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SELECTED MATHEMATICAL REVIEWS

related to the paper in the previous section by
ANDREW GRANVILLE

MR0424730 (54 #12689) 10H30

Halberstam, H.; Richert, H.-E.

Sieve methods. (English)

London Mathematical Society Monographs, 4.
Academic Press [*A subsidiary of Harcourt Brace Jovanovich, Publishers*],
London–New York, 1974, xiv+364 pp.

A notable gap in the literature of number theory has been the lack of any complete and connected account of the small sieve method and its applications. It is the aim of this book to fill this gap. The volume has been long awaited by sieve theorists and has attained the unusual distinction of having been referred to in several research papers that appeared some years before the book itself was actually published.

The authors give a self-contained account of the Brun and Selberg sieve methods and a number of applications, with emphasis on the estimation of the number of almost primes (i.e. numbers with a limited number of prime factors) occurring in polynomial sequences $\{F(n); n \le x\}$ and $\{F(p); p \le x\}$. Other applications, together with historical remarks, and mention of work in progress, are dealt with briefly in the notes to each chapter, which are a particularly useful inclusion.

The style of presentation is extremely clear and no pains are spared to give full verifications that the often numerous conditions of theorems are indeed satisfied in the particular problems to which they are applied. As the authors admit, the expert may find this somewhat tedious. Nevertheless some research papers have contained errors due precisely to a failure to perform such verifications. The book has a very full bibliography arranged by both author and subject.

After their introductory remarks, the authors, in the first chapter, develop those properties of the sequences mentioned above that will be needed for the applications. They describe also the sieve of Eratosthenes-Legendre and use it to prove the weak result $\pi(x) = O(x/\log\log x)$.

In Chapter 2 it is shown how Brun refined the basic idea of the Eratosthenes-Legendre sieve to yield considerably improved results. The treatment given, which is due to the authors, formulates a general condition that must be satisfied by any sieving function of the type considered by Brun. They then go on to give an ab initio treatment of Brun's pure sieve and use it to obtain the upper bound $O(x(\log\log x)^2(\log x)^{-2})$ for the number of prime twins not exceeding x. Returning to the more intricate approach described at the beginning of the chapter, a more complicated choice of sieving function is introduced, and this is used to obtain a first result in the corresponding lower-bound problem, viz. that there are infinitely many n such that both n and $n+2$ have at most 7 prime factors, or, in a useful notation, denoting by P_r a number with at most r prime factors, $P_7 + 2 = P_7$, infinitely often. Further applications and results of the "Fundamental Lemma" type are also discussed. The sieve of Rosser is briefly described in the last section, but for the full details, even of special cases, of this, one must still refer to the work of H. Iwaniec [Acta Arith. **19** (1971), 1–30; MR0296043 (45 #5104); ibid. **21** (1972), 203–234; MR0304331 (46 #3466)].

The next four chapters are concerned with the Selberg upperbound sieve. Chapter 3 describes the basic method and proves the Brun-Titchmarsh inequality in the form due to J. van Lint and the second author [ibid. **11** (1965), 209–216; MR0188174 (32 #5613)]. It is also applied to obtain upper bound results in the prime-twin and Goldbach problems. The next two chapters (4 and 5) are devoted to further applications, such as upper bounds for the number of integers n for which $F(n)$ is prime. Chapter 6 describes an extension of the Selberg upper-bound method which plays an important role in deriving the Selberg lower-bound.

The first step in obtaining the lower bound is treated in Chapter 7, which is based on the work of N. Ankeny and H. Onishi [ibid. **10** (1964–5), 31–62; MR0167467 (29 #4740)].

Chapter 8 deals with the method of W. Jurkat and the second author [ibid. **11** (1965), 217–40; MR0202680 (34 #2540)], which obtains the definitive sieve bounds for the linear case (i.e. sieving by one residue class) by iteration of the Buhštab identity. In the following chapter a weighting procedure is introduced which considerably enhances the effectiveness of the bounds and enables a proof to be given that if F is an irreducible polynomial of degree g then $F(n)$ is infinitely often a P_{g+1} and $F(p)$ infinitely often a P_{2g+1}. This chapter is based on work of the second author [Mathematika **16** (1969), 1–22; MR0246850 (40 #119)].

