BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 52, Number 3, July 2015, Pages 465–496
http://dx.doi.org/10.1090/bull/1492
Article electronically published on April 6, 2015

HEILBRONN CHARACTERS

RICHARD FOOTE, HY GINSBERG, AND V. KUMAR MURTY

Dedicated to Professor Hans A. Heilbronn

Abstract. In a seminal paper in 1972 Hans Heilbronn introduced a virtual character associated to representations of Galois extensions of number fields and Artin's Conjecture on the holomorphy of L-series. His construction has evolved in both application and scope, and may now be applied to produce what are called Heilbronn characters of arbitrary finite groups. This article surveys the inception and development of this concept, weaving together its number-theoretic and group-theoretic dimensions, and culminates in a description of the recent classification of unfaithful minimal Heilbronn characters. Connections with other areas of mathematics, variations on these themes, and possible future directions are also explored.

1. Introduction

In 1972, stimulated by a visit to the University of Toronto by L. J. Goldstein who gave a talk on work by himself and his doctoral student Judy Sunley ([Gs75, Su75]), Professor Hans Heilbronn wrote and lectured on the beautiful four-page paper *On the real zeros of Dedekind ζ-functions* [He73]. Its main result is that if E is a finite Galois extension of a number field F, and F_2 is the composite of all quadratic extensions of F lying in E, then any real simple zero of $\zeta_E(s)$ is a zero of $\zeta_{F_2}(s)$, where $\zeta_K(s)$ denotes the Dedekind zeta function of any number field K. In his paper (and the conference proceedings [He72]) Heilbronn introduced a "bookkeeping device" for relating the zeros of the zeta function of K at a complex point s_0 to zeros of zeta functions attached to the subfields of K at the same point s_0, and vice versa. This simple yet seminal idea has blossomed into an entire line of research based on what are now called *Heilbronn characters*, with applications to Artin's Conjecture on the holomorphy of L-series, and concomitant implications in algebraic number theory. Moreover, the notion of a Heilbronn character, which originally relied on orders of zeros and poles of L-series attached to representations of Galois groups, has been "axiomatized" so that it may be abstracted to arbitrary finite groups. Some of the research in this area has then focused on the work of classifying, to the extent possible, Heilbronn characters, or more precisely *minimal Heilbronn characters*, for arbitrary finite groups. In this vein there are significant results for solvable groups; and with the completion of the Classification of the Finite Simple Groups in hand [GLS94], an essential determination of all the finite groups that possess what are called *unfaithful minimal Heilbronn characters* has just been completed. It therefore seems an auspicious moment to survey this field

Received by the editors June 18, 2014, and, in revised form, March 17, 2015.
2010 *Mathematics Subject Classification.* Primary 20C15, 11R42.

in a self-contained, thorough, and accessible fashion, following mostly an historical development, up to the current state of the art.

Our survey begins in Section 2 with the definition of Heilbronn characters associated to Galois extensions of number fields together with their basic properties. Short proofs of these properties are also included at the end, as these help to illustrate both the power and elegance of this line of research. The character theory in Section 2 is interwoven with number-theoretic background that illuminates the importance of Artin L-series to fundamental classical problems in number theory, many of which are still unsolved. These intertwined themes are further developed in Sections 3 and 4, where the first inklings of minimal Heilbronn characters for solvable groups emerge. A short, self-contained proof of a generalization of Heilbronn's original result is given at Theorem 3.2. With this background and motivation in hand, Section 5 introduces the notion of abstract Heilbronn characters for arbitrary finite groups, thereby extricating the original concept from its Galois-theoretic dependence. This generalization is exploited, using the Classification of the Finite Simple Groups, in Sections 6 and 7, where the aforementioned determination of groups possessing an unfaithful minimal Heilbronn character is explicated.

As with any great idea, the notion of Heilbronn characters impelled the development of ancillary results that were employed as tools in their study. In particular, the classification of minimal Heilbronn characters ultimately required knowledge of all finite groups possessing what are called *strongly closed p-subgroups*. This independent group-theoretic classification has far-reaching applications to areas as diverse as the homotopy structure of classifying spaces of finite groups, modular representation theory, and the burgeoning and very active interdisciplinary area of *fusion systems*. In Section 8 we briefly provide an introduction to strongly closed subgroups and these ramifications as well.

Returning to number-theoretic themes, Sections 9 and 10 survey applications of Heilbronn characters and variations of Heilbronn characters to other types of L-functions and other settings, such as elliptic curves. The article concludes with brief remarks on some possible future directions.

Finally, as one of the authors (Foote) was present at Heilbronn's first lecture on this subject, and many mathematicians remember Professor Heilbronn with enduring fondness, it is our honor to dedicate this survey to him.

2. Definition and basic properties of Heilbronn characters

In this section we set up the machinery and motivation for defining Heilbronn characters associated to Galois extensions of number fields. Eschewing for the moment our historical approach, for expediency we list the exact properties of Artin L-series that translate into the defining properties of Heilbronn characters. Historically only the definition and some of the properties of these (virtual) characters were introduced and exploited in Heilbronn's original work. The term "Heilbronn character" was coined by Sandy Rhoades in her doctoral dissertation [Rh93a]. The "axiomatic" setting evolved from work by Foote and Murty [FM89], and was first written out explicitly in [Fo90].

In order to be completely precise, we first define the Artin L-series attached to a (finite-dimensional, complex) representation of a Galois group for a number field extension. A more leisurely introduction to L-series, their properties and uses, appears in [He67]. For those unfamiliar with this theory, it suffices for the purpose

of this survey that Artin L-series are a family of meromorphic functions on the whole complex plane associated to representations of a finite Galois group. We shall see that various L-series are related functorially via inflation-restriction maps between subgroups of Galois groups; it is primarily these relations, not the specific L-series themselves, that determine the definition of Heilbronn characters. (But, of course, in the number-theoretic applications, the L-series themselves are essential.)

Because some of the initial consequences of the definitions are so elementary and far-reaching, we include very brief proofs of these, including both Stark's Theorem and a generalization of the classical Aramata–Brauer Theorem (both are stated later). Readers wishing to follow only the survey may peruse these proofs lightly.

Let E/F be a Galois extension of number fields with Galois group G, and let $T : G \to GL(V)$ be a finite-dimensional complex representation of G with character ϕ. The Artin L-series, $L(s, \phi, E/F)$, is defined as follows:

$$L(s, \phi, E/F) = \prod_{\mathcal{P}} \left[\det \left(1 - N_{F/Q}(\mathcal{P})^{-s} T|_{V^I}(\mathrm{Frob}_{\mathcal{P}})\right)\right]^{-1}, \qquad s \in \mathbb{C},$$

where the product is over all primes $\mathcal{P}$ in F, V^I is the subspace of V fixed by the inertia group I of a prime in E over $\mathcal{P}$ and $\mathrm{Frob}_{\mathcal{P}}$ is a Frobenius element of G at that prime over $\mathcal{P}$ in E. The determinant is independent of the choice of prime above $\mathcal{P}$, and so the L-function is well defined. When the extension associated to the L-series is clear, we shall simply denote the series by $L(s, \phi)$. By results of Hecke, Artin, and Brauer, this Euler product (which is seen to converge in the right half-plane $\mathrm{Re}\ s > 1$) has a meromorphic continuation to the entire complex plane. *Artin's Conjecture is that if ϕ does not contain the principal character of G, then $L(s, \phi)$ is an entire function.*

The following properties of Artin L-series will ultimately serve as the function-theoretic "axioms" for reformulating Artin's Conjecture in the language of the character theory of G:

(L1) $L(s, \psi_1 + \psi_2) = L(s, \psi_1)L(s, \psi_2)$, where ψ_1, ψ_2 are characters of G.

(L2) If E_0 is the fixed field of $\ker \psi$, then $L(s, \psi, E/F) = L(s, \psi', E_0/F)$, where ψ' is the character ψ considered as a character of $G/\ker \psi$. (The kernel of any character ψ is, by definition, the kernel of a representation affording it.)

(L3) $L(s, \lambda, E/E^H) = L(s, \mathrm{Ind}_H^G(\lambda), E/F)$, where H is a subgroup of G, λ is a character of H, E^H is the fixed field of H, and $\mathrm{Ind}_H^G(\lambda)$ is the character λ of H induced to a character of G.

(L4) If χ is a nonprincipal linear character, $L(s, \chi)$ is entire; if I_G is the principal character of G, $L(s, I_G) = \zeta_F(s)$ is analytic everywhere except for a simple pole at $s = 1$ (here $\zeta_F(s)$ is the Dedekind zeta function of F).

Results (L1) to (L3) are proved in [He67] and (L4) is a consequence of Artin Reciprocity and the analytic continuation of Hecke L-series (see [Ar27, Hk17]). The Artin Reciprocity Law is a grand generalization of the Law of Quadratic Reciprocity. In general terms, it asserts that the L-function associated to a one-dimensional Artin character is the same as the L-function associated to another type of character, this one built entirely out of data in the ground field (that is, with no reference to the extension). For example, for a Galois extension E of $F = \mathbb{Q}$, it asserts that for any one-dimensional complex character χ of $\mathrm{Gal}(E/\mathbb{Q})$, there is an integer N

and a character

$$r_\chi : (\mathbb{Z}/N\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times$$

so that for any prime p that does not ramify in E, and any prime $\mathcal{P}$ of E dividing p, we have the equality

$$\chi(\mathrm{Frob}_{\mathcal{P}}) = r_\chi(p),$$

where the right-hand side means the value of the character r_χ at the image of p in $(\mathbb{Z}/N\mathbb{Z})^\times$. In terms of L-functions, one has the Dirichlet series

$$L(s, r_\chi) = \prod_{p \nmid N} \left(1 - \frac{r_\chi(p)}{p^s}\right)^{-1},$$

and the statement is that

$$L(s, \chi, E/\mathbb{Q}) = L(s, r_\chi).$$

The L-series on the right belongs to a class that has been extensively studied in the context of the distribution of primes in arithmetic progressions. In particular, we know that it has an analytic continuation for all values of s except possibly for a simple pole at $s = 1$ which occurs if and only if r_χ is the trivial character.

If the ground field F is not $\mathbb{Q}$, one replaces $(\mathbb{Z}/N\mathbb{Z})^\times$ with $(\mathcal{O}_F/\mathfrak{f})^\times$, where $\mathcal{O}_F$ denotes the ring of integers of F (the integral closure of $\mathbb{Z}$ in F) and $\mathfrak{f}$ is an integral ideal of $\mathcal{O}_F$ canonically associated to the Galois character χ. In this case, the L-function $L(s, r_\chi)$ is a special case of a family studied by Hecke, and again one knows the analytic continuation for all s with a possible simple pole at $s = 1$ which occurs if and only if r_χ is the trivial character.

Consider again the case $F = \mathbb{Q}$. If r_χ is real-valued, then the fixed field of the kernel of χ is a quadratic extension K (say) of $\mathbb{Q}$. In this case, a classical formula of Dirichlet relates the special value $L(1, r_\chi)$ to the arithmetic of K. More precisely, it says that

$$L(1, r_\chi) = \begin{cases} 2\pi h_K / w_K \sqrt{|d_K|} & \text{if } K \text{ is imaginary quadratic,} \\ h_K \log \epsilon_K / \sqrt{|d_K|} & \text{if } K \text{ is real quadratic.} \end{cases} \tag{1}$$

Here, h_K, w_K, ϵ_K, d_K denote, respectively, the class number of K, the number of roots of unity in K, a fundamental unit of K (a "smallest" generator of the group of units, which is known to be an abelian group of rank 1), and the discriminant of K.

A problem of significant interest from the time of Gauss was to understand the growth of the class number h_K as K varies. In particular, Gauss conjectured that $h_K = 1$ for only a finite number of imaginary quadratic fields, and $h_K = 1$ for infinitely many real quadratic fields. The first assertion is now a theorem due to the efforts of many authors, but culminating in the effective determination by Baker and Stark of all imaginary quadratic K such that $h_K = 1$, namely $K = \mathbb{Q}(\sqrt{-D})$ with

$$D = 3, 4, 7, 8, 11, 19, 43, 67, 163.$$

Gauss's Conjecture for real quadratic fields is still open.

Assuming K is an imaginary quadratic field, if one were to approach Gauss's Conjecture from the beautiful formula (1) of Dirichlet, we see that to make h_K

large, we need $L(1, r_\chi)$ to be large. In fact, if we had a bound of the form

$$L(1, r_\chi) \gg \frac{1}{\log |d_K|} \tag{2}$$

this would imply that

$$h_K \gg \frac{\sqrt{|d_K|}}{\log |d_K|}. \tag{3}$$

Hecke was able to prove that if $L(s, r_\chi)$ has no real zeros on the interval

$$(1 - \frac{c}{\log |d_K|}, 1) \tag{4}$$

for some constant $c > 0$, then the bound (2) in fact holds. In particular, if the Riemann Hypothesis for $L(s, r_\chi)$ were assumed, then the class number bound (3) does hold, and $h_K \to \infty$ as $|d_K| \to \infty$. On the other hand, Deuring and Heilbronn [He34] proved that if the Riemann Hypothesis were *false* for some Dirichlet L-function, then also $h_K \to \infty$. These complementary results thus proved Gauss's Conjecture in the imaginary quadratic case, albeit ineffectively. The approach of Baker [Baker66] and Stark [St67] was to use transcendental number theory, and they were able to obtain an effective determination of the fields K for which $h_K = 1$. With more work, they could also do the case $h_K = 2$ [Baker71], [St72]. By developing a more sophisticated version of the Deuring–Heilbronn approach and using L-functions associated to elliptic curves, Goldfeld (combined with the work of Gross and Zagier [Goldfeld76], [GZ86]) showed effectively that in fact

$$h_K \gg \frac{\log |d_K|}{(\log \log |d_K|)^2}.$$

This is of course enough to show that $h_K \to \infty$ effectively, but it is still quite far from the expected bound (3).

