contact structures were also classified on several classes of 3-manifolds; see e.g. [26, 42, 53, 54]. V. Colin, E. Giroux, and K. Honda in [11] proved that any atoroidal contact 3-manifold admits at most finitely many non-isotopic tight contact structures. Symplectic Field Theory provided powerful invariants for distinguishing contact structures on manifolds of all dimensions; see [29], [85], [63], et al.

The flexible side also had a number of success stories. In particular, the existence h-principle for a closed 5-manifold was gradually established, beginning from the work of H. Geiges [36, 37] and H. Geiges and C.B. Thomas, [39, 40] and followed by the work of R. Casals, D. M. Pancholi, and F. Presas [10] and J. Etnyre [33]. For manifolds of dimension > 5 the results were more scarce. The work [24] implied existence of contact structures on all closed $(2n + 1)$-dimensional, $n > 1$, manifolds which bound manifolds of homotopy type of $(n + 1)$-dimensional cell complexes. F. Bourgeois [7] proved, using work of E. Giroux [41], that for any closed contact
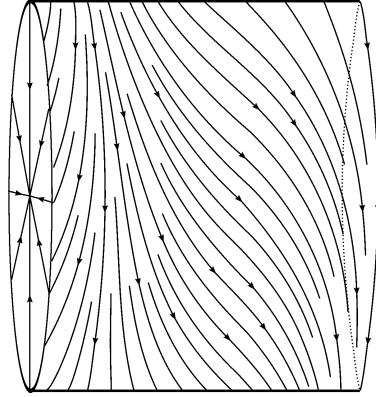
FIGURE 5.2. An overtwisted disk with its characteristic foliation

manifold M and any surface Σ with genus at least one, the product $M \times \Sigma$ admits a contact structure. This positively answered a long-standing problem about existence of contact structures on tori of dimension $2n + 1 > 5$.

The following recently proved results significantly advanced the flexible side of the contact story in higher dimensions.

Theorem 5.23 ([6])**.** *Let M be a $(2n+1)$-manifold, and let $A \subset M$ be a closed set. Suppose that M and $M \setminus A$ are connected. Let ξ be an almost contact structure on M which is genuine on $\mathcal{O}p\, A \subset M$. Then ξ is homotopic relative to A to a genuine contact structure. In particular, any almost contact structure on a closed manifold is homotopic to a genuine contact structure.*

It turned out that the *overtwisting phenomenon* can be generalized to higher dimensions as well. Without giving a precise definition, we will just say here that a contact manifold (M, ξ) is called *overtwisted* if it admits a contact embedding of a piecewise smooth $2n$-disc D_{ot} with a certain model germ ζ_{ot} of a contact structure; see Figure 5.2.

Theorem 5.24 ([6])**.** *On any closed manifold M any almost contact structure is homotopic to an overtwisted contact structure which is unique up to isotopy.*

To formulate a more precise result consider a $(2n+1)$-dimensional manifold M, a closed subset A, and an almost contact structure ξ_0 on M that is a genuine contact structure on A, and define $\mathrm{Cont}_{\mathrm{ot}}(M; A, \xi_0)$ to be the space of contact structures on M that are overtwisted on $M \setminus A$ and that coincide with ξ_0 on $\mathcal{O}p\, A$. The notation $\mathrm{cont}(M; A, \xi_0)$ stands for the corresponding *formal* object, i.e., the space of *almost* contact structures that agree with ξ_0 on $\mathcal{O}p\, A$. We denote by j the inclusion map $j : \mathrm{Cont}_{\mathrm{ot}}(M; A, \xi_0) \to \mathrm{cont}(M; A, \xi_0)$. We denote by $\mathrm{Cont}_{\mathrm{ot}}(M; A, \xi_0, \phi)$ and $\mathrm{cont}_{\mathrm{ot}}(M; A, \xi_0, \phi)$ the subspaces of $\mathrm{Cont}_{\mathrm{ot}}(M; A, \xi_0)$ and $\mathrm{cont}(M; A, \xi_0)$ of contact and almost contact structures for which a fixed embedding $\phi : D_{\mathrm{ot}} \to M \setminus A$ is a contact embedding $(D_{\mathrm{ot}}, \zeta_{\mathrm{ot}}) \to (M, \xi)$.

Theorem 5.25 ([6])**.** *If $M \setminus A$ is connected, then the inclusion map induces an isomorphism*

$$j_* : \pi_0(\mathrm{Cont}_{\mathrm{ot}}(M; A, \xi_0)) \to \pi_0(\mathrm{cont}(M; A, \xi_0)),$$

and moreover the map

$$j : \mathrm{Cont}_{\mathrm{ot}}(M; A, \xi_0, \phi) \to \mathrm{cont}_{\mathrm{ot}}(M; A, \xi_0, \phi)$$

is a (weak) homotopy equivalence.

