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SELECTED MATHEMATICAL REVIEWS

related to the paper in the previous section by
YAKOV ELIASHBERG

MR1668563 (2000a:57070) 57R17; 32E10, 57R65

Gompf, Robert E.

Handlebody construction of Stein surfaces.

Annals of Mathematics. Second Series **148** (1998), *no.* 2, 619–693.

In this paper the author studies the topology of 4-dimensional compact Stein manifolds and their boundaries that are contact 3-manifolds. Such a study of compact Stein manifolds was first undertaken in a more general setting by Y. Eliashberg, who showed that for $n > 2$ a smooth almost-complex open $2n$-manifold admits a Stein structure if and only if it is the interior of a (possibly infinite) handlebody without handles of index $\geq n$, and that a similar result holds for $n = 2$ with some restriction on attaching 2-handles.

Recall that a contact structure on a closed oriented 3-manifold M is a totally nonintegrable 2-plane field $\xi \subset TM$, that is, if the plane field is given by the kernel of a 1-form α then $d\alpha \wedge \alpha$ is never equal to zero. Hence $d\alpha \wedge \alpha$ gives an orientation on M; if it agrees with the original orientation of M then ξ is called a positive contact structure (otherwise negative). Also, a knot $K \subset M$ is called Legendrian if all its tangents lie in ξ. Such knots inherit a natural framing from ξ which is called the Thurston-Bennequin framing. Being Legendrian is not a restriction to a knot type because all knots can be isotoped to Legendrian knots.

Now, consider the simplest kind of Stein manifold $B^4 \subset \mathbf{C}^2$. Restriction of the complex lines of $\mathbf{C}^2$ to the boundary gives a contact structure on $S^3 = \partial B^4$. Eliashberg's theorem says that the Stein structure on B^4 can be extended across 1-handles to $X_1 = 1B^4\cup$(1-handles). Furthermore, the Stein structure on X_1 can be extended across 2-handles to $X_2 = X_1\cup$(2-handles), provided 2-handles are attached to a Legendrian link with the framing of each component at least one less than its Thurston-Bennequin framing. Contact 3-manifolds which are boundaries of such Stein manifolds X_2 are called "fillable". It is known that "fillable" contact 3-manifolds are "tight" contact 3-manifolds in the sense of Eliashberg.

The author surveys these results and by using framed link descriptions of 4-manifolds he gives many interesting applications. For example, he proves that there is an uncountable collection of exotic $\mathbf{R}^4$'s admitting Stein structures. He proves that various interesting 4-manifolds admit Stein structures, and that as a consequence the 3-manifolds which occur as their boundaries have tight contact structures. For example, he shows that every Seifert fibered space bounds a Stein 4-manifold after possibly reversing its orientation; some of them could bound a Stein manifold with both orientations, but he conjectures that only one orientation of $\Sigma(2,3,5)$ can be a Stein boundary (this conjecture was recently proven by P. Lisca [Geom. Topol. **2** (1998), 103–116 (electronic); MR1633282 (99f:57038); Turkish J. Math. **23** (1999), no. 1, 151–159]). The author associates some invariants of contact 3-manifolds; in particular he gives a complete set of invariants of 2-plane fields on 3-manifolds. One consequence of this is the following: Let (M_i, ξ_i) be two contact 3-manifolds ($i = 1, 2$) which are boundaries of Stein 4-manifolds obtained by attaching 2-handles to B^4 along Legendrian knots K_i as above. Suppose that there is an orientation preserving diffeomorphism $\varphi\colon M_1 \to M_2$ such that $\varphi_*(\xi_1)$ is homotopic to ξ_2. Then the knots K_1 and K_2 have the same Thurston-Bennequin numbers (framings), and up to sign they have the same rotation numbers (the rotation number of a Legendrian knot K is just the relative first Chern class of the dual canonical line bundle of $\mathbf{C}^2$ restricted to the Seifert surface of K).

To sum up, this is a useful paper which not only puts Eliashberg's results in a nice perspective, but also provides many interesting new results.

Selman Akbulut

From MathSciNet, October 2014

MR1909245 (2003g:53164) 53D99; 57R17, 58E99

Eliashberg, Y.; Mishachev, N.