In the following chapter, which follows the work of the authors [ibid. **19** (1972), 25–50], the same weighting procedure is applied to the nonlinear case, corresponding to F being reducible in the above type of application. Since the sieve bounds known in this case are not of comparable quality to those known in the linear case, the results are correspondingly lacking in precision. Also one has to have recourse to numerical work to make them effective. Results of tolerable quality are nevertheless obtained, and the authors have formulated their basic theorem in a way that will make it easier for future researchers to make use of any improvements in the quality of the sieve bounds. Such improvements are indeed already on the way.

The final chapter, added after the rest of the book had gone to press, deals with the remarkable theorem of C. Chen [Sci. Sinica **16** (1973), no. 2, 157–76] to the effect that any sufficiently large even number is the sum of a prime and a P_2, which represents the closest approach to the Goldbach conjecture (the previously described method giving P_3 in place of P_2).

J. W. Porter

From MathSciNet, January 2015

MR0834613 (88b:11058) 11N13

Bombieri, E.; Friedlander, J. B.; Iwaniec, H.

Primes in arithmetic progressions to large moduli.

Acta Mathematica **156** (1986), *no.* 3-4, 203–251.

This important paper extends earlier attempts to improve on the Bombieri-Vinogradov theorem by E. Fouvry and Iwaniec [Mathematika **27** (1980), no. 2, 135–152; MR0610700 (82h:10057); Acta Arith. **42** (1983), no. 2, 197–218; MR0719249 (84k:10035); Fouvry, ibid. **41** (1982), no. 4, 359–382; MR0677549 (84b:10065); Acta Math. **152** (1984), no. 3-4, 219–244; MR0741055 (85m:11052)]. The most striking results are the following. (1) Let $a \neq 0$, $\epsilon > 0$ and $Q = x^{4/7-\epsilon}$. For any well factorable function $\lambda(q)$ of level Q and any $A > 0$ we have

$$\sum_{(q,a)=1} \lambda(q) \left(\psi(x;q,a) - \frac{x}{\phi(q)} \right) \ll_{\epsilon,a,A} x(\log x)^{-A}.$$

As a corollary one has: (2) Let $\pi_2(x)$ be the number of twin primes p, $p+2$ with $p \le x$. Then

$$\pi_2(x) \le (\tfrac{7}{2}+\epsilon) 2 \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right) x(\log x)^{-2},$$

for any $\epsilon > 0$ and $x \ge x(\epsilon)$. In a rather different direction, the error term in the Titchmarsh divisor problem is improved: (3) There are certain explicit constants c_1 and c_2 such that

$$\sum_{p \le x} d(p-1) = c_1 x \log x + c_2 x + O_A(x(\log x)^{-A}),$$

for any $A > 0$. Previously (1) had been obtained with $Q = x^{17/32-\epsilon}$, so that the corresponding version of (2) had $\frac{64}{17}$ in place of $\frac{7}{2}$. The estimate (3) has been obtained independently by Fouvry.

The proofs depend on estimates for averages of incomplete Kloosterman sums. These are obtained from the work of J.-M. Deshouillers and Iwaniec [Invent. Math. **70** (1982/83), no. 2, 219–288; MR0684172 (84m:10015)]. Many of the intermediate results obtained in the paper have other applications.

D. R. Heath-Brown
From MathSciNet, January 2015

MR2265008 (2007k:11150) 11N05; 11-02, 11N36

Soundararajan, K.

Small gaps between prime numbers: the work of Goldston-Pintz-Yıldırım.

Bulletin of the American Mathematical Society (New Series) **44** (2007), *no.* 1, 1–18.

This is a survey article about the result of [D. A. Goldston, J. Pintz and C. Y. Yıldırım, "Primes in tuples. I.", preprint, `arxiv.org/abs/math/0508185`, Ann. of Math. (2), to appear], which states that

$$\liminf_{n\to\infty} \frac{p_{n+1}-p_n}{\log p_n} = 0.$$

There is a description of distribution problems for the primes, giving Cramér's model, and various "classical" results. This is followed by the Hardy-Littlewood k-tuples conjecture, and a description of Selberg's sieve. This leads to the Goldston-Pintz-Yıldırım result, first in the context of the Elliott-Halberstam conjecture and finally unconditionally.