What about when K is a real quadratic field? As we stated above, Gauss's Conjecture is still open in this case. However, the method described above of Hecke shows that the absence of zeros of $L(s, r_\chi)$ in the region (4) implies that

$$h_k \log \epsilon_K \gg \frac{\sqrt{|d_K|}}{\log |d_K|}.$$

Another way of saying this is that if the L-function does not have zeros very close to $s = 1$, then the value at $s = 1$ can be shown to be large. Yet another way of saying this is to consider the Dedekind zeta function $\zeta_K(s)$ of K. This function has a simple pole at $s = 1$ with residue $L(1, r_\chi)$, and we have a factorization $\zeta_K(s) = \zeta(s)L(s, r_\chi)$. It is known that the Riemann zeta function $\zeta(s)$ does not have real zeros with $\mathrm{Re}(s) > 0$, and so, if $\zeta_K(s)$ does not have a zero very close to $s = 1$, then the residue at $s = 1$ is large. This formulation combines both the real and the imaginary quadratic cases.

The Brauer–Siegel Theorem is a generalization of this to certain families of number fields. It asserts that if E ranges over a sequence of number fields with the property that

$$\frac{1}{[E : \mathbb{Q}]} \log |d_E| \to \infty,$$

then

$$\log (\mathrm{res}_{s=1} \zeta_E(s)) \to 0.$$

The result is not effective and this plays an important role in our narrative.

Returning to general Artin L-series, it is known (see [He67]) that all Artin L-series $L(s,\chi)$ are analytic at $s=1$ for all nonprincipal irreducible characters χ of G; henceforth, we consider only complex points $s_0 \neq 1$. Brauer's Theorem [Is94, 8.4(b)] gives that every irreducible character is an integral linear combination of characters induced from linear characters of nilpotent subgroups; this together with (L1) to (L4) establish that Artin L-series are meromorphic functions on all of $\mathbb{C}$, for all Galois extensions of number fields.

Fix some point $s_0 \in \mathbb{C} - \{1\}$. In [He73] Heilbronn introduced the following virtual (or generalized) character of any Galois group G:

$$\theta_G = \sum_{\chi \in \mathrm{Irr}\,(G)} n(G,\chi)\chi, \tag{5}$$

where $n(G,\phi)$ is the order of zero or pole of the meromorphic function $L(s,\phi)$ at $s=s_0$ for any character ϕ of G (not necessarily irreducible). Indeed, it follows directly from property (L1) that

$$n(G,\phi) = \langle\, \theta_G\,,\, \phi\,\rangle = \mathrm{ord}_{s=s_0} L(s,\phi), \tag{6}$$

where $\langle\, \alpha\,,\, \beta\,\rangle$ is the usual Hermitian product of complex class functions on G. So this Artin L-series is analytic at s_0 if and only if the inner product above is nonnegative. This permits a character-theoretic formulation of Artin's Conjecture:

(7) *all Artin L-series for the extension E/F are analytic at s_0 if and only if θ_G is a character of G or identically zero.*

Note that this definition encompasses all subgroups H and quotient groups G/N of G as well, since each such is the Galois group of an extension of number fields (namely, of extensions E/E^H and E^N/F, respectively). Thus the virtual characters θ_H and $\theta_{G/N}$ are well defined with respect to s_0 as well. The family of Heilbronn virtual characters for a fixed s_0 satisfy certain compatibility properties with respect to subgroups and quotient groups of G in the following sense (these are proved as Proposition 2.1 following):

(H1) $\theta_G|_H = \theta_H$, for all subgroups H of G.
(H2) If N is a normal subgroup of G, then $\theta_{G/N}$ is the sum of all constituents $n(G,\psi)\psi$, where ψ is an irreducible character of G whose kernel contains N.
(H3) If λ is a linear character of a subgroup H of G, then $\langle\, \theta_G\,,\, \mathrm{Ind}_H^G(\lambda)\,\rangle \geq 0$.

Recall that characters of G that are induced from linear (i.e., degree 1) characters are called *monomial* characters of G. If every irreducible character of G is monomial, G is called an *M-group*. Thus Artin's Conjecture is true (at all points) if G is an M-group. Classical results from group theory show that every nilpotent group is an M-group and that all M-groups are solvable (see [Fe67, Section 10]).

Although properties (H1) to (H3) delineate that which we shall see are the distinguishing features of "abstract Heilbronn characters", before defining the latter we continue the Artin-theoretic motivation by recording some important consequences of the properties of Heilbronn characters associated to Galois extensions and L-series. From properties (L4) and (H1) it follows immediately that

(H4) $\theta_G|_A$ is a character of every abelian subgroup A of G (or is identically zero on A); more generally, $\theta_G|_P$ is a character or zero for every nilpotent subgroup P.

Thus by restricting θ_G to the identity subgroup, evaluating at the identity of the group G (denoted henceforth by 1_{G}) and using the second part of property (L4), one sees that

(H5) $\theta_G(1_{\mathrm{G}}) = \mathrm{ord}_{s=s_0}\zeta_E(s)$.

Since the Dedekind zeta function has no pole at s_0, $\theta_G(1_{\mathrm{G}}) \geq 0$. Furthermore, by restricting θ_G to every cyclic subgroup, one sees that $\theta_G(1_{\mathrm{G}}) = 0$ if and only if θ_G is identically zero (in which case Artin's Conjecture is true at s_0). In particular, these considerations show

(H6) If the Dedekind zeta function of E is nonzero at s_0, then Artin's Conjecture is true at s_0, i.e., all Artin L-series $L(s, \psi, E/F)$ are analytic at s_0.

Indeed, the same considerations show that the set of zeros and poles of all Artin L-series $L(s, \phi, E/F)$ is contained in the set of zeros of the Dedekind zeta function $\zeta_E(s)$ for all characters ϕ of G. Moreover, the *order* of zero of the zeta function $\zeta_E(s)$ is a critical parameter in restricting the possibility that some L-function may have a pole at s_0. We shall elaborate on and exploit this in the ensuing sections.

2.1. **Proofs of elementary properties of Heilbronn characters.** This subsection provides very brief, self-contained proofs of most of the results cited previously. The arguments serve to illuminate some elementary manipulations of Heilbronn characters to obtain powerful results, as first glimpsed in Heilbronn's original paper, thereby highlighting the power and elegance of this approach.

Proposition 2.1. *Let G be the Galois group of some finite Galois extension E/F of number fields, and let θ_G be the Heilbronn character of G at some point $s_0 \in \mathbb{C}-\{1\}$. Let r be the order of zero of the Dedekind zeta function of E at s_0 (so $r \geq 0$). The following hold:*

(1) *$\theta_G|_H = \theta_H$ for all subgroups H of G.*
(2) *$\theta_G|_P$ is a character of P for all nilpotent subgroups P of G.*
(3) *$\theta_G(1_G) = r$.*
(4) *$|\,\theta_G(g)\,| \leq r$ for all $g \in G$.*
(5) *$||\,\theta_H\,||_H^2 = \langle\,\theta_H\,,\,\theta_H\,\rangle_H \leq r^2$ for every subgroup H of G, with equality holding if and only if $|\,\theta_H(h)\,| = r$ for every $h \in H$.*
(6) *$||\,\theta_G\,||^2 = \sum_\chi n(G,\chi)^2$.*

Proof. (1) We must show that $\langle\,\theta_H\,,\,\psi\,\rangle_H = \langle\,\theta_G|_H\,,\,\psi\,\rangle_H$ for every irreducible character ψ of H. The left-hand side is, by Frobenius Reciprocity, equal to $\langle\,\theta_G\,,\,\mathrm{Ind}_H^G(\psi)\,\rangle_G$. As observed earlier, it is immediate from property (L1) of L-series that displayed property (6) above holds. Applying this to $\phi = \mathrm{Ind}_H^G(\psi)$ and invoking property (L3) then shows the left-hand side equals the right, as needed.

(2) By (1), $\theta_G|_P = \theta_P$ for P a nilpotent subgroup of G. As noted earlier, every irreducible character of a nilpotent group is monomial, so by properties (L3) and (L4) applied to the Galois group P, we see that each irreducible character of P must appear with nonnegative multiplicity in θ_P, i.e., θ_P is a character or zero.

(3) This follows by restricting θ_G to the identity subgroup to obtain $\theta_{\langle\,1_{\mathrm{G}}\,\rangle}$. The result now follows from (L4) applied to the extension F/F with Galois group $\langle\,1_{\mathrm{G}}\,\rangle$.

(4) For each $g \in G$, θ_G restricted to the abelian (hence nilpotent) subgroup $\langle\,g\,\rangle$ is a character of degree r. Thus $\theta_G(g)$ is a sum of r roots of unity, whence (4) holds.

(5) This follows immediately from (4) and the definition of the inner product of θ_H with itself.

(6) This follows immediately from the definition of θ_G and the inner product. $\square$

As a consequence of these results we see that $\theta_G(1_{\mathrm{G}}) \geq 0$, and θ_G is the zero function if and only if $\theta_G(1_{\mathrm{G}}) = 0$. We will henceforth be concerned with the case $\theta_G \neq 0$ and so $\theta_G(1_{\mathrm{G}}) \geq 1$.

We shall see in the upcoming sections how Proposition 2.1 gives short and elegant proofs—which we therefore also include in this survey—of both classical and new results in the theory of L-series.

3. The Heilbronn and Stark Theorems

The ideas in Heilbronn's paper next came to light in the beautiful and important paper by H. Stark [St74]. His motivation was to look for an effective version of the Brauer–Siegel Theorem. At first, this would seem impossible as the discussion above points out the difficulties even in the case of quadratic fields. However, Stark's insight was that this is essentially the *only* difficulty. The elementary development of the previous section enables us to include a very brief, complete proof of Stark's Theorem. Moreover, we provide a proof of a generalization of Heilbronn's seminal result stated at the outset of the paper (although historically the latter preceded and impelled the former).

Property (H6) of Heilbronn characters shows that the order of vanishing of the Dedekind zeta function of the top field E in a Galois extension E/F plays an important role in the determination of whether Artin L-series are analytic at $s_0 \in \mathbb{C}$. Moreover, Proposition 2.1 shows that this parameter is encoded in the Heilbronn character θ_G as its "degree" $\theta_G(1_{\mathrm{G}})$. A critical ingredient in Stark's work is the extension of (H6) to simple zeros:

Theorem 3.1. *If the Dedekind zeta function of E has a simple zero at s_0, i.e., $\mathrm{ord}_{s=s_0}\zeta_E(s) = 1$, then Artin's Conjecture is true at s_0, i.e., all Artin L-series $L(s, \psi, E/F)$ are analytic at s_0.*

Proof. By Proposition 2.1, the hypothesis that $\zeta_E(s)$ has a simple zero at s_0 is equivalent to $\theta_G(1_{\mathrm{G}}) = 1$. Proposition 2.1(5) applied to $H = G$ gives that $||\,\theta_G\,||^2 \leq 1$. Thus the virtual character θ_G has exactly one irreducible constituent. Since $\theta_G(1_{\mathrm{G}}) = 1 > 0$, it must be an irreducible character (i.e., have positive coefficient), hence Artin's Conjecture is true at s_0. $\square$

In fact, the argument tells us somewhat more. Not only is θ_G a character, it must be a one-dimensional character (because $\theta_G(1_{\mathrm{G}}) = r = 1$). Moreover, if s_0 is real, then χ must be real-valued: indeed, if $L(s_0, \chi, E/F) = 0$, then $L(\overline{s_0}, \overline{\chi}, E/F) = 0$ also, and both χ and $\overline{\chi}$ occur as constituents of θ_G; hence $\chi = \overline{\chi}$.

Since a real linear character χ is a homomorphism of G into $\{\pm 1\}$, its kernel is a subgroup H of index at most 2 in G. By Proposition 2.1(1), for any subgroup H_1 of H, the Heilbronn character $\theta_{H_1} = \theta_G|_{H_1}$ is the principal character of H_1. Interpreted number-theoretically this says if E_1 is the fixed field of H_1, then the Dedekind zeta function of E_1 has a simple zero at s_0 (and all $L(s, \psi, E/E_1)$ are analytic and nonzero at s_0 for all nonprincipal irreducible characters ψ of H_1). This gives a generalization of Heilbronn's seminal result:

Theorem 3.2. *If the Dedekind zeta function of E has a real simple zero at s_0, then there is a subfield K of E containing F and of degree at most 2 over F such that for any intermediate field $K \subseteq E_1 \subseteq E$, the Dedekind zeta function of E_1 has a simple zero at s_0. In particular, this holds when E_1 is the composite of all quadratic extensions of F contained in E.*

These last comments are what is behind the number-theoretic applications of the above result. Indeed, by analytic considerations, Stark showed that in the region where $s = \sigma + it$ is bounded by

$$1 - \frac{1}{4\log|d_E|} < \sigma < 1, \quad |t| < \frac{1}{4\log|d_E|}, \tag{8}$$

the Dedekind zeta function $\zeta_E(s)$ has at most a simple zero, and if this zero β_0 (say) occurs, it must be real. To simplify the discussion, let us suppose that $F = \mathbb{Q}$. Applying the above result, Stark deduced that there is a subfield $\mathbb{Q} \subseteq K \subseteq E$ with $[K : \mathbb{Q}] \leq 2$ such that β_0 is a zero of $\zeta_K(s)$. Since the Riemann zeta function does not have any real zeros in $\mathrm{Re}(s) > 0$, this zero must belong to a quadratic extension, and so $[K : \mathbb{Q}] = 2$. In this case, we have an effective estimate

$$1 - \beta_0 \gg \frac{1}{\sqrt{|d_K|}}.$$

Using the elementary bound

$$|d_E| \ \geq \ |d_K|^{[E:\mathbb{Q}]/2},$$

we deduce that

$$1 - \beta_0 \gg \frac{1}{|d_E|^{1/[E:\mathbb{Q}]}}.$$

Combining this with further analytic considerations, Stark deduces the effective lower bound

$$\mathrm{Re}\, s_{s=1}\zeta_E(s) \gg \ (1 - \beta_0) \ \gg \ \frac{1}{|d_E|^{1/[E:\mathbb{Q}]}}$$

for the residue. The upper bound is in fact easy to make effective, so this actually gives an effective version of the Brauer–Siegel Theorem, at least in the case that $E/\mathbb{Q}$ is Galois.