Introduction to the *h*-principle. (English)

Graduate Studies in Mathematics, 48.

American Mathematical Society, *Providence, RI*, 2002, xviii+206 pp., $30.00, ISBN 0-8218-3227-1

The book under review, as the title implies, is an introduction to the h-principle. Anyone who has spent time trying to mine the beautiful depths of Gromov's book [M. L. Gromov, *Partial differential relations*, Springer, Berlin, 1986; MR0864505 (90a:58201)] or the research literature on the h-principle will certainly welcome this book.

The basic idea of the h-principle is as follows: any differential equation (or relation) can be interpreted as a subset S of an appropriate jet space and a solution is simply an appropriate function whose jet lies in S. Gromov's strategy for solving differential relations was to first find a section of the jet bundle whose image is in S and then try to show that there is a function whose jet agrees with this section. The first part of this program frequently has an algebraic (or sometime geometric) flavor, while the second part is usually more analytic. A differential equation (or relation) satisfies an h-principle if the second part of the above strategy follows automatically (though not necessarily easily) from the first part. Said another way, an equation (or relation) satisfies an h-principle if its solvability is determined by some algebraic (or geometric) data associated to the problem.

There are many ways to try to show that a problem satisfies an h-principle; this book considers two: holonomic approximations and convex integration. Holonomic approximations are a relatively recent variant of Gromov's continuous sheaves methods developed by the authors [in *Essays on geometry and related topics, Vol. 1, 2*, 271–285, Enseignement Math., Geneva, 2001; MR1929330 (2003k:58006)].

The h-principle has been a useful way to prove, or interpret prior proofs of, results in topology and geometry. This book describes many of these applications with a specific emphasis on symplectic and contact geometry and various embedding and immersion theorems. In addition one can find a good introduction to the literature.

John B. Etnyre

From MathSciNet, October 2014

MR2286033 (2007k:57053) 57R17; 53D35

Niederkrüger, Klaus

The plastikstufe—a generalization of the overtwisted disk to higher dimensions.

Algebraic & Geometric Topology **6** (2006), 2473–2508.

In a series of papers in the 1980's and 1990's, building on work of Bennequin and Gromov, Y. M. Eliashberg [Ann. Inst. Fourier (Grenoble) **42** (1992), no. 1-2, 165–192; MR1162559 (93k:57029)] established a fundamental dichotomy in 3-dimensional contact geometry. Specifically, he observed that contact structures were either overtwisted—that is, contained an embedded disk, called an overtwisted disk, that was tangent to the contact planes along the boundary—or tight—that is, not overtwisted. Overtwisted contact structures are classified and are determined by simple algebraic topological data [Y. M. Eliashberg, Invent. Math. **98** (1989), no. 3, 623–637; MR1022310 (90k:53064)]. On the other hand, tight contact structures still remain somewhat mysterious: they do not always exist on manifolds, they are classified only on a few simple families of manifolds and they seem to be connected with subtle properties of a manifold that supports them. One of the key differences between tight and overtwisted contact structures is that overtwisted contact structures cannot be the weak convex boundary of a symplectic manifold [Y. M. Eliashberg, in *Geometry of low-dimensional manifolds, 2 (Durham, 1989)*, 45–67, Cambridge Univ. Press, Cambridge, 1990; MR1171908 (93g:53060)]. In other words, the existence of an overtwisted disk is an obstruction for a contact structure to be symplectically fillable.

The paper under review proposes to generalize the notion of overtwisted disk to higher dimensions. A similar such generalization was suggested in M. Gromov's landmark paper [Invent. Math. **82** (1985), no. 2, 307–347; MR1554036], but has not been developed. The author suggests the generalization of an overtwisted disk to a contact structure ξ on a $(2n-1)$-dimensional manifold M is plastikstufe. Plastikstufe is, loosely speaking, a family of overtwisted disks. More rigorously, a plastikstufe with singular set S is an embedding of $S \times D^2$ into M such that S is an $(n-2)$-dimensional manifold, ξ induces a singular foliation on $S \times D^2$, $\partial(S \times D^2)$ is the only closed leaf of the singular foliation, $S \times \{0\}$ is the singular set of the foliation (which is elliptic) and the other leaves of the foliation are diffeomorphic to $S \times (0,1)$. Note that in 3 dimensions S must be a point and $S \times D^2$ is just a disk with singular foliation having a single elliptic point in the center, ∂D^2 is tangent to ξ and the other leaves spiral from the elliptic point to the boundary. Such a disk is called the standard overtwisted disk. In general one can take the product structure on $S \times D^2$ so that any point in S crossed with D^2 has the same foliation as the standard overtwisted disk. Thus we may think of a plastikstufe as a family of overtwisted disks parameterized by S.