As an application of Theorem 5.24 we provide an explicit classification of over-twisted contact structures on spheres S^{2n+1}. Note that almost contact structures on the sphere S^{2n+1} are classified by the homotopy group $\pi_{2n+1}(SO(2n+2)/U(n+1))$. Hence, Theorem 5.24 implies that the elements of this group also enumerate isotopy classes of overtwisted contact structures on S^{2n+1}.

The following lemma computes this group.

Lemma 5.26 (Bruno Harris, [52])**.**

$$\pi_{2n+1}(SO(2n+2)/U(n+1)) = \begin{cases} \mathbb{Z}/n!\mathbb{Z}, & n = 4k; \\ \mathbb{Z}, & n = 4k+1; \\ \mathbb{Z}/\frac{n!}{2}\mathbb{Z}, & n = 4k+2; \\ \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}, & n = 4k+3. \end{cases}$$

In particular, on spheres S^{8k+1}, $k > 0$, there are exactly $(4k)!$ different over-twisted contact structures, on spheres S^{8k+5}, $k \geq 0$, there are $\frac{(4k+2)!}{2}$ different overtwisted contact structures, while on all other spheres there are infinitely many. For instance, there is a unique overtwisted contact structure on S^5.

It is interesting to note that S^5 has infinitely many *tight*, i.e., non-overtwisted contact structures. Besides the standard contact structure, these are examples given by Brieskorn spheres (see [85]). The full classification of tight contact structures on any manifold of dimension > 3 is an open problem.

We note that there were many proposals for defining the overtwisting phenomenon in dimension greater than three. The overtwistedness in the above sense is stronger than any other possible notions. Namely, any exotic phenomenon, e.g., a *plastikstufe* [71], can be found in any overtwisted contact manifold. Indeed, suppose we are given some exotic model (A, ζ) which is compact contact manifold with boundary which formally embeds into (M, ξ_{ot}). Given any smooth embedding $A \hookrightarrow M$, there is an almost contact structure on M which is equal to ζ on A and homotopic to ξ_{ot} through almost contact structures. Theorem 5.23 implies this almost contact structure can be homotoped relative to A to a contact structure ξ on M. Theorem 5.25 and Gray's theorem [44] implies that ζ is isotopic to ξ_{ot}.

In particular, the known results about contact manifolds with a *plastikstufe* apply to overtwisted manifolds as well:

- *overtwisted contact manifolds are not symplectically fillable* [71];
- *the Weinstein conjecture holds for any contact form defining an overtwisted contact structure on a closed manifold* [1];
- *any Legendrian submanifold whose complement is overtwisted is loose. Conversely, any loose Legendrian in an overtwisted ambient manifold has an overtwisted complement* [66].

We also point out the following corollary concerning iso-contact embeddings into an overtwisted contact manifold

- *Let (N, ζ) and (M, ξ) be two contact manifolds, so that (M, ξ) is overtwisted and either N is open or $\dim N < \dim M$. Let $\Phi : TN \to TM$ be an iso-contact bundle homomorphism (i.e., $\Phi_* \zeta \subset \xi$ respects the linear conformal*

symplectic structure) covering an embedding $f : N \to M$ such that $df, \Phi :$
$TN \to TM$ are homotopic as injective bundle homomorphisms. Then f is
isotopic to an iso-contact embedding $\widetilde{f} : (N, \zeta) \to (M, \xi)$.

6. NEXT STEPS OF SYMPLECTIC FLEXIBILITY

6.1. **Holomorphic curves or nothing dichotomy.** Since Gromov's seminal pa-
per [46] holomorphic curve technique, sometimes applied in a very subtle and unex-
pected way, brought tremendous progress on the side of symplectic rigidity. In fact,
all other methods and techniques, such as generating functions, Floer homological
techniques etc., are different variations on the theme of holomorphic curves. The
multiple attempts to introduce genuinely different methods so far did not bring to
life anything promising.

Hence, I propose the "*holomorphic curves or nothing*" principle: *If one cannot
disprove a flexible h-principle type conjecture using holomorphic curve techniques
and its derivatives, then it should be true.* In fact, this principle guided the authors
in the work [6] because there was no feasible way to use holomorphic curves in
overtwisted contact manifods to establish any rigidity results.

Here are a few very optimistic conjectures implied by the above principle.

6.2. **Symplectic structures on closed manifolds.** Let us recall Problem 3.1(1):
*When does a germ of a symplectic or contact structure along the boundary of an
n-ball B extend to B?* In the contact case Theorem 5.25 provides a flexible answer
to this question.

