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Unipotent and nilpotent classes in simple algebraic groups and Lie algebras, by Martin W. Liebeck and Gary M. Seitz, Mathematical Surveys and Monographs, 180, American Mathematical Society, Providence, RI, 2012, xii+380 pp., ISBN 978-0-8218-6920-8, US$76.80

One of the most fruitful ways to study a group is to consider its actions on various objects, e.g., sets, linear spaces, varieties, algebras, graphs, etc. In particular, the conjugation action of a group G on itself is a fundamental object of study. The orbits are the conjugacy classes. If G is an algebraic group, then there is also some geometric structure that one can study. The action of G on its Lie algebra $\mathfrak{g}$ is also quite an important representation.

For the rest of this review fix an algebraically closed field k of characteristic $p \geq 0$ and a simple linear algebraic group G with Lie algebra $\mathfrak{g}$ over k.

The orbits of a simple algebraic group G in the two actions described above have been an object of study since the beginning of Lie theory. Let us consider the case of conjugacy classes first. Any element $g \in G$ has a Jordan decomposition $g = su = us$, where s is semisimple and u is unipotent (this generalizes the usual notion for an invertible matrix). The semisimple classes are relatively well understood: they are parametrized by the orbits of the Weyl group of G on a maximal torus. The conjugacy classes with a given semisimple part s correspond to the unipotent classes in $C_G(s)$—a nice reductive group (if G is simply connected, then $C_G(s)$ is in fact a connected reductive group). Thus, the crucial case to study is that of unipotent classes.

In the case of $\mathrm{SL}_n(k)$, the unipotent classes are parametrized by partitions of n. In particular, there are only finitely many unipotent conjugacy classes. The finiteness of the number of unipotent classes was proved for all reductive groups by Dynkin and Konstant (see [7]) in characteristic 0, and it was conjectured to hold in all characteristics by Steinberg [19]. Richardson [17] proved an analogous result when p is good for G (in particular if $p > 5$). His proof was considerably easier but did not apply in bad characteristic. It turns out that the description of the unipotent conjugacy classes in good characteristic is the same as for characteristic 0.

The full result was finally proved by Lusztig [12]. Lusztig's proof used the Deligne–Lusztig theory of complex characters of the finite groups of Lie type. The idea is as follows: it is enough to show that there are only finitely many unipotent classes when k is the algebraic closure of the finite field $\mathbb{F}_p$ and thus to show there is a uniform upper bound on the number of conjugacy classes of p-elements in $G(p^a)$, i.e., a bound independent of p^a. Lusztig does this by writing down a family of characters of bounded cardinality of $G(p^a)$ which separates unipotent classes, whence the result. The bound he gets this way is not sharp.

For classical groups (i.e., linear, symplectic and orthogonal groups) in characteristic not 2, it turns out that the conjugacy class of a unipotent element is determined by its Jordan form, and so the issue is only to determine what are the possible Jordan forms. The answer is that Jordan blocks of odd size come in pairs in

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symplectic group and Jordan blocks of even size come in pairs in orthogonal groups [21]. The problem is considerably more difficult in characteristic 2— even for involutions. For example, there are two conjugacy classes of involutions in $\mathrm{Sp}_4(k)$ for k of characteristic 2 for elements with two Jordan blocks of size 2. Hesselink [3] solved the problem for the classical groups in characteristic 2. Lusztig (see [13] and the references therein) has had a series of papers explaining how the cases of bad characteristic can be understood.

There is a vast literature on the subject studying various aspects of the problem. In particular, there are extensive tables for the conjugacy classes of unipotent elements in the exceptional algebraic groups (i.e., G_2, F_4, E_6, E_7, and E_8 by various authors). The book under review includes such tables (which are quite useful).

An important invariant is the size of the conjugacy class. Of course, the dimension of a conjugacy class is just the codimension of its centralizer. The dimensions of the centralizers have long been studied as well (see [3] for the classical groups and various papers for the exceptional groups), but not in a completely satisfactory way. One wants to know the unipotent radical of the centralizer C, its reductive quotient, and the component group of C.