In some cases this effective upper bound can be used to get an effective lower bound on class numbers. For this, consider the case where E is a CM-field (that is, a totally complex quadratic extension of a totally real field). Denote by E^+ the maximal totally real subfield. Then $[E : E^+] = 2$ and there is a positive integer f such that $|d_E| = d_{E^+}^2 f$. Moreover,

$$\frac{\mathrm{Re}\, s_{s=1}\zeta_E(s)}{\mathrm{Re}\, s_{s=1}\zeta_{E^+}(s)} \ \leq \ \frac{(2\pi)^n h_E}{|d_{E^+} f|^{\frac{1}{2}} h_{E^+}}.$$

Using the above bounds, one has an effective lower bound for the left-hand side. Thus one deduces an effective lower bound for the quotient h_E/h_{E^+}, and hence also for h_E.

All of these results require the hypothesis that E/F is Galois. If we drop this hypothesis, then the problem becomes considerably more complicated. Consider a zero ρ of $\zeta_E(s)$ in the region

$$1 - \frac{c}{n\log|d_E|} \ \leq \ \sigma \ \leq 1, \quad |t| \ \leq \ \frac{c}{n\log|d_E|},$$

for some constant $c > 0$ and $n = [E : F]$. Let M be the compositum of E and one of its conjugates, say E^ϕ. Then we have the elementary estimate

$$\log |d_M| \ \leq \ 2n \log |d_E|.$$

Thus the zero ρ lies in the region

$$1 - \frac{2c}{\log |d_M|} \ \leq \ \sigma \ \leq 1, \qquad |t| \ \leq \ \frac{2c}{\log |d_M|}.$$

If we choose $0 < c < 1/8$, then this zero lies in the Stark region (8), and so is a simple zero of $\zeta_M(s)$. Now if we assume Artin's holomorphy conjecture for L/F, where L is some extension of E that is Galois over K, then

$$\frac{\zeta_M(s)\zeta_N(s)}{\zeta_E(s)\zeta_{E^\phi}(s)}$$

is entire, where $N = E \cap E^\phi$. It follows that $\zeta_N(s)$ has a zero at the same point. Moreover, as the discriminant of N is at most that of E, the same bound as above applies to ρ with d_E replaced with d_N. In particular, ρ is a zero of $\zeta_N(s)$ that lies in the Stark region (8), and so is a simple zero. Now repeating this process, if necessary, will yield an at most quadratic extension of F where this zero occurs.

In general we do not have Artin's Conjecture. To compensate for this we have to narrow the Stark region further and also impose an additional restriction on the Galois closure of E/F. For example, in [KM1], [KM2] we assume that the normal closure is solvable. To state the result precisely, we need to introduce some arithmetical functions. For a positive integer n, set

$$e(n) = \max_{p^\alpha || n} \alpha.$$

Also define the function $\gamma(r) = 12^{r-1}e^{1/3}$. Now set

$$\delta(n) = (e(n) + 1)^2 \gamma(e(n)).$$

Let $c > 0$ be sufficiently small, and suppose that $\zeta_E(s)$ has a zero ρ in the region

$$1 - \frac{c}{n^{e(n)}\delta(n) \log |d_E|} \ \leq \ \sigma \ \leq 1, \qquad |t| \ \leq \ \frac{c}{n^{e(n)}\delta(n) \log |d_E|}.$$

Then such a zero is necessarily real and simple, and there is a field N with $F \subseteq N \subseteq E$ and $[N : F] \ \leq 2$ with $\zeta_N(\rho) = 0$. This is Theorem 2.1 of [KM2].

Using this result and arguing as before, one can obtain effective lower bounds for the class number of a CM-field whose Galois closure is solvable. Stark has conjectured that such effective lower bounds should exist for any CM-field, but this is still an open problem.

4. Heilbronn characters for solvable groups and the Aramata–Brauer Theorem

The ideas of Heilbronn and Stark remained dormant for many years until, in 1987, K. Murty began to consider whether Artin's Conjecture for the Galois extension E/F could be proved at $s_0 \in \mathbb{C} - \{1\}$ under the assumptions that $\zeta_E(s)$ has a zero of order 2 at s_0 and the degree of E/F is odd. Murty could then invoke the Feit–Thompson Theorem to assert that $G = \mathrm{Gal}(E/F)$ is a solvable group. Fortuitously, Murty and Foote were both participants in the Quebec-Vermont Number Theory Seminar, and so they began to work together on this problem. Their collaboration resulted in the paper [FM89], which contained a number of advances.

First and foremost, it made explicit the properties of Heilbronn characters established in Proposition 2.1 (although some proofs were slightly different). Using these, it gave a short proof of a (new) generalization of the classical Aramata-Brauer Theorem. We include a slight digression to help motivate this theorem.

Properties (L3) and (L4) in Section 2 yield the familiar factorization of the zeta function of E into L-series corresponding to the decomposition of the character of the regular representation of G, $\mathrm{reg} = \mathrm{Ind}_{\langle 1_G \rangle}^{G}(I)$, where I is the principal character of the trivial subgroup $\langle 1_G \rangle$:

$$\zeta_E(s) = L(s, \mathrm{reg}, E/F) = \prod_{\chi \in \mathrm{Irr}(G)} L(s, \chi, E/F)^{\chi(1_G)}.$$

From this we see that for each irreducible character χ of G, $L(s,\chi)$ is a factor of $\zeta_E(s)$ appearing with multiplicity equal to the degree of χ. In particular, if I_G denotes the principal character of G, then by (L2) and (L4) we see that $\zeta_F(s) = L(s, I_G, E/F) = L(s, I, F/F)$ is a factor of $\zeta_E(s)$ with multiplicity 1. Although $\zeta_E(s)$ is analytic at s_0, it is not clear that the same is true after removing some of its L-function factors, as these may be contributing zeros at s_0 that would cancel poles from other L-factors. The classical Aramata–Brauer Theorem shows that we may at least remove the single factor $\zeta_F(s)$ from $\zeta_E(s)$ and still retain analyticity of the resulting product: $\zeta_E(s)/\zeta_F(s)$ is analytic at s_0. The results of Subsection 2.1 show, more generally, that *any* single L-function factor may be removed from the above factorization of $\zeta_E(s)$ and still retain analyticity. Indeed, even more generally we have:

Theorem 4.1. *The functions $\zeta_E(s)L(s,\chi,E/F)$ and $\zeta_E(s)/L(s,\chi,E/F)$ are analytic at s_0 for every irreducible character χ of G.*

Proof. Using the notation of Proposition 2.1 and the properties of the character norm, we see from its part (5) that for any irreducible character χ of G,

$$n(G,\chi)^2 \leq ||\,\theta_G\,||^2 \leq r^2.$$

Thus the absolute value of the order of zero or pole of $L(s,\chi)$ is less than or equal to the order of zero of $\zeta_E(s)$ at s_0, and so the theorem follows. □

The next innovation of [FM89], the notion of *minimal counterexamples* to Artin's Conjecture, was inherent in the last main result of that paper, which is Theorem 4.2 following. Before stating the theorem, we extract and highlight that concept. If indeed Artin's Conjecture is false at the point s_0 for some Galois extension E/F, then it must fail in some intermediate Galois extension E_1/F_1 of minimal degree, where $F \subseteq F_1 \subseteq E_1 \subseteq E$. Translating into the language of Heilbronn characters, let $G_1 = \mathrm{Gal}(E_1/F_1)$ be the corresponding section of the group G. (A *section* of G is any A/B where A and B are subgroups of G with $B \trianglelefteq A$.) Then because of properties (L2) and (L3), the Heilbronn character θ_{G_1} has the additional properties:

(MH1) θ_{G_1} is not a character of G_1, but $\theta_{G_1}|_H$ is a character of H for all *proper* subgroups H of G_1; and

(MH2) for every irreducible character ψ of G_1 such that $\langle \theta_{G_1}, \psi \rangle_{G_1} < 0$, ψ is the character of a faithful representation of G_1, and ψ is not induced from any character of any proper subgroup of G_1.

The advantage of working with *minimal* counterexamples to Artin's Conjecture in the above sense and their ensuing "minimal" Heilbronn characters (satisfying

(MH1) and (MH2)) is that G_1 must then possess a faithful irreducible character; in particular by Schur's Lemma, *the center of* G_1 *is cyclic.* Furthermore, it is an easy consequence of Clifford's Theorem on restricting irreducible characters to normal subgroups that *every abelian normal subgroup of* G_1 *is central (hence cyclic)*; see [Is94, 6.13]. In the case when G_1 is solvable—hence rich in normal subgroups—these conditions impose significant constraints on the structure of a *minimal counterexample to Artin's Conjecture.* Relying heavily on these considerations, Foote and Murty prove the following:

Theorem 4.2. *Let E/F be a Galois extension of number fields with solvable Galois group G, and let $p_1, p_2, \ldots, p_n$ be the distinct prime divisors of $|\,G\,|$ with $p_1 < p_2 < \cdots < p_n$. If $\zeta_E(s)$ has a zero of order r at $s = s_0$, where $r \leq p_2 - 2$, then $L(s,\chi)$ is analytic at s_0 for all irreducible characters χ of G.*

Since all irreducible characters of nilpotent groups are monomial, if only one prime divides $|\,G\,|$, all $L(s,\chi)$ are analytic at s_0 in this case as well. As a consequence of this theorem and the Feit–Thompson Theorem on the solvability of groups of odd order the authors obtain:

Corollary 4.3. *Under the hypotheses of Theorem* 4.2*, if G has odd order, then all $L(s,\chi)$ are analytic at every point where $\zeta_E(s)$ has a zero of order ≤ 3.*

In a subsequent paper [Fo90], for the first time properties (L1) to (L4) of Artin L-series are explicitly listed as the fundamental basis for an "axiomatic" approach to Heilbronn characters. Each is then translated into a corresponding property of the set of all characters of all sections of a group G. The main result of this paper generalizes Theorem 4.2 as follows. Define $\mathcal{M}_G$ to be the minimum, taken over all subgroups H of G, of the degrees of all nonmonomial irreducible characters of H (where $\mathcal{M}_G = \infty$ if every irreducible character of every subgroup is monomial). It is easy to see that $\mathcal{M}_G \geq p_2 - 1$ in the case when G is solvable as in Theorem 4.2.

Theorem 4.4. *Let E/F be a Galois extension of number fields with solvable Galois group G, and let $s_0 \in \mathbb{C} - \{1\}$. If $\mathrm{ord}_{s=s_0}\zeta_E(s) < \mathcal{M}_G$, then all Artin L-series $L(s,\chi,E/F)$ are analytic at s_0 for every irreducible character χ of G.*

These two papers appear to be the best possible results for characterizing the "minimal obstructions" to Artin's Conjecture for the family of solvable Galois extensions. Moreover, they lead naturally to a more axiomatic approach that frees the notion of Heilbronn character from being wedded to a Galois-theoretic context. We investigate this line of thinking next.

5. Abstract Heilbronn characters for arbitrary finite groups

We now have the motivation and background for defining "abstract" Heilbronn characters of arbitrary finite groups. Such virtual characters should satisfy at least some of the above (H)-properties in order to be candidates for Heilbronn characters associated to Artin L-series. It appears that the most general yet tractable family of virtual characters is delineated by the following:

Definition 5.1. Let G be any finite group. A *Heilbronn character* of G is a virtual character θ such that $\langle\,\theta\,,\,\mathrm{Ind}_H^G(\lambda)\,\rangle \geq 0$ for every degree 1 character λ of every subgroup H of G.

In other words, the inner product of a Heilbronn character θ with every monomial character of G must be nonnegative. Of course every character of G is necessarily a Heilbronn character, so we shall be concerned primarily with Heilbronn (virtual) characters that are *not* characters.

We now observe that if we begin with an arbitrary ("abstract") Heilbronn character θ of any finite group G, we obtain "compatible" Heilbronn characters of all sections of G, namely ones that satisfy conditions (H1) to (H4) of a Galois-theoretic Heilbronn character, as follows: Given the setup of Definition 5.1, one now simply *defines* θ_H to be the restriction of θ to H for every subgroup H of G. Thus property (H1) is tautologically imposed by this definition. Furthermore, if λ is a linear character of some subgroup K of H, then by Frobenius Reciprocity and the transitivity of induction,

$$\langle\, \theta_H \,,\, \mathrm{Ind}_K^H(\lambda)\,\rangle_H = \langle\, \theta_G \,,\, \mathrm{Ind}_K^G(\lambda)\,\rangle_G \geq 0,$$

that is, θ_H satisfies the defining condition for being a Heilbronn character of H. Next, if N is a normal subgroup of G, *define*

$$\theta_{G/N} = \sum_{\chi} \langle\, \theta \,,\, \chi\,\rangle \chi,$$

where the sum is over all irreducible characters χ of G that contain N in their kernel. Evidently by (L2), if θ_G is the Heilbronn character obtained in the number-theoretic fashion, this definition of the "quotient character" $\theta_{G/N}$ gives precisely the Heilbronn character attached to the Galois extension E^N/F. Likewise in the abstract setting, one easily checks that $\theta_{G/N}$ satisfies Definition 5.1 for being a Heilbronn character of the abstract group G/N. Moreover, one easily sees that the conclusions to Proposition 2.1 hold for "abstract" Heilbronn characters in place of those obtained from representations of Galois groups. In this light, *the theory of Heilbronn characters applies independently to all finite groups.*

The term "Heilbronn character" first appears in Sandy Rhoades's doctoral dissertation [Rh93a], written after the papers [FM89, Fo90]. More generally, for any set of characters $\mathcal{F}$ of a finite group G, she defines a *Heilbronn character with respect to* $\mathcal{F}$ to be any virtual character of G whose inner product with all elements of $\mathcal{F}$ is nonnegative. For example, when $\mathcal{F}$ consists of all irreducibles, a Heilbronn character with respect to $\mathcal{F}$ is just an ordinary character (or zero). When $\mathcal{F}$ is not specified explicitly, we shall (as originally defined) assume $\mathcal{F}$ consists of all the monomial characters of G. In the first part of her thesis, Rhoades illuminates, in a more general setting, the "dual" approaches to Artin's Conjecture: on one hand via direct applications of Brauer Induction and on the other via Heilbronn characters (see [Rh93]). The proof of Rhoades's Theorem is a nice application of results about positive polyhedral cones in the theory of convexity and optimization [SW75].