The main theorem of the paper is that if a contact structure admits a plastikstufe then it cannot be symplectically filled by a semi-positive symplectic manifold; moreover, if the contact manifold is of dimension less than or equal to 5 then it has no symplectic filling at all. As a corollary to this result the author observes that many exotic contact structures can be constructed on Euclidean spaces.

The strategy of proof for the main theorem is somewhat similar to Eliashberg and Gromov's proof in dimension 3. Specifically, if a contact manifold is symplectically fillable and contains a plastikstufe then one chooses an almost complex structure

J compatible with the symplectic structure and studies the moduli space of J-holomorphic disks with marked point on the boundary and with boundary mapping to the plastikstufe. A detailed analysis of this moduli space leads to the conclusion that the homology of the plastikstufe with the singular set removed is trivial in dimension $n-1$. Since this is clearly not the case one cannot have a symplectic filling of a contact manifold that admits a plastikstufe.

Unfortunately it is not clear at this point if platikstufe will provide the same fundamental dichotomy as overtwisted disks did for 3-dimensional contact manifolds, but they at least provide a place to begin to study fillability properties in higher dimensions.

John B. Etnyre
From MathSciNet, October 2014

MR3012475 53-02; 32E10, 32Q65, 53C15, 53D05, 58E05

Cieliebak, Kai; Eliashberg, Yakov

From Stein to Weinstein and back. (English)

American Mathematical Society Colloquium Publications, 59.
American Mathematical Society, *Providence, RI*, 2012, xii+364 pp.,
ISBN 978-0-8218-8533-8

In 1990, Y. M. Eliashberg [Internat. J. Math. **1** (1990), no. 1, 29–46; MR1044658 (91k:32012)] published the startling result that a manifold of even dimension greater than four admits the structure of a Stein manifold (i.e. a complex manifold that can be properly and holomorphically embedded into some $\mathbb{C}^N$) if and only if certain obviously necessary homotopy-theoretic conditions are satisfied. The proof was based on a combination of Morse-theoretic ideas with several powerful h-principles emerging from the "soft" side of symplectic geometry, and it thus initiated the study of Stein manifolds from a symplectic topological perspective. In subsequent years, work of Eliashberg and M. Gromov [in *Several complex variables and complex geometry, Part 2 (Santa Cruz, CA, 1989)*, 135–162, Proc. Sympos. Pure Math., 52, Part 2, Amer. Math. Soc., Providence, RI, 1991; MR1128541 (93f:58073); Ann. of Math. (2) **136** (1992), no. 1, 123–135; MR1173927 (93g:32037)], A. D. Weinstein [Hokkaido Math. J. **20** (1991), no. 2, 241–251; MR1114405 (92g:53028)], R. E. Gompf [Ann. of Math. (2) **148** (1998), no. 2, 619–693; MR1668563 (2000a:57070)] and others revealed further deep insights into the symplectic nature of Stein manifolds: in particular, it became a well-known folk theorem that the essentially complex geometric objects we call "Stein structures" are in some sense equivalent to certain Morse-theoretic symplectic data called "Weinstein structures", so that the two notions can be (and in the symplectic literature often are) used interchangeably for most purposes. The impact that these results have had on symplectic topology over the last 20 years cannot be overstated, though their proofs have until now seemed largely inaccessible to all but a few experts. Much more recently, the exciting discovery by E. Murphy ["Loose Legendrian embeddings in high dimensional contact manifolds", preprint, arXiv:1201.2245] of a flexible class of Legendrian submanifolds has brought the subject to maturity, leading in particular to a dramatically improved understanding of flexibility in Stein manifolds.

The book under review provides the first comprehensive account of this story, including detailed proofs of several important results that have been considered "standard" among symplectic topologists for many years but were only fully understood by a very few people, as well as new results about flexible Stein structures that have quickly come to be regarded as fundamental in the subject.