One of the attractive features of this book are the new results on the structures of centralizers. In the case of classical groups, the authors use results of Hesselink [3] to write down representatives for the unipotent classes (which are complicated in the case of characteristic 2). These representatives are a bit different from those described by Hesselink. They compute the centralizers giving more detailed information about the structure of these groups (including the structure of the component group). For the exceptional groups, they start by studying E_8. In particular, they write down a set of representatives of the unipotent classes. They show these are distinct and compute their centralizers, including the component groups. By Lang's theorem, they can see how many $\mathbb{F}_{p^a}$ points there are for each of the conjugacy classes. They then compute the total number of unipotent elements contained in the classes they have written down—the number they obtain is p^{240a}. By a result of Steinberg [20] that is the total number of unipotent elements, whence all classes are represented and in particular there are only finitely many unipotent classes. They deal with the other exceptional groups by viewing them as subgroups of E_8 and determining which E_8 classes intersect the smaller groups and how the classes break up in the smaller groups.

This approach provides precise, detailed information about unipotent classes in the finite groups of Lie type. Since most finite simple groups are of Lie type, this is very important.

The unipotent conjugacy classes and component groups of the centralizers for the groups of type E were described by Mizuno [15].

Similarly, the book considers the action of G on its Lie algebra $\mathfrak{g}$. In good characteristic there are Springer correspondences between the unipotent variety of G and the variety of nilpotent elements in $\mathfrak{g}$, and so the analysis is essentially the same. For bad characteristic, this is no longer the case, and it is quite delicate to find all the orbits and the stabilizers. For E_8 (in any characteristic) this was originally done by Holt and Spaltenstein [4]. See also [6]. As noted above, the finiteness of the number of orbits in all characteristics is a result of Lusztig; see [18].

This book is a very useful addition to the literature. It gives an independent proof of the finiteness of the number of unipotent classes and nilpotent orbits.

It puts many of the results about unipotent and nilpotent classes in an easily accessible place. In particular, they produce a large number of very useful tables summarizing the results both for the algebraic and finite groups. Moreover, the authors obtain new results about centralizers which are very useful—particularly when dealing with the finite groups of Lie type. One result that comes out easily from the description of unipotent classes is that if x and y are unipotent elements in a simple algebraic group with $\langle x\rangle = \langle y\rangle$, then x and y are conjugate (i.e., unipotent elements are rational). This had originally been proved by Lusztig [14]. Another consequence is that if G is a simple algebraic group and $u \in G$ is a nontrivial unipotent element, then a double coset $C_G(u)xC_G(u)$ is never dense in G. This has been used by Prasad [16] in studying quasi-reductive groups (a more general result had been obtained independently in [2]). The results and tables in the book have already been used by several authors.

Of course, there are many aspects of unipotent elements which are not considered here. The geometry of the nilpotent cone or the unipotent variety is not studied (see above for some references, and also see Humphreys [5]). There is no discussion of the natural partial ordering on unipotent classes induced by taking closures of conjugacy classes. Here the book of Spaltenstein [18] is the the best source. There is no mention of the Springer correspondence, relating unipotent classes to representations of the Weyl group which plays a crucial role in the representation theory of the finite groups of Lie type. The structure of the unipotent radical of the centralizer of a unipotent element is not studied in depth (other than the dimension). For example, there is no mention of the result that an element $g \in G$ has abelian centralizer if and only if it is regular (see [11] for the fact that regular elements have abelian centralizers, and see Kurtzke [8] and Lawther [9] for the converse). This is not a criticism—the authors have chosen to address certain aspects of the subject and do so in a very readable and extremely useful way.

The case of unipotent classes in automorphism groups of simple algebraic groups is addressed here only for the orthogonal groups. The other cases are dealt with in [10]. One still has finiteness results in this case (see [18] – the result is due to Lusztig). In fact, there is an easy proof of the finiteness of outer unipotent classes given the finiteness of inner unipotent classes [1].

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