Theorem 5.2. *Let $\mathcal{F}$ be a nonempty set of characters of the finite group G, and let ψ be a nonzero virtual character of G. Then ψ can be written as a positive rational linear combination of characters from $\mathcal{F}$ if and only if ψ is a Heilbronn character with respect to $\mathcal{F}$.*

Although Theorem 5.2 seems to show that no new verifications of Artin's Conjecture can be obtained via Heilbronn characters alone that might not, in some way, be established via direct applications of Brauer Induction, the efficacy of Heilbronn characters remains undiminished. In addition to vastly simplifying, and

indeed generalizing, such classical results as the Aramata-Brauer Theorem, Heilbronn characters provide new insight and number-theoretic "leverage", as Stark's Theorem and its consequences illustrate. Moreover, they effectively parameterize (via $\theta(1_{\mathrm{G}})$) "minimal counterexamples" to both Artin's Conjecture and, as we shall see shortly, "minimal Heilbronn characters" of arbitrary finite groups. We have also already seen how, given only some knowledge of orders of vanishing of zeta functions, Heilbronn characters extend beyond the family of M-groups the frontier of Galois groups for which Artin's Conjecture is known to be true. Moreover, abstract Heilbronn characters now yield insights about Brauer Induction in the general theory of finite groups, as the ensuing sections will render even more manifest.

We shall be especially concerned with the theory of *minimal* Heilbronn characters, and *unfaithful* Heilbronn characters, which we now define. The notion of a minimal Heilbronn character is precisely the translation to an abstract group setting of properties (MH1) and (MH2) of a minimal counterexample to Artin's Conjecture for a Galois group.

Definition 5.3. Let G be any finite group. A *minimal Heilbronn character* of G is a Heilbronn character θ such that

(MHi) θ is not a character of G, but $\theta|_H$ is a character of H for every proper subgroup H of G, and

(MHii) if χ is an irreducible character of G with $\langle\, \theta\,,\, \chi\,\rangle < 0$, then χ is faithful, nonlinear, and not induced from any proper subgroup of G.

The *kernel* of any Heilbronn character θ of G is $\{g \in G \mid \theta(g) = \theta(1_{\mathrm{G}})\}$.

A Heilbronn character θ is called *faithful* if its kernel is the identity subgroup (and θ is called *unfaithful* otherwise).

Assuming (MHi), it is an exercise that condition (MHii) can be replaced by the following condition, so one sees that (MHi) and (MHii′) give an equivalent definition of minimal:

(MHii′) *For every nontrivial normal subgroup N of G, $\theta_{G/N}$ is a character of G/N.*

The terms *degree* and *kernel* of a Heilbronn character θ are abuses of terminology, since θ is a *difference* of characters, so these have no strictly representation-theoretic interpretation. In particular, the kernel is *not* generally a subgroup of G, but rather a union of conjugacy classes.

For the sake of completeness we conclude this section with some complementary results. By Theorem 5.2 if χ is an irreducible character of G such that $\langle\, \chi\,,\, \psi\,\rangle \geq 0$ for all monomial characters ψ of G, then ψ is a positive *rational* linear combination of monomial characters. It follows immediately then that $k\chi$ is monomial for some positive integer k. By Brauer's Theorem every L-function is meromorphic, so if some positive integer power of it is analytic, then the L-function itself is analytic. By a result of Ferguson and Isaacs, [FI89], if G is solvable and any positive integer multiple of an irreducible character χ is monomial, then χ itself is monomial.

A recent paper by J. König [Ko09] proves that if G is a finite group for which every irreducible character has some positive integer multiple that is a monomial character, then G is necessarily solvable. This proof relies on the Classification of the Finite Simple Groups.

In a similar vein, Arthur and Clozel in [Cl87] defined the notion of *accessible* characters and proved for solvable groups that Artin's Conjecture holds for accessible characters. However, in [Da88] E. Dade proved that every accessible character

of a solvable group is monomial, hence again no new instances of the validity of Artin's Conjecture arise via their construction.

Finally, it is worth recording that the minimal non-M-groups—finite groups that are not M-groups but all of whose proper subgroups and quotient groups are M-groups—were classified by van der Waall in [vdW78].

6. Zeros of order 2 and Heilbronn characters of nonsolvable groups

Just after the Foote–Murty work on Heilbronn characters of small degree for solvable groups, again motivated by both Stark and Foote–Murty, Foote and Wales considered the general (nonsolvable) case of a zero of order 2 for $\zeta_E(s)$ at some $s_0 \in \mathbb{C} - \{1\}$. As before, by Proposition 2.1 the corresponding Heilbronn character θ_G of the Galois group G has degree 2. Moreover, by Proposition 2.1(5), $||\,\theta_G\,||^2 \leq 4$; and so if θ_G is not already a character (in which case Artin's Conjecture holds at s_0), θ_G can have at most four irreducible constituents, all with multiplicity ± 1, and at least one with each sign. These conditions seem difficult to deal with for arbitrary groups; however, the abstract Heilbronn character approach allows one to classify the *minimal* obstructions to Artin's Conjecture [FW90]:

Theorem 6.1. *Let E/F be a Galois extension of number fields with Galois group G, and let $s_0 \in \mathbb{C} - \{1\}$. Assume the Dedekind zeta function of E has a zero of order ≤ 2 at s_0. If Artin's Conjecture is false at s_0 for some character of G, then there exist intermediate fields $F \subseteq F_1 \subseteq E_1 \subseteq E$ with E_1/F_1 a Galois extension with Galois group isomorphic to $SL_2(p)$ or $\widehat{SL}_2(3)$ such that Artin's Conjecture is likewise false at s_0 for some (irreducible) character of this Galois group.*

Here $\widehat{SL}_2(3)$ is any nontrivial semidirect product of the quaternion group of order 8 by a cyclic 3-group. (Note that $\mathrm{Gal}(E_1/F_1)$ is a section of the group G.) In the terminology of Section 5, Theorem 6.1 is essentially equivalent to proving that if the counterexample Galois group G has a *Heilbronn character of degree* 2, then G contains a section isomorphic to $SL_2(p)$ for some odd prime p or $\widehat{SL}_2(3)$. Indeed, this paper classifies all finite groups that possess a minimal faithful Heilbronn character of degree 2, and so obtains a characterization applicable to general finite groups (and likewise gives new insight into the subtleties of Brauer Induction). This proof relies on the Classification of the Finite Simple Groups, but in a very focused way. We sketch the Foote–Wales strategy because it became the template for subsequent generalizations and offshoots.

For the remainder of this section assume θ is a minimal (abstract) Heilbronn character of degree 2 of the arbitrary finite group G. As in the case where G is solvable, the presence of nontrivial proper normal subgroups together with the minimality conditions provide enough leverage via Clifford's Theorem to reduce to the case where G is a quasisimple group (a perfect central extension of a nonabelian simple group); this reduction takes some effort, especially eliminating the case where G has a normal simple subgroup of prime index. Now by the Feit–Thompson Theorem G has a nontrivial Sylow 2-subgroup T. Let

$$S = \{x \in T \mid \theta(x) = 2\} = \ker \theta|_T.$$

Note that since T is a proper subgroup of G, the restriction $\theta|_T$ is an ordinary character of T and so S is a subgroup of T (i.e., here "kernel" has the usual

representation-theoretic significance). A critical observation is that

(9) S is a *strongly closed* subgroup of T with respect to G.

By definition this means the following.

Definition 6.2. Let G be any finite group, and let S and T be any subgroups of G with $S \leq T$. We say S is strongly closed in T with respect to G if for every $s \in S$, whenever $gsg^{-1} \in T$ for any $g \in G$ we must have $gsg^{-1} \in S$.

If S is a p-group for some prime p, we say S is strongly closed (without reference to T) if it is strongly closed with respect to G in some (hence every) Sylow p-subgroup containing it.

In other words, the G-conjugacy class of each s intersected with T must be contained in S for each $s \in S$. The reason (9) holds is because θ is a *global* class function on G, so its value is the same on each G-conjugate of any element s. Strong closure is a powerful "fusion-theoretic" condition on p-subgroups of G, and we explore this in greater detail in Section 8. Using this property together with a transfer theorem due to Goldschmidt [Go75, Th. B], the authors show that $T/S \cong Q_8$. Lengthy inspection of the list of all finite simple groups, using numerous structure theorems to expedite the elimination of all but the target groups, ultimately completes the proof.

In summary, the restriction of a minimal Heilbronn character of a finite group G to its (proper) Sylow p-subgroups produces strongly closed p-subgroups; these in turn provide powerful fusion-theoretic information restricting the global nature of G.

Foote and Wales also show that their result is best possible by constructing minimal Heilbronn characters for each of the groups listed in their conclusion. In particular, for every prime $p \geq 5$, $SL_2(p)$ has irreducible characters χ and ψ of degrees $p+1$ and $p-1$, respectively, such that $\chi - \psi$ is a minimal Heilbronn character of degree 2.

Finally, in the second part of her doctoral dissertation [Rh93a], Rhoades shows that if one weakens the hypothesis in Theorem 6.1 to assume a zero of order at most 3, the same conclusion holds, i.e., there are no new minimal Heilbronn character obstruction groups in degree 3. (Note that one could always add a character to a minimal Heilbronn character to obtain another minimal Heilbronn character.)

7. Classification of unfaithful minimal Heilbronn characters

As noted, an essential component of the proof of the classification of "minimal Heilbronn groups" for degree 2 Heilbronn characters is the leverage provided by the strongly closed subgroup S of a Sylow 2-subgroup T of G where, as before, $S = \ker \theta|_T$. It turns out that the ordinary character $\theta|_T$ is, in fact, a *faithful* character of T of degree 2, i.e., $S = \langle 1_{\mathrm{G}} \rangle$. The 2-groups that possess faithful representations of degree 2 have an abelian subgroup of rank at most 2 and index at most 2.

If one could prove $S = \langle 1_{\mathrm{G}} \rangle$ without relying on the full Classification of the Finite Simple Groups, much less complicated classifications could then be invoked to determine the minimal obstructions in Theorem 6.1. In other words, knowing that θ restricts to a *faithful* ordinary character of degree $\theta(1_{\mathrm{G}})$ on a Sylow 2-subgroup gives significant information about a minimal counterexample G. This observation instigated the next development [Fo97a].

Theorem 7.1. *Let E/F be a Galois extension of number fields with Galois group G. If E/F is a minimal counterexample to Artin's Conjecture at s_0, then $\theta_G|_T$ is a faithful character for any Sylow 2-subgroup T of G. In particular, any Sylow 2-subgroup of G has a faithful representation of degree r, where $r = \theta_G(1_G)$ is the order of the zero of the Dedekind zeta function of E at s_0.*

Again, it is important to note that this theorem is a consequence of the corresponding general theorem about abstract minimal Heilbronn characters θ of arbitrary finite groups (which we omit stating, as it is, *mutatis mutandis*, the above result). As in the Foote and Wales paper, the proof reduces to the bedrock case where G is a quasisimple group. Likewise for $T \in \mathrm{Syl}_2(G)$ and $S = \ker \theta|_T$, we see immediately that S is strongly closed in T with respect to G. At this point Foote quotes the complete classification of all finite groups possessing a strongly closed 2-subgroup (see Theorem 8.1), which was specifically proved in order to deal with this setup (although it has significant independent ramifications, as we shall explore in Section 8). If $S = \langle 1_{\mathrm{G}} \rangle$, then $\theta|_T$ is faithful, and the conclusion holds. If $\langle 1_{\mathrm{G}} \rangle < S < T$, then the strongly closed classification yields that G is a quasisimple group with a BN-pair of rank 1, and we shall see that such groups easily lead to a contradiction. It remains to handle when $S = T$, i.e., $\theta|_T$ is a multiple of the principal character. This difficult case—about which the strongly closed classification gives no information—is eliminated by a graph-theoretic argument which we now sketch, as it is a cornerstone of generalizations that we shall describe later.

Recall that $\ker \theta = \{g \in G \mid \theta(g) = \theta(1_{\mathrm{G}})\}$ is *not* generally a subgroup (here G is quasisimple, so its only proper normal subgroups are contained in its center); however, since $\theta|_H$ is an ordinary character of every proper subgroup H of G,

$$\ker \theta|_H \text{ is a normal subgroup of } H \text{ for every } \textit{proper} \text{ subgroup } H \text{ of } G.$$

The strategy is to "tie together" all these proper kernels to obtain the contradiction that θ is constant on all of G. To this end, define a relation on G-stable subsets of G as follows: Let $\mathcal{C}$ be a union of conjugacy classes of G, let $\mathcal{D}$ be a single conjugacy class, and write $\mathcal{C} \to \mathcal{D}$ if there is a proper subgroup H of G generated by some elements of $\mathcal{C}$ such that $\mathcal{D} \cap H \neq \emptyset$. Observe that if $\mathcal{C} \subseteq \ker \theta$ and $\mathcal{C} \to \mathcal{D}$, then because $\theta|_H$ is an ordinary character of H but H is generated by elements in the kernel of this character, every element of H is in the kernel of $\theta|_H$. In particular, $\theta(d) = \theta(1_{\mathrm{G}})$ for some $d \in \mathcal{D}$, and so the entire conjugacy class $\mathcal{D}$ lies in the set $\ker \theta$. Whenever this happens, we can then replace $\mathcal{C}$ by the ostensibly larger set $\mathcal{C} \cup \mathcal{D} = \mathcal{C}'$ and seek new conjugacy classes $\mathcal{D}'$ such that $\mathcal{C}' \to \mathcal{D}'$. The method of proof in [Fo97a] is that if $S = T$, then we may begin this process by taking $\mathcal{C}$ to be the set of all 2-elements in G. Ultimately, Foote shows that every conjugacy class is "swallowed-up" via the $\to$ generation process, hence $\ker \theta = G$, a contradiction.