In order to summarize the main results, recall first that by H. Grauert's characterization of Stein manifolds [Ann. of Math. (2) **68** (1958), 460–472; MR0098847 (20 #5299)], a *Stein structure* on an open manifold V of real dimension $2n$ can be defined as an integrable complex structure J such that (V, J) admits a smooth function $\phi V \to \mathbb{R}$ that is *exhausting* (i.e. proper and bounded below) and plurisubharmonic (or J*-convex*, as it is usually called in this book). Since plurisubharmonic Morse functions never have critical points of index greater than n, every Stein manifold has the homotopy type of an n-dimensional CW-complex. The original result of Eliashberg, proved in [op. cit.; MR1044658 (91k:32012)] and stated in this book as Theorem 1.5, says that for any almost complex manifold (V, J) that has real dimension $2n > 4$ and admits an exhausting Morse function without critical points of index greater than n, J is homotopic through almost complex structures to a Stein structure; in fact, after this homotopy one can find an exhausting plurisubharmonic Morse function with the same level sets as φ. This theorem also has a variant for compact Stein cobordisms (W, J), where one assumes the J-convex function $\varphi\, W \to \mathbb{R}$ is constant and regular on each boundary component—such cobordisms are also called *Stein domains* whenever the negative part of the boundary is empty.

In modern terms, Eliashberg's existence result for Stein structures can be understood to follow from two results of a more obviously symplectic nature, and these two results serve as the focal point of the book. A *Weinstein structure* (ω, X, φ) on an open $2n$-dimensional manifold V is defined to consist of a symplectic form ω, together with an exhausting (generalized) Morse function $\varphi V \to \mathbb{R}$ and a gradient-like Liouville vector field X. These conditions ensure that level sets of φ are naturally contact type hypersurfaces, so in particular any Weinstein cobordism can also be regarded as an exact symplectic cobordism between two contact manifolds. Any Stein structure J with accompanying J-convex function φ naturally gives rise to a Weinstein structure $(-d(d\varphi \circ J), \nabla\varphi, \varphi)$, whose homotopy class is independent of φ. (Note that for compact cobordisms, this last statement is more or less obvious, but the notions of Stein and Weinstein homotopy on noncompact open manifolds are somewhat subtler—this issue is discussed at length in §11.6.)

The two main theorems on existence and homotopy (both of which also have variants for domains and cobordisms) can now be expressed as follows:

Theorem 13.2 (existence): Suppose V is a manifold of dimension $2n > 4$ with an exhausting Morse function φ that has no critical points of index greater than n, and η is a nondegenerate 2-form on V. Then V admits a Weinstein structure (ω, X, φ) such that ω and η are homotopic through nondegenerate 2-forms.

Theorem 1.1 ("from Weinstein to Stein"): Any Weinstein structure is homotopic to one that arises from a Stein structure. Moreover, two Stein structures are Stein homotopic if and only if the induced Weinstein structures are Weinstein homotopic.

Note that while the first of these two theorems is again only valid in higher dimensions, the second has no such restriction. Theorem 1.1 can be understood as saying that the natural map from the space of Stein structures to the space of Weinstein structures induces an isomorphism on π_0. The authors conjecture in fact that this map should be a weak homotopy equivalence—proving this will require a subtler analysis accounting for singularities that can appear in parametric families of Morse functions.

Another important theme in this book is flexibility. It has been understood since Eliashberg's work in the 1990s that *subcritical* Stein structures (i.e. those for which the J-convex function has no index n critical points) satisfy an h-principle in dimensions greater than four: two subcritical Stein structures on the same manifold are homotopic if and only if their underlying almost complex structures are homotopic. This fact is a deep consequence of a more basic h-principle due to M. Gromov [*Partial differential relations*, Ergeb. Math. Grenzgeb. (3), 9, Springer, Berlin, 1986; MR0864505 (90a:58201)] for (subcritical) isotropic embeddings in contact manifolds, which serve as the attaching spheres for subcritical Weinstein handles. While no such h-principle holds for isotropic embeddings in the critical dimension (i.e. Legendrian submanifolds), a major step forward occurred in 2011, when Murphy [op. cit.] discovered a special class of so-called "loose" Legendrian embeddings which do satisfy an h-principle. This provides a way to construct a correspondingly special class of so-called *flexible* Weinstein structures, defined as those for which all critical Weinstein handles are attached along Legendrian spheres that are loose. This leads to:

Theorem 1.8 (flexibility): The Weinstein structure constructed by Theorem 13.2 (see above) may be assumed flexible without loss of generality, and two flexible Weinstein structures on a manifold of dimension $2n > 4$ are Weinstein homotopic if and only if their symplectic structures are homotopic through nondegenerate 2-forms.