Details of the calculations in the sieve argument are not given, but the reader is presented with enough information to gain a good impression of what is new, and what problems lie ahead.

D. R. Heath-Brown
From MathSciNet, January 2015

MR2552109; 2011c:11146 11N05; 11N36

Goldston, Daniel A.; Pintz, János; Yıldırım, Cem Y.

Primes in tuples. I. (English)

Annals of Mathematics (2) **170** (2009), *no.* 2, 819–862.

This paper is a significant advance in our understanding of the distribution of small gaps between primes. It deals with questions that are motivated by the twin prime conjecture, which states that there are infinitely many positive integers n such that n and $n+2$ are both prime. This conjecture generalizes naturally to k-tuples

$$(n+h_1, n+h_2, \ldots, n+h_k).$$

Let $\mathcal{H} = (h_1, \ldots, h_k)$ and $\nu_p(\mathcal{H})$ be the number of distinct residue classes modulo p in $\mathcal{H}$. If $\nu_p(\mathcal{H}) < p$ for all primes p, we say that $\mathcal{H}$ is admissible. For any natural number n, we say that $(n+h_1, n+h_2, \ldots, n+h_k)$ is an admissible tuple. It is a long-standing conjecture that any admissible k-tuple contains all primes infinitely often. The twin prime conjecture is the special case $\mathcal{H} = (0, 2)$.

We say that the primes have level of distribution θ if for any $\epsilon > 0$,

$$\sum_{q \le N^{\theta-\epsilon}} \max_{\substack{a \\ (a,q)=1}} \left| \sum_{\substack{p \le N \\ p \equiv a \pmod q}} \log p - \frac{N}{\phi(q)} \right| \ll \frac{N}{(\log N)^A}.$$

Unconditionally, we know that the primes have level of distribution $1/2$; this is the Bombieri-Vinogradov Theorem. P. D. T. A. Elliott and H. Halberstam [in *Symposia Mathematica, Vol. IV (INDAM, Rome, 1968/69)*, 59–72, Academic Press, London, 1970; MR0276195 (43 #1943)] conjectured that the primes have level of distribution 1. In this paper, the authors prove the following.

Theorem 1. Suppose that the primes have level of distribution $\theta > 1/2$. Then there exists an explicitly calculable constant $C(\theta)$ depending only on θ such that any admissible k-tuple with $k \ge C(\theta)$ contains at least two primes infinitely often. In particular, if $\theta \ge 0.971$, one may take $k = 6$.

Since the 6-tuple $(n+7, n+11, n+13, n+17, n+19, n+23)$ is admissible, the Elliott-Halberstam conjecture implies that $p_{n+1} - p_n \le 16$ infinitely often. Unconditionally, the authors prove:

Theorem 2. We have

$$\Delta_1 := \liminf_{n\to\infty} \frac{p_{n+1} - p_n}{\log p_n} = 0.$$

To put this in historical context, observe that the Prime Number Theorem immediately implies that $\Delta_1 \le 1$. Hardy and Littlewood, in the unpublished paper in their *Partitio Numerorum* series, proved, conditionally on GRH, that $\Delta_1 \le 2/3$. P. Erdős [Duke Math. J. **6** (1940), 438–441; MR0001759 (1,292h)] gave the first unconditional proof that $\Delta_1 < 1$. There were many subsequent improvements; the best prior result was $\Delta_1 < 0.2484\ldots$, due to H. Maier [Michigan Math. J. **35** (1988), no. 3, 323–344; MR0978303 (90e:11126)].

The authors also prove new bounds for $\Delta_\nu := \liminf_{n\to\infty} p_{n+\nu} - p_n$.

Theorem 3. Suppose the primes have level of distribution θ. Then for $\nu \ge 2$, $\Delta_\nu \le (\sqrt{\nu} - \sqrt{2\theta})^2$.

Note, in particular, $\Delta_\nu \le (\sqrt{\nu} - 1)^2$ unconditionally. If the Elliott-Halberstam conjecture is true, then $\Delta_2 = 0$.