In the symplectic case there is, first of all, an additional *volume obstruction*.
Namely, given a germ ω of a symplectic structure along a boundary ∂D^{2n} of a ball
D^{2n}, $n > 1$, we observe that the form ω is exact, $\omega = d\eta$, and if ω extends as a
symplectic form to D^{2n}, then by Stokes' theorem

$$\int_{\partial D^{2n}} \eta \wedge \omega^{n-1} = \int_{D^{2n}} \omega^n > 0,$$

where the left-hand side is independent of the choice of a primitive η. But even
with this modification, an extension h-principle for symplectic structures fails in
all dimensions > 1, as a holomorphic curve argument, similar to Gromov's proof of
his famous non-squeezing theorem, [46], shows. It seems that this argument never
appeared in print, so we sketch it here. In the standard symplectic space

$$(\mathbb{R}^{2n}, \omega_{\mathrm{st}} = dp_1 \wedge dq_1 + \cdots + dp_n \wedge dq_n),$$

consider three parallelepipeds

$$A := \{|p_i|, |q_i| \le 1,\ i = 1, \ldots, n-1,\ |q_n| \le 1,\ 0 \le p_n \le 1\},$$

$$B := \{|p_i|, |q_i| \le 2,\ i = 1, \ldots, n-1,\ |q_n| \le 2,\ 0 \le p_n \le \frac{1}{4}\ \},$$

and

$$C := \{|p_i|, |q_i| \le 1,\ i = 1, \ldots, n-1,\ |q_n| \le 1,\ 0 \le p_n \le \frac{1}{8}\}.$$

Note that Volume $A = 2^{2n-1} <$ Volume $B = 4^{2n-2}$, while A has a larger Gromov's
width: $w(A) = 2 > w(B) = 1$. Hence, Gromov's non-squeezing theorem implies
that there is no symplectic embedding $A \to B$. The parallelepipeds $A, B,$ and C
share a common boundary piece $E := \{p_n = 0, |p_i|, |q_i| \le 1,\ i = 1, \ldots, n-1,, 0 \le
p_n \le 1\}$, so that the remaining parts of the boundary of A and B form an *immersed*

piecewise smooth $(2n-1)$-sphere $\widetilde{\Sigma} := (\partial A \setminus \operatorname{Int} E) \cup (\partial B \setminus \operatorname{Int} E)$. Similarly, the remaining parts of the boundary of C and B form an *embedded* piecewise smooth $(2n-1)$-sphere $\Sigma := (\partial C \setminus \operatorname{Int} E) \cup (\partial B \setminus \operatorname{Int} E)$ which bounds a ball G. There exists a smooth immersion $h \colon \mathcal{O}p\,\Sigma \to \mathcal{O}p\,\widetilde{\Sigma}$ such that $h(\Sigma) = \widetilde{\Sigma}$. We claim that the symplectic structure $h^*\omega$ on $\mathcal{O}p\,\Sigma$ does not extend as a symplectic structure to G while neither homotopic nor volume obstructions exist for such an extension. Indeed, suppose that such an extension $\widetilde{\omega}$ does exist. Then one can glue back (A, ω) to $(G, \widetilde{\omega})$ using a symplectomorphism h on $\mathcal{O}p\,(\partial A \setminus E)$. The resulting symplectic manifold

$$(H, \widehat{\omega}) := (G, \widetilde{\omega}) \underset{h}{\cup} (A, \omega)$$

contains the parallelepiped (A, ω), while near its boundary ∂H it is symplecto-morphic to (B, ω). It remains to notice that the proof of Gromov's non-squeezing theorem works without any changes for symplectic embeddings into any aspherical symplectic manifold which is symplectomorphic to a polydisc near its boundary. Hence, the symplectic structure $h^*\omega$ on $\mathcal{O}p\,\Sigma$ is not extendable to G.

The existence h-principle for symplectic structures on closed manifolds is also known to be wrong in dimension 4; see [84]. However, in higher dimensions there are no known obstructions, and even no feasible approaches how such obstruction could be constructed. Could it be then that this problem in dim > 4 belongs to the flexible world? In other words,

Conjecture 6.1. *On any manifold M of dimension $n = 2k > 4$ with a cohomology class $\eta \in H^2(M)$ with $\eta^k \neq 0$ and a non-degenerate 2-form ω_0, there exists a symplectic form ω homotopic to ω_0 through non-degenerate forms, whose cohomology class $[\omega]$ can be deformed to η keeping its k-th power non-vanishing.*

As we already pointed out above, the analogous statement is wrong in dimension 4. However, could it be that it is still *virtually true*?