This result also has analogues for Weinstein domains and Weinstein cobordisms. The situation is more complicated in dimension $2n = 4$, though, as discussed in Chapters 14 and 15, some version of an h-principle does hold for 4-dimensional Weinstein cobordisms whose concave contact boundary is overtwisted. Since there is no flexible class of Legendrian knots in tight contact 3-manifolds, a 4-dimensional Weinstein domain is flexible if and only if it is subcritical, but even in the subcritical case it is not currently known whether the h-principle holds in general (see the discussion of Chapter 16 below for more on this).

The impact of these flexibility results during the two years since they were first publicized in talks by both authors has been pronounced. They have furnished major ingredients for instance in a proof by J. B. Etnyre that every almost contact 5-manifold is contact ["Contact structures on 5-manifolds", preprint, arXiv:1210.5208], S. Courte's negative answer to the question of whether a contact manifold is determined by its symplectization [Geom. Topol. **18** (2014), no. 1, 1–15; MR3158770], and a hint of flexibility for higher-dimensional contact manifolds discovered by Murphy et al. [Geom. Topol. **17** (2013), no. 3, 1791–1814; MR3073936].

Before further outlining the contents, I would like to offer a tip for first-time readers: the authors have produced two shorter expository articles ["Stein structures: existence and flexibility", preprint, arXiv:1305.1619; "Flexible Weinstein manifolds", preprint, arXiv:1305.1635] that sketch the main ideas of the proofs in the book, and it is worth reading through these to get the big picture before delving into the (often quite intricate) details. The first, based on two lecture series given by K. Cieliebak in 2012 at the IAS and at a summer school in Budapest, is a very digestible explanation of the existence theorem and flexibility in the subcritical case, while the second goes into more detail about loose Legendrians and flexible Weinstein structures.

Here is a rough outline of the organization of the book.

After an introductory chapter that states the main results, the book is divided broadly into five parts, with the first four devoted to the proofs of the results discussed above. Part 1 is the most complex-analytic, including many basic results and technical lemmas about J-convex functions on integrable complex manifolds. Chapter 5 in particular constitutes a brief digression into the theory of several complex variables. Only a few of the results in this chapter are actually needed in the rest of the book, but they provide an intriguing glimpse into a world that may be largely unfamiliar to many symplectic readers, and those who (like this reviewer) have an interest in symplectic fillings of contact manifolds will find the discussion of CR structures and holomorphic fillings in §5.10 especially illuminating.

Part 2 is concerned with the proof of Eliashberg's theorem on the existence of Stein structures, including a review chapter on basic notions from symplectic and contact geometry and another chapter on h-principles before proving the main theorem in Chapter 8. Chapter 7 on h-principles deserves special mention, as it serves as a valuable complement to the popular monograph of Eliashberg and N. M. Mishachev [*Introduction to the h-principle*, Grad. Stud. Math., 48, Amer. Math. Soc., Providence, RI, 2002; MR1909245 (2003g:53164)], and also includes a thorough discussion (though not a complete proof) of Murphy's h-principle for loose Legendrians.

Parts 3 and 4 are devoted mainly to the results involving Weinstein structures. Since these results are based to a large extent on ideas originating in Morse-Smale theory, Part 3 provides some preparatory Morse-theoretic results, including (for illustrative purposes) a sketch of the proof of the h-cobordism theorem, and generalizations of these results for J-convex functions. Part 4 then focuses on Weinstein structures, including in Chapter 11 the main definitions of fundamental notions such as Weinstein (and Stein) homotopies on noncompact manifolds, and flexible Weinstein structures. The proofs of the main results on existence of Weinstein structures and the "Weinstein implies Stein" theorem are spread throughout Chapters 12 through 15, using a number of Morse-theoretic lemmas that are Weinstein analogues of results proved for J-convex functions in earlier chapters. Chapter 14 also proves the h-principle for flexible Weinstein structures, and uses it to give a detailed proof of Cieliebak's famous theorem that "subcritical Weinstein manifolds are split" (originally proved in the 2002 preprint ["Subcritical Stein manifolds are split", preprint, arXiv:math/0204351], which was never published).