To illustrate, consider when $G \cong \mathrm{Sz}(2^n)$ is a Suzuki simple group. In this group, assuming only $S \neq \langle 1_{\mathrm{G}} \rangle$, we get that S contains some involution in G. Thus $\ker \theta$ contains the (single) class of all involutions. If an involution s inverts an odd order element x of G, then x lies in the dihedral group $\langle s, x \rangle = \langle s, xsx^{-1} \rangle$, and so all such odd order elements lie in $\ker \theta$. In $\mathrm{Sz}(2^n)$ this includes all odd order elements. Finally, the Sylow 2-subgroup T is normalized by an odd order element h (a Cartan element of G) such that $T = [T, h]$. Thus T is contained in $\ker \theta|_H$ because $H = \langle T, h \rangle$ is a proper subgroup generated by conjugates of h. This shows θ is constant on all of G, a contradiction. Indeed, an analogous short

"chain of arrows" may be constructed to eliminate each of the BN-rank 1 groups mentioned above where $\langle 1_G \rangle < S < T$. More elaborate chains are needed for other quasisimple groups.

The $SL_2(p)$ examples in [FW90], for p a prime ≥ 5, show that if G is a quasisimple group possessing a minimal Heilbronn character θ, then θ need not be faithful. More specifically, the minimal Heilbronn characters of degree 2 on these groups all restrict to twice the principal character on a Sylow p-subgroup. This somewhat discouraging observation, together with the lack of a classification of strongly closed p-subgroups for p odd, impeded progress along these lines for over ten years.

In early 2007, Ramón Flores contacted Foote seeking information about strongly closed p-subgroups for odd primes p in order to extend some work he was doing in the area of homotopy, completely unrelated to Heilbronn characters. Flores, along with J. Scherer [FS07], gave a complete description of the $B\mathbb{Z}/2\mathbb{Z}$-cellularization of classifying spaces BG for all finite groups G, by classifying the possible homotopy types of $\mathbf{CW}_{B\mathbb{Z}/2\mathbb{Z}}BG$. Their classification relied, in an essential way, on the classification of finite groups possessing strongly closed 2-subgroups. Flores wished to extend his results to odd primes. As a result, he and Foote collaborated to first give a complete classification of all finite groups possessing a strongly closed p-subgroup for all odd p [FF09], and then used this to prove the corresponding $B\mathbb{Z}/p\mathbb{Z}$-cell structure classification for odd primes p too [FF11]. For quasisimple groups, the families of strongly closed "obstructions" obtained for p odd is much more diverse than for $p = 2$, and includes a wide range of Chevalley simple groups as well as some sporadic simple groups (see Theorem 8.2 following). This unexpected collaboration, impelled by problems in completely different fields, then provided new impetus to the study of Heilbronn characters (and to other areas, as will be outlined in Section 8).

With the aforementioned group-theoretic classification in hand, Hy Ginsberg undertook the project of extending Theorem 7.1 to all odd primes. More precisely, he set out to characterize exactly which groups possess a minimal Heilbronn character that *fails* to restrict to a faithful character on some Sylow p-subgroup. In other words, Ginsberg sought to classify groups that possess an unfaithful minimal Heilbronn character. The main result of his doctoral dissertation [Gi10, Gi11] follows:

Theorem 7.2. *Suppose G is a finite group possessing an unfaithful minimal Heilbronn character θ. Then θ restricts to an unfaithful character of some Sylow subgroup of G, and if P is a Sylow p-subgroup of G on which θ is unfaithful, then all of the following hold:*

(i) *p is odd;*
(ii) *G is quasisimple with a cyclic center of order prime to p;*
(iii) *P is cyclic;*
(iv) *$N_G(P)$ is a maximal subgroup of G; and*
(v) *either $N_G(P)$ is the unique maximal subgroup of G containing $\Omega_1(P)$ (the subgroup of order p in P) or $G/Z(G) \cong L_2(q)$ for q an odd prime with p dividing $q-1$. (In the latter case $\Omega_1(P)$ is also contained in a Borel subgroup $N_G(Q)$ for some Sylow q-subgroup Q of G.)*

Conversely, suppose G is a finite group, and for some prime p and Sylow p-subgroup P of G conditions (i) *to* (v) *above hold. If P_1 is any nontrivial subgroup of P, then G has a minimal Heilbronn character whose restriction to P has kernel equal to P_1.*

This result established highly restrictive necessary and sufficient conditions for a general finite group to have an unfaithful minimal Heilbronn character.

Ginsberg's determination of the necessary conditions follows the same overall strategy as in the $p = 2$ classification, first reducing to the quasisimple case, then handling the groups that possess "nontrivial" strongly closed p-subgroups (quoting the Flores–Foote classification). He unravels the "arrow" relationship in simple groups, starting from the configuration that the initial "$\mathcal{C}$" is a set of p-elements of G, for appropriate odd prime p. He achieves some savings by quoting the overall $p = 2$ classification whenever some nontrivial 2-element lies in the kernel of the minimal Heilbronn character restricted to some proper subgroup. However, he had considerable work to do since there are an array of different quasisimple groups that satisfy his conditions (i.e., he is not simply heading for a contradiction, as described in the $p = 2$ case).

To prove that conditions (i) to (v) are also sufficient, Ginsberg showed how to construct the desired unfaithful minimal Heilbronn characters on such a quasisimple group. He took a faithful irreducible character χ of G and then redefined χ on p-elements in such a way that it has any specified kernel P_1 when restricted to the Sylow p-subgroup P. He then showed that the resulting class function θ restricts to an ordinary character on every proper subgroup, hence is a virtual character (hence an unfaithful minimal Heilbronn character) by Brauer's Characterization of Characters. The "uniqueness" condition (v) on P—which says that P is "isolated" from the rest of G in some respects—is crucial to showing that the redefinition of χ does not destroy the property that the modified χ remains a character when restricted to all proper subgroups.

Ginsberg classifies the unfaithful minimal Heilbronn characters for the simple groups $L_2(q)$ in (v) in [Gi13a].

Finally, Ginsberg recently essentially classified all quasisimple groups satisfying conditions (i) to (v) in [Gi13b].

Theorem 7.3. *Assume G is a finite quasisimple group, p is an odd prime dividing the order of G, a Sylow p-subgroup P of G is cyclic, and $N_G(P)$ is the unique maximal subgroup of G containing $\Omega_1(P)$. Then $G/Z(G)$ is isomorphic to one of the following groups:*

(i) *An alternating group A_p of prime degree, with $p \neq 11$ and $p \neq 23$. The cyclotomic polynomial $\Phi_d(q)$ does not equal p for any prime d and prime power q.*
(ii) *A linear group $L_2(p)$.*
(iii) *A linear group $L_n(q)$, $q = r^t$ for some prime $r \neq p$, with n a prime, t odd, and $\mathrm{ord}_p(r) = nt$. The prime p does not divide the order of any subgroup in the collection $\mathcal{S}$ of almost simple subgroups of G.*
(iv) *A unitary group $U_n(q)$, $q = r^t$ for some prime $r \neq p$, with n an odd prime and $\mathrm{ord}_p(-r) = nt$. The prime p does not divide the order of any subgroup in the collection $\mathcal{S}$ of almost simple subgroups of G.*
(v) *A Suzuki group ${}^2B_2(q)$ with $\mathrm{ord}_p(q_0) = 4t$ whenever $q_0 > 2$ and $q = q_0^t$ for $t = 1$ or t prime.*
(vi) *A Ree group ${}^2G_2(q)$ with $\mathrm{ord}_p(q_0) = 6t$ whenever $q = q_0^t$ for $t = 1$ or t prime.*
(vii) *${}^3D_4(q)$ with $\mathrm{ord}_p(q_0) = 12t$ whenever $q = q_0^t$ for $t = 1$ or t prime.*

(viii) *A Ree group* ${}^2F_4(q)$ *with* $\mathrm{ord}_p(q_0) = 12t$ *whenever* $q = q_0^t$ *for* $t = 1$ *or* t *prime.*

(ix) $E_8(q)$ *with* $\mathrm{ord}_p(q_0) = 15t$, $24t$, *or* $30t$ *whenever* $q = q_0^t$ *for* $t = 1$ *or* t *prime.*

(x) *A sporadic group:* J_1 *with* $p = 19$*;* M_{23} *with* $p = 23$*;* Ly *with* $p = 37$ *or* $p = 67$*;* J_4 *with* $p = 29$ *or* $p = 43$*;* Fi'_{24} *with* $p = 29$*; or* B *with* $p = 47$.

Moreover, except in case (ix)*, the given conditions are also sufficient.*

The collection of almost simple subgroups of a classical group G is discussed in [KL90]; $\mathrm{ord}_p(q)$ denotes the multiplicative order of q modulo p. The sufficiency condition in this theorem is a consequence of the Bang–Zsigmondy Theorem on the existence of primitive prime divisors of $a^n - 1$ [Ba86, Ro97].

This result together with Theorem 7.2 now give an essentially complete classification of the groups that possess unfaithful minimal Heilbronn characters, and hence a corresponding classification for minimal counterexamples to Artin's Conjecture.

Corollary 7.4. *Let* E/F *be a Galois extension of number fields with Galois group* G*. If* E/F *is a minimal counterexample to Artin's Conjecture at* s_0*, then either the Heilbronn character of* G *at* s_0 *is faithful, i.e.,* $\ker \theta_G = \langle 1_G \rangle$*, or there is an odd prime* p *such that* $\theta|_P$ *is not faithful for some Sylow* p*-subgroup* P *of* G*, and the pair* (G, p) *is as described in the conclusion to Theorem* 7.3*, or* $G \cong L_2(q)$ *for* q *an odd prime with* p *dividing* $q-1$*. In particular, in the former case every proper subgroup of* G *has a faithful representation of degree* r*, where* $r = \theta_G(1_G)$ *is the order of the zero of the Dedekind zeta function of* E *at* s_0*.*

Note that the conclusion θ_G is faithful, imposes significant restrictions on the structure of G. In this case every proper subgroup of G has a faithful representation of degree r; in particular, the rank of every abelian subgroup of G is at most r.

8. Strongly closed subgroups of finite groups

Because the theory of strongly closed subgroups plays such an important role in the study of Heilbronn characters, and is also central to finite group theory as well as to some other areas of mathematics, this section is a brief diversion that highlights some major results employed in—and partially impelled by—Heilbronn character research. Subsequent sections do not rely on it, so this section may be skipped or postponed by readers wishing to return expeditiously to L-functions.

As noted earlier, the classification of all finite groups possessing a strongly closed p-subgroup was first carried out for $p = 2$ [Fo97b] with the specific intention of using it to prove Theorem 7.1 on minimal Heilbronn characters—the latter theorem appears in the same journal issue as the strongly closed classification. About ten years later, Flores instigated the p odd strongly closed subgroup classification for applications in homotopy theory (having used the $p = 2$ result in his earlier work). The p odd classification was then employed by Ginsberg to determine all unfaithful minimal Heilbronn characters, thereby coming full circle. The theory of strongly closed subgroups has a rich and extensive history within the overall theory of finite groups, and it would be too long and take us too far afield to survey it. We therefore include this brief section, following closely the introduction in [FF09], to give an *aperçu* of the area and its larger scope.

For any finite group G and subgroup T we say two elements of T are *fused* in G if they are conjugate in G but not necessarily in T. This concept has played

a central role in group theory and representation theory, particularly in the case when T is a Sylow p-subgroup of G for p a prime.

Recall from Definition 6.2 that a subgroup S of T is called *strongly closed* in T with respect to G if for every $a \in S$, every element of T that is fused in G to a lies in S; in other words, $a^G \cap T \subseteq S$, where a^G denotes the G-conjugacy class of a. It is easy to verify that if S is a p-subgroup, then S is strongly closed in a Sylow p-subgroup if and only if it is strongly closed in $N_G(S)$, so the notion of strong closure for a p-subgroup does not depend on the Sylow subgroup containing it. For a p-group S we therefore simply say S is strongly closed. Seminal works in the theory of strongly closed 2-subgroups are the celebrated Glauberman Z^*-Theorem [Gl66] and Goldschmidt's theorem on strongly closed abelian 2-subgroups [Go74]. The Z^*-Theorem proves that if S is strongly closed and of order 2, then $\overline{S} \leq Z(\overline{G})$, where the overbars denote passage to $G/O_{2'}(G)$. Goldschmidt extended this by showing that if S is a strongly closed abelian 2-subgroup, then $\langle \overline{S}^{\overline{G}} \rangle$ is a central product of an abelian 2-group and quasisimple groups that either have a BN-pair of rank 1 or have abelian Sylow 2-subgroups. These two theorems, in particular, played fundamental roles in the study of finite groups, especially in the Classification of the Finite Simple Groups.