The central idea behind the proof of Theorem 1 is to consider the sum

$$\mathcal{S} = \sum_{N<n\le 2N} \left(\sum_{i=1}^{k} \theta(n+h_i) - \log 3N \right) \Lambda_R(n;\mathcal{H})^2 \tag{1}$$

for an appropriate choice of $\Lambda_R(n;\mathcal{H})$. Theorem 1 follows by showing that $\mathcal{S}$ is positive for N sufficiently large. The choice of Λ_R is

$$\Lambda_R(n;\mathcal{H}) = \frac{1}{(k+\ell)!} \sum_{\substack{d\mid(n+h_1)\cdots(n+h_k)\\ d\le R}} \mu(d)(\log R/d)^{k+\ell},$$

where ℓ is a positive integer that depends on θ. This choice is motivated by the Selberg upper bound sieve. In the corresponding upper bound sieve, the optimal choice is $\ell = 0$. However, in the present context, it is crucial to take some $\ell > 0$.

The authors attribute the idea of using $\mathcal{S}$ in this context to A. Granville and K. Soundarajaran. However, they also note that this construction is similar to one used by A. Selberg [*Collected papers. Vol. II*, Springer, Berlin, 1991; MR1295844 (95g:01032)] in his 1951 proof that $n(n+2)$ will infinitely often have at most five prime factors. This construction was later extended by D. R. Heath-Brown [Mathematika **44** (1997), no. 2, 245–266; MR1600529 (99a:11106)] to general almost-prime tuples.

The proof of Theorem 2 uses the basic construction (1) together with an averaging argument and a theorem of P. X. Gallagher [Mathematika **23** (1976), no. 1, 4–9; MR0409385 (53 #13140); corrigendum, Mathematika **28** (1981), no. 1, 86; MR0632799 (82j:10072)] on averages of singular series. Theorem 3 uses a variant of (1).

The ideas and methods of this paper have been extended to many other contexts; we mention a few of these. In the second paper of this series, the authors [Acta Math. **204** (2010), no. 1, 1–47; MR2011f:11121] gave a more precise version of Theorem 2; they proved

$$\liminf_{n\to\infty} \frac{p_{n+1}-p_n}{(\log p_n)^{1/2}(\log\log p_n)^2} < \infty.$$

In the third paper of this series [Funct. Approx. Comment. Math. **35** (2006), 79–89; MR2271608 (2008f:11102)], they incorporated Maier's matrix method [op. cit.] into the proof of Theorem 3 and established the bound $\Delta_\nu \le e^{-\gamma}(\sqrt{\nu}-1)^2$. The authors, together with the reviewer [Proc. Lond. Math. Soc. (3) **98** (2009), no. 3, 741–774; MR2500871 (2010a:11179)], considered analogous results for semiprimes, i.e., for numbers that are products of exactly two prime factors. They proved that

$$\liminf_{n\to\infty} \frac{q_{n+1}-q_n}{\log q_n} \le 6,$$

where q_n denotes the nth semiprime. F. Thorne [Int. Math. Res. Not. IMRN **2008**, no. 5, Art. ID rnm 156; MR2418287 (2009m:11149)] considered similar questions in the context of elliptic curves and quadratic number fields.

The arguments here come tantalizingly close to proving bounded gaps between primes, i.e., that $\liminf_{n\to\infty} p_{n+1}-p_n < \infty$. Any improvement in the level of distribution θ beyond 1/2 probably lies very deep. There are stronger versions of the Bombieri-Vinogradov theorem; see work of E. Bombieri, J. B. Friedlander and H. Iwaniec [J. Amer. Math. Soc. **2** (1989), no. 2, 215–224; MR0976723 (89m:11087)]. However, such results do not appear useful in the current context. Soundararajan [Bull. Amer. Math. Soc. (N.S.) **44** (2007), no. 1, 1–18; MR2265008 (2007k:11150)] showed that the current argument cannot give bounded gaps for $\theta = 1/2$, but there may be more efficient arguments. In any case, the results here give hope that bounded gaps can be achieved in the near future.

S. W. Graham

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