The fifth and final part of the book consists of two chapters on topics that are somewhat distinct from the main results in the introduction, but no less important. Chapter 16 discusses Stein surfaces, which are specifically excluded from the main existence and flexibility theorems in the book (most of these require $2n > 4$), but exhibit a wealth of rigidity phenomena that can be proved using pseudoholomorphic curves. The most important of these is probably the following, whose proof was originally sketched in [Y. M. Eliashberg, in *Geometry of low-dimensional manifolds, 2 (Durham, 1989)*, 45–67, London Math. Soc. Lecture Note Ser., 151, Cambridge Univ. Press, Cambridge, 1990; MR1171908 (93g:53060)]:

Theorem 16.6: Every Stein filling of S^3 is deformation equivalent to the standard Stein structure on the unit ball.

The book gives an almost but not completely self-contained proof of this theorem; it relies on a technical result about filling by holomorphic disks whose proof is only sketched, but readers comfortable with holomorphic curves will be able to fill in the missing details with a little help from standard references such as [D.

McDuff and D. A. Salamon, *J-holomorphic curves and symplectic topology*, second edition, Amer. Math. Soc. Colloq. Publ., 52, Amer. Math. Soc., Providence, RI, 2012; MR2954391] (a more complete proof of closely related results has also been published by H. Geiges and K. Zehmisch [J. Topol. Anal. **2** (2010), no. 4, 543–579; MR2748217 (2012d:53270)]). It is worth pointing out the relationship between this result and flexibility: if the h-principle were known for subcritical Stein surfaces, one could conclude from the above that all Stein structures on the 4-ball are Stein homotopic, a stronger condition than Stein deformation equivalence. One can show in fact that Theorem 16.6 implies this if and only if the group of orientation-preserving diffeomorphisms of the 4-ball is connected—the latter, unfortunately, is a notoriously difficult open question in differential topology. Similarly, another argument based on filling by holomorphic disks shows (Theorem 16.7) that any Stein filling of a contact connected sum of two 3-manifolds can be decomposed into a Stein 1-handle attached to Stein fillings of those two manifolds. This is used to show that Stein structures on fillings of subcritically fillable contact 3-manifolds are unique up to deformation equivalence—again, not enough is known about the diffeomorphism groups of these 4-manifolds to turn this into a proper flexibility result. The last section of Chapter 16 proves corresponding results about uniqueness (up to homotopy) of finite type Stein structures on open 4-manifolds, including the nonexistence of such structures on $S^2 \times \mathbb{R}^2$, which provides a counterexample to the $n = 2$ case of Eliashberg's existence theorem. The proofs are to a large extent minor variations on the earlier proofs for compact Stein domains, but an additional ingredient is needed to gain control over the topology of level sets near infinity, and the ingredient used here is G. Perelman's solution to the geometrization conjecture ["Finite extinction time for the solutions to the Ricci flow on certain three-manifolds", preprint, arXiv:math/0307245]. It would be interesting to know whether these proofs can still be carried out without relying on such deep results.

Chapter 17 discusses exotic Stein structures, such as the discoveries by M. McLean [Geom. Topol. **13** (2009), no. 4, 1877–1944; MR2497314 (2011d:53224)], M. Abouzaid and P. Seidel ["Altering symplectic manifolds by homologous recombination", preprint, arXiv:1007.3281] and others of infinitely many Stein structures on $\mathbb{C}^n$ for $n \geq 3$ that are not deformation equivalent to the standard one. Combining this with their flexibility results and the symplectic homology computations of F. Bourgeois, T. Ekholm and Eliashberg [Geom. Topol. **16** (2012), no. 1, 301–389; MR2916289], the authors conclude their exposition with a brief proof of the following beautiful result:

Theorem 17.2: Suppose V is a manifold of dimension $2n > 4$ that admits a finite type Stein structure J. Then V admits infinitely many finite type Stein structures such that all are homotopic to J through almost complex structures, but no two of them are Stein deformation equivalent to each other.