The concept of strong closure has important ramifications beyond finite group theory. In particular, it is intimately connected to Puig's formulation of *fusion systems* (or Frobenius categories), which evolved from the modular representation theory of finite groups (each p-block of a finite group has an associate fusion system). A fusion system on any finite p-group T is a collection of injections between subgroups of T satisfying various axioms (see [As07]). This construct subsumes the special case when T is a Sylow p-subgroup of some finite group G, and the collection of maps consists of all conjugation maps $H \to gHg^{-1}$ whenever H and gHg^{-1} are both subgroups of T for any $g \in G$. In other words, in the latter context a fusion system "throws out" the ambient group G and retains only the (abstract) maps induced on various subgroups of T by G-conjugation. (There are examples of fusion systems on a specific p-group T that cannot arise from just conjugation maps by embedding T as a Sylow subgroup of some finite group G; so the category of fusion systems properly contains the group-theoretic families of examples.) The concept of strong closure extends in an obvious way to abstract fusion systems and plays a critical role therein: if $\mathcal{F}$ is a fusion system on a p-group T, then the "homomorphic images" of $\mathcal{F}$ are in bijective correspondence with the strongly closed subgroups of T. Fusion systems were further refined by Broto, Levi, and Oliver in [BLO03] to create the class of p-local finite groups (see also [As07, BLO07, BCGLO07, Li06]). Oliver then used this approach to prove that the homotopy type of the p-completed classifying space of a finite group G is uniquely determined by the (saturated) fusion system (G, T), where T is a Sylow p-subgroup of G. Thus strong closure and its extensions to fusion systems and p-local finite group theory also has significant ramifications in deep and currently very active areas of modular representation theory and algebraic topology.

To describe the main classification, we introduce some notation. Henceforth p is any prime, T is a Sylow p-subgroup of the finite group G, and S is a subgroup of T. First note that strongly closed subgroups abound in finite group theory: Namely, if N is any normal subgroup of G, then $T \cap N$ is a Sylow p-subgroup of N and is also strongly closed in G. Thus the presence of strongly closed subgroups suggests,

but does not guarantee, the presence of normal subgroups. Indeed, the aim of the classification is to characterize the "obstructions" to this phenomenon. To do so we must first quotient out the largest "natural" factor.

In general let R be any p-subgroup of G. If N_1 and N_2 are normal subgroups of G with $R \cap N_i \in \mathrm{Syl}_p(N_i)$ for both $i = 1, 2$, then $R \cap N_1N_2$ is a Sylow p-subgroup of N_1N_2. Thus there is a unique largest normal subgroup N of G for which $R \cap N \in \mathrm{Syl}_p(N)$; denote this subgroup by $\mathcal{O}_R(G)$. Thus

$$R \text{ is a Sylow } p\text{-subgroup of } \langle R^G \rangle \text{ if and only if } R \leq \mathcal{O}_R(G).$$

Note that $|\, O_{p'}(G/\mathcal{O}_R(G))\,| = 1$; in particular, if $R = \langle 1_{\mathrm{G}} \rangle$ is the identity subgroup, then $\mathcal{O}_R(G) = O_{p'}(G)$. In general, $R\mathcal{O}_R(G)/\mathcal{O}_R(G)$ does not contain the Sylow p-subgroup of any nontrivial normal subgroup of $G/\mathcal{O}_R(G)$; in other words, $\mathcal{O}_{\overline{R}}(\overline{G}) = \langle 1_{\overline{\mathrm{G}}} \rangle$, where overbars denote passage to $G/\mathcal{O}_R(G)$. Observe that strong closure passes to quotient groups, so when analyzing groups where $R \not\leq \mathcal{O}_R(G)$, we may factor out $\mathcal{O}_R(G)$. With this in mind, the classification for strongly closed 2-subgroups from [Fo97a] is as follows:

Theorem 8.1. *Let G be a finite group that possesses a strongly closed 2-subgroup S. Assume S is not a Sylow 2-subgroup of $\langle S^G \rangle$, and let $\overline{G} = G/\mathcal{O}_S(G)$. Then $\overline{S} \neq \overline{\langle 1_G \rangle}$ and $\langle \overline{S}^{\overline{G}} \rangle = L_1 \times L_2 \times \cdots \times L_r$, where each L_i is isomorphic to $U_3(2^{n_i})$ or $Sz(2^{n_i})$ for some n_i, and $\overline{S} \cap L_i$ is the center of a Sylow 2-subgroup of L_i.*

The classification for p odd [FF11] yields a more diverse set of "obstructions" with added "decorations" as well.

Theorem 8.2. *Let p be an odd prime, and let G be a finite group that possesses a strongly closed p-subgroup S. Assume S is not a Sylow p-subgroup of $\langle S^G \rangle$, and let $\overline{G} = G/\mathcal{O}_S(G)$. Then $\overline{S} \neq \overline{\langle 1_G \rangle}$ and*

$$\langle \overline{S}^{\overline{G}} \rangle = (L_1 \times L_2 \times \cdots \times L_r)(D \cdot S_F), \tag{10}$$

where $r \geq 1$, each L_i is a simple group, and $S_i = \overline{S} \cap L_i$ is a homocyclic abelian group. Furthermore, $D = [D, S_F]$ is a (possibly trivial) p'-group normalizing each L_i, and S_F is a (possibly trivial) abelian subgroup of $\overline{S}$ of rank at most r normalizing D and each L_i and inducing outer automorphisms on each L_i, and the extension $(S_1 \cdots S_r) : S_F$ splits. Each L_i belongs to one of the following families:

(i) *L_i is a group of Lie type in characteristic $\neq p$ whose Sylow p-subgroup is abelian but not elementary abelian. In this case the Sylow p-subgroup of L_i is homocyclic of the same rank as S_i but larger exponent than S_i; here $D/(D \cap L_iC_{\overline{G}}(L_i))$ is a cyclic p'-subgroup of the outer diagonal automorphism group of L_i, and $S_F/C_{S_F}(L_i)$ acts as a cyclic group of field automorphisms on L_i.*

(ii) *$L_i \cong U_3(p^n)$ or $\mathrm{Re}(3^n)$ is a group of BN-rank 1 ($p = 3$ with n odd and ≥ 2 in the latter family). In the unitary case S_i is the center of a Sylow p-subgroup of L_i (elementary abelian of order p^n), and in the Ree group case S_i is either the center or the commutator subgroup of a Sylow 3-subgroup (elementary abelian of order 3^n or 3^{2n}, respectively). In both families D and S_F act trivially on L_i.*

(iii) *$L_i \cong G_2(q)$ with $(q, 3) = 1$. Here $|\, S_i \,| = 3$ and both D and S_F act trivially on L_i.*

(iv) *L_i is one of the following sporadic groups, where in each case S_i has prime order, and both D and S_F act trivially on L_i:*
$(p = 3)$ **:** J_2;
$(p = 5)$ **:** Co_3, Co_2, HS, Mc;
$(p = 11)$**:** J_4.

(v) *$L_i \cong J_3$, $p = 3$, and S_i is either the center or the commutator subgroup of a Sylow 3-subgroup (elementary abelian of order 9 or 27, respectively). Here D and S_F act trivially on L_i.*

The proof of Theorem 8.2 relies on the Classification of the Finite Simple Groups, which in turn relies heavily on the Glauberman–Goldschmidt results cited earlier; so Theorem 8.2 does not give an independent verification of the latter.

We note that for p odd, "most" simple groups possess a nontrivial strongly closed p-subgroup that is proper in a Sylow p-subgroup; that is, conclusion (i) of Theorem 8.2 is the "generic obstruction" in the following sense. Let $\mathcal{L}_n(q)$ denote a simple group of Lie type and BN-rank n over the finite field $\mathbb{F}_q$ with $(q, p) = 1$. For all but the finitely many primes dividing the order of the Weyl group of the untwisted version of $\mathcal{L}_n(q)$, the Sylow p-subgroups of $\mathcal{L}_n(q)$ are homocyclic abelian. Furthermore, the order of $\mathcal{L}_n(q)$ can be expressed as a power of q times factors of the form $\Phi_m(q)^{r_m}$ for various $m, r_m \in \mathbb{N}$, where $\Phi_m(x)$ is the mth cyclotomic polynomial. Then it can be shown that if m_0 is the multiplicative order of q (mod p), then p divides $\Phi_{m_0}(q)$ and the abelian Sylow p-subgroup of $\mathcal{L}_n(q)$ is homocyclic of rank r_{m_0} and exponent $|\,\Phi_{m_0}(q)\,|_p$ (see [GL83, 10.1]). In particular it is not elementary abelian whenever $p^2 \mid \Phi_{m_0}(q)$. For example, this is the case in the groups $PSL_{n+1}(q)$ whenever $p > n + 1$ and p^2 divides $q^m - 1$ for some $m \leq n + 1$. Thus for fixed n and all but finitely many p, this can always be arranged by taking q suitably large.

9. Applications to other L-functions

The formalism of Heilbronn characters described in the previous sections can be applied to L-functions other than Artin L-functions. In [RK1] an application to the L-function of an elliptic curve is investigated. We briefly describe that here. Consider an elliptic curve C defined over the number field F. This is a complete genus 1 curve with a model over F and having an F-rational point. As is well known, this means that we can define an F-rational group structure on the set of points $C(\overline{F})$. By the Mordell–Weil Theorem, for any finite extension E/F, the set of E-rational points $C(E)$ is a finitely generated abelian group, and so we may speak of its rank. We certainly have the inequality

$$\text{rank } C(E) \geq \text{rank } C(F). \tag{11}$$

Associated to C and E there is an L-function given as an Euler product

$$L(s, C, E) = \prod_w L_w(s, C_w, k_w).$$

Here the product is over finite primes w of E, and k_w denotes the residue field at w. Moreover, C_w is the "reduction" of C modulo w. Apart from a finite number of primes w, the factor can be described as follows:

$$L_w(s, C_w, k_w) = \left(1 - a_w(\mathbb{N}w)^{-s} + (\mathbb{N}w)^{1-2s}\right)^{-1}.$$

Here the rational integer a_w is given by the formula

$$a_w = \mathbb{N}w + 1 - |C(k_w)|,$$

and $\mathbb{N}w$ denotes the norm of w. Because of the Hasse bound

$$|a_w| \leq 2(\mathbb{N}w)^{1/2},$$

the product above converges absolutely for $\mathrm{Re}(s) > 3/2$. It is conjectured to have an analytic continuation for all s, and in many cases this is now known (beginning with the spectacular work of Wiles and continuing with the work of his students and others).

Assuming the analytic continuation, the conjecture of Birch and Swinnerton-Dyer asserts (in its weak form) that

$$\mathrm{ord}_{s=1} L(s, C, E) = \mathrm{rank}\ C(E).$$

Combining this with the inequality (11), this conjecture predicts that

$$\mathrm{ord}_{s=1} L(s, C, E) \geq \mathrm{ord}_{s=1} L(s, C, F).$$

More generally, we even expect that this inequality should hold at all points, in other words

$$\text{the quotient } L(s, C, E)/L(s, C, F) \text{ is entire.}$$

This is the elliptic analogue of Dedekind's Conjecture. In [RK1, Theorem 1], it is shown that if C has complex multiplication, then the analogue of the Aramata–Brauer Theorem holds; in other words, the quotient is entire for E/F Galois. An elliptic curve C is said to have complex multiplication if its ring of endomorphisms, $\mathrm{End}\ C$, is strictly larger than $\mathbb{Z}$. In this case $\mathrm{End}\ C$ is in fact an order in an imaginary quadratic field K (say). From the work of Deuring, it is known that the L-function can be expressed in terms of L-functions associated to Hecke characters (of infinite order) of K.

For a non-Archimedean prime v of K, restricting a Hecke character ψ of K to the multiplicative group $K_v^\times$ of the completion at v, gives a character ψ_v, and the Hecke L-function associated to ψ is given by an Euler product

$$L(s, \psi, K) = \prod_v \left(1 - \frac{\psi_v(\pi_v)}{(Nv)^s}\right)^{-1},$$

where the product is over the non-Archimedean primes of K at which ψ is unramified, and π_v denotes a uniformizing parameter of $K_v^\times$. The work of Hecke (recast in the language of idèles by Tate in his thesis) shows that this L-function has an analytic continuation as an entire function of s with a suitable functional equation.

Deuring [De] proved that if $K \subseteq F$, then there are Hecke characters ψ_1, ψ_2 of F of infinite order such that

$$L(s, C, F) = L(s, \psi_1, F)L(s, \psi_2, F). \tag{12}$$

If $K \not\subseteq F$, then $M = KF$ is a quadratic extension of F and there exists a Hecke character ψ of M such that

$$L(s, C, F) = L(s, \psi, M).$$

To explain how the ideas described in this article can be used to prove the above-mentioned analogue of the Aramata–Brauer Theorem for CM elliptic curves,

consider the special case $K \subseteq F$. We need the concept of an Artin–Hecke L-function. If χ is a character of $G = \mathrm{Gal}(E/F)$ with underlying space V and ψ is a Hecke character of F, we define

$$L(s, \psi \otimes \chi, E/F) = \prod_v \det(1 - \psi(v)\chi(\mathrm{Frob}_v)|_{V^I}(Nv)^{-s})^{-1},$$

where the product is over (non-Archimedean) primes of F. Artin–Hecke L-functions satisfy the formalism given in properties (L1)–(L4) of Section 2. In particular, if $\chi = \mathrm{Ind}_H^G \phi$, then

$$L(s, \psi \otimes \chi, E/F) = L(s, \psi \circ \mathbb{N}_{E^H/F} \otimes \phi, E/E^H), \tag{13}$$

where for any extension M of F, we denote by $\mathbb{N}_{M/F}$ the norm map from M to F.

By Brauer Induction and class field theory, the Artin–Hecke L-series are known to have a meromorphic continuation for all s. Moreover, the identity (13) above shows that if the Artin character χ is the induction of a one-dimensional character, say $\chi = \mathrm{Ind}_H^G \phi$, then the L-function $L(s, \psi \otimes \chi, E/F)$ is in fact holomorphic except possibly at $s = 1$, where there may be a simple pole that occurs if and only if the character $\psi \circ N_{E^H/F} \otimes \phi$ is the trivial character. In particular, if ψ is of infinite order, $L(s, \psi \otimes \chi, E/F)$ is entire.