Conjecture 6.2. *Given any 4-manifold M with a cohomology class $\eta \in H^2(M; \mathbb{Z})$ with $\eta^2 \neq 0$ and a non-degenerate 2-form ω_0, one can find an orientable surface $F \subset M$ which realizes a homology class dual to $N\eta$ for a sufficiently large N, and such that a certain branch cover of M along F admits a symplectic structure for which the branching locus is symplectic, whose cohomology class is the pullback of η and which is homotopic to the pullback of the 2-form ω_0 through non-degenerate forms.*

6.3. **Symplectic caps.** Holomorphic curve methods also seem to be helpless for *symplectic manifolds with an overtwisted concave contact boundary*. We say that a compact symplectic manifold (X, ω) has a *concave contact boundary* (Y, ξ) if near Y the form ω is exact: $\omega = d\lambda$, the contact structure ξ is defined by the form $\lambda|_Y$, and the Liouville vector field Z which is ω-dual to λ, $\iota(Z)\omega = \lambda$, is inward transverse to $Y = \partial X$.

Conjecture 6.3. *Symplectic manifolds with a concave overtwisted contact boundary satisfy an h-principle. In particular, suppose we are given*

- *a compact $2n$-dimensional manifold X with boundary Y;*
- *a non-degenerate (not necessarily closed) 2-form ω which is exact near Y;*
- *an inward transverse to Y Liouville vector field Z for $\omega|_{\mathcal{O}p\,Y}$, i.e., $\omega = d(\lambda := \iota(Z)\omega)$ on $\mathcal{O}p\,Y$, such that the contact structure $\xi = \{\lambda|_Y = 0\}$ is overtwisted;*

- *a closed 2-form η on X such that $\eta|_{OpY} = \omega|_{\mathcal{O}p\,Y}$ and*

$$\int_X \eta^n > \int_Y \lambda \wedge (d\lambda)^{n-1}.$$

Then there exists a homotopy ω_t, $t \in [0,1]$, connecting $\omega_0 = \omega$ and a symplectic form ω_1, such that ω_t is non-degenerate on X, and $\omega_t = \omega$ on $\mathcal{O}p\,Y$ for all $t \in [0,1]$.

Conjecture 6.3 would imply the following weaker version of Conjecture 6.1 which potentially could hold also in dimension 4.

Conjecture 6.4. *Under assumptions of Conjecture* 6.1, *the required symplectic form ω on X could be constructed with just one singular point, so that near this point ω is symplectomorphic to the negative part of the symplectization of an overtwisted contact structure on S^{2n-1} from the standard almost contact homotopy class.*

We note that according to Theorem 5.24 such a contact structure is unique. A similar h-principle may even hold for symplectic cobordisms with an overtwisted concave boundary (or even more optimistically, for the symplectic extension problem). In particular, it seems plausible that *there exists a* symplectic concordance, *i.e., a symplectic structure on $S^{2n-1} \times [0,1]$, $n > 1$, between the overtwisted contact structure on the concave side $S^{2n-1} \times 0$ and the standard one on the convex one $S^{2n-1} \times 1$.*

Note that for $n = 2$, a symplectic cobordism (rather that a concordance) does exist between an overtwisted contact sphere on the concave side and the standard contact sphere on the convex one; see, e.g., [34].

6.4. **How far could Arnold conjectures go?** The most optimistic form of the Arnold conjecture concerning the lower bound for the number of intersection points of an exact Lagrangian submanifold $L \subset T^*M$ of a closed manifold M with its 0-section asserts that the number of intersection points, assuming the transversality of the intersection, is bounded below by the minimal number $\mathrm{Morse}(M)$ of critical points of a Morse function on M. However, the best lower bound that is known in the case when L is Lagrangian isotopic to the 0-section is in terms of the *stable* Morse number $\mathrm{Morse}_{\mathrm{st}}(M)$, i.e., the minimal number of critical points of a function on $M \times \mathbb{R}^q$ which, outside a compact set, coincides with the pullback of a non-degenerate quadratic form on $\mathbb{R}^q$. The numbers $\mathrm{Morse}(M)$ and $\mathrm{Morse}_{\mathrm{st}}(M)$ are known to be different (see, e.g., [14]). It does not seem feasible to the author that holomorphic curve methods could be used to prove an estimate in terms of $\mathrm{Morse}(M)$. Hence, the "*holomorphic curves or nothing*" dichotomy suggests that *if* $\mathrm{Morse}(M) > \mathrm{Morse}_{\mathrm{st}}(M)$, *then there should exist an exact Lagrangian submanifold $L \subset T^*(M)$, possibly even Hamiltonian isotopic to the 0-section which intersects M transversely in less than $\mathrm{Morse}(M)$ points.*

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