The book concludes with three appendices, of which the first two review material from algebraic topology, and the third provides biographical notes on several of the mathematicians who have played major roles in the study of Stein manifolds. Unlike Chapters 2 through 17, Appendix C can easily be read just before going to sleep for the night, and I highly recommend it.

Chris M. Wendl

From MathSciNet, October 2014

MR3158770 53D10; 53D05

Courte, Sylvain

Contact manifolds with symplectomorphic symplectizations.

Geometry & Topology **18** (2014), *no.* 1, 1–15.

Given a contact manifold (M,ξ), consider the one-dimensional subbundle of the cotangent bundle of M formed by all non-zero cotangent vectors that vanish on ξ and induce the right coorientation. The total space $S_\xi M$ of this principal ${\mathbb{R}_+}^*$-bundle is diffeomorphic to $\mathbb{R}\times M$ and is called the symplectization of (M,ξ). Let λ_ξ be the 1-form on $S_\xi M$ induced by the canonical 1-form on T^*M. Then $\omega_\xi = d\lambda_\xi$ is a symplectic form on $S_\xi M$ and the Liouville vector field X_ξ (i.e., the vector field defined by the relation $\iota_{X_\xi}\omega_\xi = \lambda_\xi$) is the infinitesimal generator of the ${\mathbb{R}_+}^*$-action.

The paper under review partially answers the following important and longstanding open question. Given two contact manifolds (M,ξ) and (M',ξ') for which $S_\xi M$ and $S_{\xi'}M'$ are symplectomorphic, are (M,ξ) and (M',ξ') necessarily contactomorphic? The main result of the paper under review shows that this is not the case, not even if we assume that $S_\xi M$ and $S_{\xi'}M'$ are exact symplectomorphic, i.e., that there is a diffeomorphism $\Psi : S_\xi M \to S_{\xi'}M'$ such that $\Psi^*\lambda_{\xi'} - \lambda_\xi$ is exact (note, however, that if we have $\Psi^*\lambda_{\xi'} = \lambda_\xi$ then (M,ξ) and (M',ξ') are easily seen to be contactomorphic). More precisely, Courte shows that there exist contact manifolds (M,ξ) and (M',ξ') of dimension greater than or equal to 5 such that $S_\xi M$ and $S_{\xi'}M'$ are exact symplectomorphic but M and M' are not even diffeomorphic. On the other hand, it remains an open question whether there can be two non-contactomorphic contact structures ξ and ξ' on the same manifold M such that $S_\xi M$ and $S_{\xi'}M'$ are (exact) symplectomorphic. This question is particularly difficult to answer because most known contact invariants (for example those coming from SFT) cannot distinguish contact manifolds that have exact symplectomorphic symplectization.

The problem just described can be seen as a symplectic analogue of the following question from differential topology. Can we have two closed orientable non-diffeomorphic manifolds M and M' such that $\mathbb{R}\times M$ and $\mathbb{R}\times M'$ are diffeomorphic? The answer to this question is yes. Indeed, on the one hand, whenever M and M' have dimension greater than or equal to 5 and are h-cobordant, $\mathbb{R}\times M$ and $\mathbb{R}\times M'$ are diffeomorphic. On the other hand, we do have h-cobordant manifolds that are not diffeomorphic (for example Milnor showed that this is the case for $M = L(7,1)\times S^{2n}$ and $M' = L(7,2)\times S^{2n}$). The fact that $\mathbb{R}\times M$ and $\mathbb{R}\times M'$ are diffeomorphic if M and M' are h-cobordant closed orientable manifolds of dimension greater than or equal to 5 can be seen as follows. Let $(W;M,M')$ be an h-cobordism between M and M'. As a consequence of the s-cobordism theorem there is an inverse h-cobordism $(W';M',M)$, i.e., an h-cobordism such that the compositions $W\odot W'$ and $W'\odot W$ are diffeomorphic to $[0,1]\times M$ and $[0,1]\times M'$, respectively. Consider now the infinite sum $V = \cdots\odot W\odot W'\odot W\odot W'\odot\cdots$. Since we can see this sum both as $V = \bigodot_{j\in\mathbb{Z}}(W\odot W') \cong \mathbb{R}\times M$ and as $V = \bigodot_{j\in\mathbb{Z}}(W'\odot W) \cong \mathbb{R}\times M'$ (Mazur trick), we have that $\mathbb{R}\times M$ and $\mathbb{R}\times M'$ must be diffeomorphic.