In any case, for any subgroup H of G and any character ϕ of H, we may define the integer

$$n(H, \phi) = \mathrm{ord}_{s=s_0} L(s, \psi_1 \circ \mathbb{N}_{E^H/F} \otimes \phi, E/E^H) L(s, \psi_2 \circ \mathbb{N}_{E^H/F} \otimes \phi, E/E^H),$$

where ψ_1, ψ_2 are the Hecke characters from Deuring's theorem (12). Then one may consider the Heilbronn (virtual) character

$$\theta_{G,C} \;=\; \sum_\chi n(G, \chi)\chi,$$

where as usual, the sum is over the irreducible characters χ of G. It is easy to verify that the properties of Heilbronn characters described in Proposition 2.1 apply to this character, suitably interpreted. Indeed, we have

$$L(s, C, E) \;=\; L(s, \psi_1 \circ \mathbb{N}_{E/F}, E) L(s, \psi_2 \circ \mathbb{N}_{E/F}, E).$$

Moreover, for any Hecke character ψ of F and any finite Galois extension E/F with Galois group G, we have

$$L(s, \psi \circ \mathbb{N}_{E/F}, E) \;=\; L(s, \psi \otimes \mathrm{Ind}_{1_G}^G I_G, F) \;=\; L(s, \psi \otimes \mathrm{reg}, F).$$

In particular, we see that $\theta_{G,C}(1)$ is the order at s_0 of $L(s, C, E)$. Moreover, using the same argument as in the proof of Proposition 2.1(1), we see that for any subgroup H of G, the restriction of $\theta_{G,C}$ to H is equal to $\theta_{H,C'}$, where C' is the base change of C to the subfield E^H fixed by H. Thus, as in Proposition 2.1(4), we see that for any $g \in G$, we have

$$|\theta_{G,C}(g)| \;\leq \theta_{G,C}(1_G).$$

Hence,

$$\sum_\chi n(G, \chi)^2 \;=\; ||\theta_{G,C}||^2 \;\leq\; \theta_{G,C}(1_G)^2.$$

It follows that

$$n(G, I_G) \leq n(G, \mathrm{reg}),$$

where I_G is the principal character of G; and so, in particular, $L(s,C,E)/L(s,C,F)$ is entire.

For curves without complex multiplication, it is shown in [RK1, Theorem 2], that the quotient is entire provided E/F is solvable. For this we have to invoke the Arthur–Clozel theory of base change. We refer the reader to [RK1] for the details.

10. Refinements and variations

Consider the original Heilbronn (virtual) character (5). Ram Murty [RM1], studied the modified Heilbronn character

$$\theta_G^* = \theta_G - n(G, I_G)I_G.$$

Suppose that for every cyclic subgroup H of G, we have $n(H, I_H) \geq n(G, I_G)$. Then the same arguments as in Section 2 show that

$$\sum_{\chi \neq I_G} |n(G,\chi)|^2 \leq (n(G,\mathrm{reg}) - n(G, I_G))^2.$$

In particular, if for every cyclic subgroup H of $G = \mathrm{Gal}(E/F)$, we have $\zeta_{E^H}(s)/\zeta_F(s)$ is analytic at $s = s_0$, then

$$\sum_{\chi \neq I_G} n(G,\chi)^2 \leq (\mathrm{ord}_{s=s_0}\zeta_E(s)/\zeta_F(s))^2.$$

The hypothesis is satisfied if E/F is solvable.

Ram Murty and A. Raghuram [RR] give a slight generalization of the above in which the trivial character is replaced by any one-dimensional character. More precisely, they prove that for any one-dimensional character χ_0 of the solvable group $G = \mathrm{Gal}(E/F)$ we have

$$\sum_{\chi \neq \chi_0} n(G,\chi)^2 \leq (\mathrm{ord}_{s=s_0}\zeta_E(s)/L(s,\chi_0,E/F))^2. \tag{14}$$

To prove this, the work of Uchida and van der Waall shows that given a subgroup H of G, there are subgroups H_i and one-dimensional characters ψ_i of H_i such that

$$\mathrm{Ind}_H^G I_H = I_G + \sum \mathrm{Ind}_{H_i}^G \psi_i.$$

Tensoring both sides by χ and writing $\psi = \chi|_H$, we have

$$\mathrm{Ind}_H^G \psi = \chi + \sum \mathrm{Ind}_{H_i}^G \psi_i',$$

where $\psi' = \psi_i \cdot \chi|_{H_i}$. It follows that the quotient

$$L(s, \mathrm{Ind}_H^G \psi, E/F)/L(s,\chi,E/F)$$

is analytic at all $s \neq 1$. Notice that this can be rephrased in the following (apparently more general) form: Let H be any subgroup of G, and let ψ be a one-dimensional character of H. Let χ be a one-dimensional character of G, and let $m(\chi,\psi)$ denote the multiplicity of χ in $\mathrm{Ind}_H^G\psi$. Then

$$L(s, \mathrm{Ind}_H^G \psi, E/F)/L(s,\chi,E/F)^{m(\chi,\psi)}$$

is analytic at all $s \neq 1$.

Using this, it follows that for any cyclic subgroup H of G, we have

$$n(H,\psi) - n(G,\chi_0)m(\chi_0,\psi) \geq 0.$$

Thus the analogue of the hypothesis of the result of Ram Murty is satisfied, and proceeding as in that paper but working with the modified Heilbronn character

$$\theta_G^{\chi_0} = \theta_G - n(G, \chi_0)\chi_0,$$

they deduce that (14) holds.

In [RR] this is further sharpened to show in fact that for all s_0,

$$\sum_{\chi(1_G)>1} n(G,\chi)^2 \leq \left(\mathrm{ord}_{s=s_0} \frac{\zeta_E(s)}{\zeta_{E^{ab}}(s)}\right). \tag{15}$$

Here E^{ab} denotes the fixed field under the commutator subgroup $[G, G]$. In particular, it follows that $\zeta_E(s)/\zeta_{E^{ab}}(s)$ cannot have any simple zeros or poles. The latter statement follows from the first because of the factorization

$$\zeta_E(s)/\zeta_{E^{ab}}(s) = \prod_{\chi(1_G)>1} L(s, \chi, E/F)^{\chi(1_G)}.$$

To prove the former statement, one considers the Heilbronn-like function

$$\theta'_G = \theta_G - \sum_{\chi(1_G)=1} n(G,\chi)\chi.$$

As usual, we have the inequality

$$\sum_{\chi(1_G)>1} n(G,\chi)^2 = \frac{1}{|G|}\sum_{g\in G} |\theta'_G(g)|^2.$$

Moreover, we have for any $g \in G$ and H the subgroup generated by g,

$$\begin{aligned} \theta'_G(g) &= \theta_H(g) - \sum_{\chi(1_G)=1} n(G,\chi)\chi(g) \\ &= \sum_{\psi\in\widehat{H}} \left(n(H,\psi) - \sum_{\chi(1_G)=1} n(G,\chi)\langle\, \chi\,,\, \mathrm{Ind}_H^G\psi\,\rangle \right)\psi(g). \end{aligned} \tag{16}$$

Suppose we could prove that all the coefficients on the right-hand side of (16) are nonnegative. We would then have for any $g \in G$,

$$|\theta'_G(g)| \leq \sum_{\psi\in\widehat{H}} \left(n(H,\psi) - \sum_{\chi(1_G)=1} n(G,\chi)\langle\, \chi\,,\, \mathrm{Ind}_H^G\psi\,\rangle \right).$$

The expression on the right can be written as

$$\mathrm{ord}_{s=s_0}\left(\frac{\prod_{\psi\in\widehat{H}} L(s,\psi,E/E^H)}{\prod_{\psi\in\widehat{H}}\prod_{\chi(1_G)=1} L(s,\chi,E/F)^{<\chi,\mathrm{Ind}_H^G\psi>}} \right).$$

The numerator is of course

$$\mathrm{ord}_{s=s_0}\zeta_E(s).$$

Since

$$\sum_{\psi\in\widehat{H}} \langle\, \chi\,,\, \mathrm{Ind}_H^G\psi\,\rangle = \sum_{\psi\in\widehat{H}} \langle\, \chi|_H\,,\, \psi\,\rangle = \langle\, \chi|_H\,,\, \mathrm{reg}_H\,\rangle = 1,$$

it follows that the denominator is

$$\prod_{\chi(1_G)=1} L(s,\chi,E/F). \tag{17}$$

Since

$$\sum_{\chi(1_{\mathrm{G}})=1} \chi = \mathrm{Ind}_{[G,G]}^{G} I_{[G,G]},$$

it follows that (17) is equal to $\zeta_{E^{ab}}(s)$.

Ram Murty and Raghuram also introduced the notion of the level of an irreducible character χ of a solvable group G. Define, as usual, $G^{(i)} = [G^{(i-1)}, G^{(i-1)}]$ and $G^{(0)} = G$. This is the derived series and, as we are assuming G is solvable, $G^{(i)} = \langle\, 1_{\mathrm{G}} \,\rangle$ for i sufficiently large. The *level* of χ is then the least i for which $\chi|_{G^{(i)}} = \chi(1_{\mathrm{G}}) I_{G^{(i)}}$.

Armed with this concept, Lansky and Wilson [LW] proved the following variation of (15): For each i let E^i denote the subfield of E fixed by $G^{(i)}$. Thus E^1 is E^{ab}. Then

$$\sum_{\chi \text{ of level } i} n(G,\chi)^2 \leq \left(\mathrm{ord}_{s=s_0} \zeta_{E^i}(s)/\zeta_{E^{i-1}}(s)\right)^2.$$

It is interesting to note that the case $i = 1$ of this result is *not* the same as (15). There seems to be some scope for further work here.

11. Conclusion

We conclude with some brief thoughts that, in the spirit of the previous section, may motivate new lines of investigation in the general realm of Brauer Induction via Heilbronn characters, with concomitant applications in number theory.

Every irreducible character χ of a finite group G can be written (nonuniquely) as a difference $\chi = \phi_1 - \phi_2$, where both ϕ_1, ϕ_2 are nonnegative integer sums of monomial characters. For fixed χ if this is done in such a way as to minimize, say, the degree of $\phi_1 + \phi_2$, what can generically be said about this (minimal) degree or the degrees of each ϕ_i? Are there tight "order of magnitude" or "asymptotic" upper bounds in terms of functions of $|\, G \,|$ (perhaps for restricted classes of G such as solvable groups, or specific families of quasisimple groups)? What then are some number-theoretic consequences if "effective" bounds can be established?

As mentioned at the outset, the virtual character θ_G was introduced by Heilbronn as a "bookkeeping device" for relating zeros and poles of various L-series; and although character theory is essential throughout the development, the degree $\theta_G(1_{\mathrm{G}})$ of the Heilbronn character has played a paramount role, especially in number-theoretic applications. In the spirit of the McKay–Thompson "Hauptmodule" series for the Monster (and other) simple groups [CN79], are there interpretations or uses of the values of θ_G at elements other than the identity for Galois Heilbronn characters? For example, what about $\theta_G(t)$ for t an involution in a group of even order? Likewise, are there representation-theoretic constructions or interpretations for (Galois) Heilbronn characters? For example, can Heilbronn characters be realized as Euler characteristics for Galois groups acting on some cohomology groups (similar to the Deligne–Lusztig construction [DL76])?

Using the compatible behavior of Heilbronn characters under inflation and restriction, one may associate a "canonical Heilbronn character" to $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, parameterized by $s_0 \in \mathbb{C}$. Can this family of virtual characters be studied using profinite techniques?

By work of Langlands and Tunnell [Tu81], Artin's Conjecture is known to be true for $L(s, \phi, E_1/F_1)$, where ϕ is a two-dimensional representation of a solvable Galois group $\mathrm{Gal}(E_1/F_1)$. By the breakthrough work of Khare and Wintenberger

([KW, Section 10.2]) we also know it for any *odd* two-dimensional representation of $\mathrm{Gal}(E_1/F_1)$. What (new) verifications of Artin's Conjecture—or new families of minimal counterexamples—accrue from generalizing Definition 5.1 to additionally impose that $\langle \theta, \mathrm{Ind}_H^G(\phi) \rangle \geq 0$ for all degree 2 characters ϕ of subgroups H of G which are odd, or which have $H/\ker\phi$ solvable (as well as for ϕ any degree 1 character)? In particular, what can one say about the (more restricted) family of "minimal unfaithful Heilbronn–Langlands characters" in this setting?

In conclusion, this survey shows how a strikingly simple yet elegant construction can lead to profound insights, new results and perspectives, and entirely new areas to explore.

About the authors

Richard Foote received his B.Sc. in 1972 from the University of Toronto, where he studied under Professor Heilbronn, and his Ph.D. from the University of Cambridge under the supervision of John G. Thompson. He is currently a professor at the University of Vermont. His research is in the area of finite group theory and its applications, and he is also known for his *Abstract Algebra* book, coauthored with David Dummit.

Hy Ginsberg earned his Ph.D. in 2010 at the University of Vermont under the supervision of Richard Foote. He is currently an assistant professor at Worcester State University, where he continues to pursue his research interests in finite group theory and character theory.

Kumar Murty received his Ph.D. from Harvard University under the supervision of John Tate. He is currently a professor and the chair at the University of Toronto. His research interests include number theory and arithmetic geometry and their applications to information technology.

References

[Ar27] Emil Artin, *Beweis des allgemeinen Reziprozitätsgesetzes* (German), Abh. Math. Sem. Univ. Hamburg **5** (1927), no. 1, 353–363, DOI 10.1007/BF02952531. MR3069486

[As07] Michael Aschbacher, Radha Kessar, and Bob Oliver, *Fusion systems in algebra and topology*, London Mathematical Society Lecture Note Series, vol. 391, Cambridge University Press, Cambridge, 2011. MR2848834 (2012m:20015)

[Baker66] A. Baker, *Linear forms in the logarithms of algebraic numbers. IV*, Mathematika **15** (1968), 204–216. MR0258756 (41 #3402)

[Baker71] A. Baker, *Imaginary quadratic fields with class number* 2, Ann. of Math. (2) **94** (1971), 139–152. MR0299583 (45 #8631)

[Ba86] A. S. Bang, *Taltheoretiske Undersølgelser*, Tidskrift f. Math. **5** (1886), 70–80 and 130–137.