By generalizing the above argument to the symplectic setting, Courte proves that if (M,ξ) is a closed contact manifold of dimension greater than or equal to 5 then for any h-cobordism $(W;M,M')$ there is a contact structure ξ' on M' such that $S_\xi M$ and $S_{\xi'}M'$ are exact symplectomorphic (and thus by taking M and M' to be h-cobordant but not diffeomorphic we obtain examples of non-contactomorphic contact manifolds with exact symplectomorphic symplectizations). The generalization of the above differential topological argument to the symplectic world is made possible by the flexibility properties of certain Weinstein cobordisms, which were discovered and described by Y. M. Eliashberg [in *Gauge theory and symplectic geometry (Montreal, PQ, 1995)*, 49–67, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 488, Kluwer Acad. Publ., Dordrecht, 1997; MR1461569 (98g:58055)], E. Murphy ["Loose Legendrian embeddings in high dimensional contact manifolds", preprint, arXiv:1201.2245] and K. Cieliebak and Eliashberg [*From Stein to Weinstein and back*, Amer. Math. Soc. Colloq. Publ., 59, Amer. Math. Soc., Providence, RI, 2012; MR3012475].

Recall that a Weinstein cobordism from a contact manifold (M,ξ) to (M',ξ') is a cobordism $(W;M,M')$ together with a Morse pair (X,φ) and a symplectic form ω on W such that $\mathcal{L}_X\omega=\omega$. We also ask that $\xi=\ker(i^*\lambda)$ and $\xi'=\ker(i'^*\lambda)$ where $\lambda=\iota_X\omega$ and i and i' are the inclusions of M and M' into W. If (W,ω,X,φ) is a Weinstein cobordism of dimension $2n$ then all critical points of φ have index smaller than or equal to n. Weinstein cobordisms such that all critical points have index strictly smaller than n are called subcritical and, as discussed by Eliashberg [op. cit.], have remarkable flexibility properties. A larger class of Weinstein cobordisms with flexibility properties was more recently described by Cieliebak and Eliashberg [op. cit.] using the notion of loose Legendrians introduced by Murphy [op. cit.]. We say that a Weinstein cobordism is flexible if it is the composition of finitely many elementary cobordisms (i.e., the Liouville vector field has no trajectories joining critical points) such that the attaching spheres of all critical handles are loose Legendrians. Using flexibility results from Cieliebak and Eliashberg for this class of Weinstein cobordisms, Courte is able to prove the following. Let (M,ξ) be a closed contact manifold of dimension greater than or equal to 5 and let $(W;M,M')$ be an h-cobordism. Then there exists a flexible Weinstein structure (ω,X,φ) on W which induces the given contact structure ξ on M. Let ξ' be the contact structure on M' induced by the Weinstein structure (ω,X,φ). Consider the inverse h-cobordism $(W';M',M)$. Again, there is a flexible Weinstein structure (ω',X',φ') on W' which induces the contact structure ξ' on M'. After some work and again using flexibility results from Cieliebak and Eliashberg, Courte proves that the contact structure on M induced by (ω',X',φ') can be assumed to be the initial ξ, and that the Mazur trick can be adapted to this situation (by translating the Weinstein cobordisms W and W' in an appropriate way) in order to prove that $S_\xi M$ and $S_{\xi'}M'$ are exact symplectomorphic.

Starting from this result, Courte also discusses the unexpected fact that the contact manifold at infinity of a Weinstein manifold (V,ω,X,φ) does depend on the choice of the Liouville vector field X. More precisely, Courte describes how one can find a Weinstein homotopy (ω,X_s,φ_s) on V, with fixed symplectic form, during which the topology of the contact manifold at infinity changes. Finally, Courte also describes a similar result for contact manifolds at infinity of Stein manifolds.

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