[BCGLO07] C. Broto, N. Castellana, J. Grodal, R. Levi, and B. Oliver, *Extensions of p-local finite groups*, Trans. Amer. Math. Soc. **359** (2007), no. 8, 3791–3858 (electronic), DOI 10.1090/S0002-9947-07-04225-0. MR2302515 (2008i:55013)

[BLO03] Carles Broto, Ran Levi, and Bob Oliver, *The homotopy theory of fusion systems*, J. Amer. Math. Soc. **16** (2003), no. 4, 779–856, DOI 10.1090/S0894-0347-03-00434-X. MR1992826 (2004k:55016)

[BLO07] Carles Broto, Ran Levi, and Bob Oliver, *Discrete models for the p-local homotopy theory of compact Lie groups and p-compact groups*, Geom. Topol. **11** (2007), 315–427, DOI 10.2140/gt.2007.11.315. MR2302494 (2008e:55021)

[Cl87] L. Clozel, *Base change for* $\mathrm{GL}(n)$, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Berkeley, Calif., 1986), Amer. Math. Soc., Providence, RI, 1987, pp. 791–797. MR934282 (89i:22034)

[CN79] J. H. Conway and S. P. Norton, *Monstrous moonshine*, Bull. London Math. Soc. **11** (1979), no. 3, 308–339, DOI 10.1112/blms/11.3.308. MR554399 (81j:20028)

[Da88] Everett C. Dade, *Accessible characters are monomial*, J. Algebra **117** (1988), no. 1, 256–266, DOI 10.1016/0021-8693(88)90253-0. MR955603 (89h:20012)

[DL76] P. Deligne and G. Lusztig, *Representations of reductive groups over finite fields*, Ann. of Math. (2) **103** (1976), no. 1, 103–161. MR0393266 (52 #14076)

[De] M. Deuring, *Die zetafunktion einer algebraischen Kurve vom Geschlechte Eins*. I, II, IIII, IV. Nachr. Akad. Wiss., Göttingen, (1953), 85–94; (1955), 13–42; (1956), 37–76; (1957), 55–80.

[Fe67] Walter Feit, *Characters of finite groups*, W. A. Benjamin, Inc., New York-Amsterdam, 1967. MR0219636 (36 #2715)

[FI89] Pamela Ferguson and I. M. Isaacs, *Induced characters which are multiples of irreducibles*, J. Algebra **124** (1989), no. 1, 149–157, DOI 10.1016/0021-8693(89)90156-7. MR1005700 (90f:20008)

[FS07] Ramón J. Flores and Jérôme Scherer, *Cellularization of classifying spaces and fusion properties of finite groups*, J. Lond. Math. Soc. (2) **76** (2007), no. 1, 41–56, DOI 10.1112/jlms/jdm031. MR2351607 (2009a:55005)

[FF09] Ramón J. Flores and Richard M. Foote, *Strongly closed subgroups of finite groups*, Adv. Math. **222** (2009), no. 2, 453–484, DOI 10.1016/j.aim.2009.05.005. MR2538017 (2011b:20075)

[FF11] Ramón J. Flores and Richard M. Foote, *The cellular structure of the classifying spaces of finite groups*, Israel J. Math. **184** (2011), 129–156, DOI 10.1007/s11856-011-0062-0. MR2823972 (2012i:55017)

[Fo90] R. Foote, *Non-monomial characters and Artin's Conjecture*, Trans. Amer. Math. Soc., 321 (1990), 261–272.

[Fo97a] Richard Foote, *Sylow* 2*-subgroups of Galois groups arising as minimal counterexamples to Artin's conjecture*, Comm. Algebra **25** (1997), no. 2, 607–616, DOI 10.1080/00927879708825877. MR1428801 (97j:11056)

[Fo97b] Richard Foote, *A characterization of finite groups containing a strongly closed* 2*-subgroup*, Comm. Algebra **25** (1997), no. 2, 593–606, DOI 10.1080/00927879708825876. MR1428800 (97k:20042)

[FM89] Richard Foote and V. Kumar Murty, *Zeros and poles of Artin L-series*, Math. Proc. Cambridge Philos. Soc. **105** (1989), no. 1, 5–11, DOI 10.1017/S0305004100001316. MR966135 (89k:11109)

[FW90] Richard Foote and David Wales, *Zeros of order* 2 *of Dedekind zeta functions and Artin's conjecture*, J. Algebra **131** (1990), no. 1, 226–257, DOI 10.1016/0021-8693(90)90173-L. MR1055006 (91b:11132)

[Gi10] H. Ginsberg, *Minimal Heilbronn Characters of Finite Groups*, Doctoral Dissertation, University of Vermont, 2010.

[Gi11] Hy Ginsberg, *Unfaithful minimal Heilbronn characters of finite groups*, J. Algebra **331** (2011), 466–481, DOI 10.1016/j.jalgebra.2010.09.043. MR2774670 (2012b:20012)

[Gi13a] Hy Ginsberg, *Unfaithful minimal Heilbronn characters of* $L_2(q)$, Proc. Edinb. Math. Soc. (2) **56** (2013), no. 1, 57–69, DOI 10.1017/S0013091512000168. MR3021405

[Gi13b] Hy Ginsberg, *Groups with strongly p-embedded subgroups and cyclic Sylow p-subgroups*, J. Group Theory **17** (2014), no. 1, 175–190, DOI 10.1515/jgt-2013-0039. MR3176657

[Gl66] George Glauberman, *Central elements in core-free groups*, J. Algebra **4** (1966), 403–420. MR0202822 (34 #2681)

[Goldfeld76] Dorian M. Goldfeld, *The class number of quadratic fields and the conjectures of Birch and Swinnerton-Dyer*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **3** (1976), no. 4, 624–663. MR0450233 (56 #8529)

[Go74] David M. Goldschmidt, 2*-fusion in finite groups*, Ann. of Math. (2) **99** (1974), 70–117. MR0335627 (49 #407)

[Go75] David M. Goldschmidt, *Strongly closed* 2*-subgroups of finite groups*, Ann. of Math. (2) **102** (1975), no. 3, 475–489. MR0393223 (52 #14033)

[Gs75] Larry Joel Goldstein, *Relative imaginary quadratic fields of class number* 1 *or* 2, Trans. Amer. Math. Soc. **165** (1972), 353–364. MR0291124 (45 #218)

[GL83] Daniel Gorenstein and Richard Lyons, *The local structure of finite groups of characteristic* 2 *type*, Mem. Amer. Math. Soc. **42** (1983), no. 276, vii+731, DOI 10.1090/memo/0276. MR690900 (84g:20025)

[GLS94] Daniel Gorenstein, Richard Lyons, and Ronald Solomon, *The classification of the finite simple groups*, Mathematical Surveys and Monographs, vol. 40, American Mathematical Society, Providence, RI, 1994. MR1303592 (95m:20014)

[GZ86] Benedict H. Gross and Don B. Zagier, *Heegner points and derivatives of L-series*, Invent. Math. **84** (1986), no. 2, 225–320, DOI 10.1007/BF01388809. MR833192 (87j:11057)

[Hk17] E. Hecke, *Über die L-funktionen und den Dirichletschen Primzahlsatz für einen beliebigen Zahlkörper*. Gesells. der Wissens. zu Göttingen, Nachrichten (1917), 299–318.

[He34] H. Heilbronn, *On the class number in imaginary quadratic fields*, Quart. J. Math. 5 (1934), 150-160.

[He67] H. Heilbronn, *Zeta-functions and L-functions*, Algebraic Number Theory (Proc. Instructional Conf., Brighton, 1965), Thompson, Washington, D.C., 1967, pp. 204–230. MR0218327 (36 #1414)

[He72] H. Heilbronn, *On real simple zeros of Dedekind ζ-functions*, Proceedings of the Number Theory Conference (Univ. Colorado, Boulder, Colo., 1972), Univ. Colorado, Boulder, Colo., 1972, pp. 108–110. MR0389785 (52 #10616)

[He73] H. Heilbronn, *On real zeros of Dedekind ζ-functions*, Canad. J. Math. **25** (1973), 870–873. MR0327719 (48 #6061)

[Is94] I. Martin Isaacs, *Character theory of finite groups*, Dover Publications, Inc., New York, 1994. Corrected reprint of the 1976 original [Academic Press, New York; MR0460423 (57 #417)]. MR1280461

[KW] Chandrashekhar Khare and Jean-Pierre Wintenberger, *Serre's modularity conjecture. I*, Invent. Math. **178** (2009), no. 3, 485–504, DOI 10.1007/s00222-009-0205-7. MR2551763 (2010k:11087)

[KL90] Peter Kleidman and Martin Liebeck, *The subgroup structure of the finite classical groups*, London Mathematical Society Lecture Note Series, vol. 129, Cambridge University Press, Cambridge, 1990. MR1057341 (91g:20001)

[Ko09] Joachim König, *Solvability of generalized monomial groups*, J. Group Theory **13** (2010), no. 2, 207–220, DOI 10.1515/JGT.2009.047. MR2607576 (2011i:20011)

[LW] Joshua M. Lansky and Kevin M. Wilson, *A variation on the solvable case of the Dedekind conjecture*, J. Ramanujan Math. Soc. **20** (2005), no. 2, 81–90. MR2169089 (2006g:11226)

[Li06] Markus Linckelmann, *Simple fusion systems and the Solomon 2-local groups*, J. Algebra **296** (2006), no. 2, 385–401, DOI 10.1016/j.jalgebra.2005.09.024. MR2201048 (2006i:20024)

[RM1] Maruti Ram Murty, *On Artin L-functions*, Class field theory—its centenary and prospect (Tokyo, 1998), Adv. Stud. Pure Math., vol. 30, Math. Soc. Japan, Tokyo, 2001, pp. 13–29. MR1846449 (2003c:11147)

[RR] M. Ram Murty and A. Raghuram, *Some variations on the Dedekind conjecture*, J. Ramanujan Math. Soc. **15** (2000), no. 4, 225–245. MR1801220 (2001m:11201)

[RK1] M. Ram Murty and V. Kumar Murty, *Base change and the Birch-Swinnerton-Dyer conjecture*, A tribute to Emil Grosswald: number theory and related analysis, Contemp. Math., vol. 143, Amer. Math. Soc., Providence, RI, 1993, pp. 481–494, DOI 10.1090/conm/143/01014. MR1210535 (94a:11095)

[KM1] V. Kumar Murty, *Stark zeros in certain towers of fields*, Math. Res. Lett. **6** (1999), no. 5-6, 511–519, DOI 10.4310/MRL.1999.v6.n5.a4. MR1739210 (2001b:11106)

[KM2] V. Kumar Murty, *Class numbers of CM-fields with solvable normal closure*, Compositio Math. **127** (2001), no. 3, 273–287, DOI 10.1023/A:1017589432526. MR1845038 (2003a:11147)

[Rh93] Sandra L. Rhoades, *A generalization of the Aramata-Brauer theorem*, Proc. Amer. Math. Soc. **119** (1993), no. 2, 357–364, DOI 10.2307/2159915. MR1166360 (93k:20015)

[Rh93a] S. Rhoades, *A Character-Theoretic Approach to Artin's Conjecture*, Doctoral Dissertation, University of Vermont, 1993.

[Ro97] Moshe Roitman, *On Zsigmondy primes*, Proc. Amer. Math. Soc. **125** (1997), no. 7, 1913–1919, DOI 10.1090/S0002-9939-97-03981-6. MR1402885 (97i:11005)

[St67] H. M. Stark, *A complete determination of the complex quadratic fields of class-number one*, Michigan Math. J. **14** (1967), 1–27. MR0222050 (36 #5102)

[St72] H. M. Stark, *A transcendence theorem for class-number problems. II*, Ann. of Math. (2) **96** (1972), 174–209. MR0309878 (46 #8983)

[St74] H. M. Stark, *Some effective cases of the Brauer-Siegel theorem*, Invent. Math. **23** (1974), 135–152. MR0342472 (49 #7218)

[SW75] Josef Stoer and Christoph Witzgall, *Convexity and optimization in finite dimensions. I*, Die Grundlehren der mathematischen Wissenschaften, Band 163, Springer-Verlag, New York-Berlin, 1970. MR0286498 (44 #3707)

[Su75] Judith Evelyn Steere Sunley, *ON THE CLASS NUMBERS OF TOTALLY IMAGINARY QUADRATIC EXTENSIONS OF TOTALLYREAL FIELDS*, ProQuest LLC, Ann Arbor, MI, 1971. Thesis (Ph.D.)–University of Maryland, College Park. MR2621002

[Tu81] Jerrold Tunnell, *Artin's conjecture for representations of octahedral type*, Bull. Amer. Math. Soc. (N.S.) **5** (1981), no. 2, 173–175, DOI 10.1090/S0273-0979-1981-14936-3. MR621884 (82j:12015)

[vdW78] Robert W. van der Waall, *On the structure of the minimal non-M-groups*, Nederl. Akad. Wetensch. Indag. Math. **40** (1978), no. 3, 398–405. MR507832 (80c:20025)

Department of Mathematics and Statistics, University of Vermont, 16 Colchester Avenue, Burlington, Vermont 05405

E-mail address: foote@math.uvm.edu

Department of Mathematics, Worcester State University, 486 Chandler Street, Worcester Massachusetts 01602

E-mail address: hginsberg@worcester.edu

Department of Mathematics, University of Toronto, 40 St. George Street, Room 6290, Toronto, Ontario M5S 2E4, Canada

E-mail address: murty@math.utoronto.ca