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Taming the unknown. A history of algebra from antiquity to the early twentieth century, by Victor J. Katz and Karen Hunger Parshall, Princeton University Press, Princeton and Oxford, 2014, xvi+485 pp., ISBN 978-0-691-14905-9, US $49.50

1. Algebra now

Algebra has been a part of mathematics since ancient times, but only in the 20th century did it come to play a crucial role in other areas of mathematics. Algebraic number theory, algebraic geometry, and algebraic topology now cast a big shadow on their parent disciplines of number theory, geometry, and topology. And algebraic topology gave birth to category theory, the algebraic view of mathematics that is now a dominant way of thinking in almost all areas. Even analysis, in some ways the antithesis of algebra, has been affected. As long ago as 1882, algebra captured some important territory from analysis when Dedekind and Weber absorbed the Abel and Riemann–Roch theorems into their theory of algebraic function fields, now part of the foundations of algebraic geometry.

Not all mathematicians are happy with these developments. In a speech in 2000, entitled *Mathematics in the 20th Century*, Michael Atiyah said:

> Algebra is the offer made by the devil to the mathematician. The devil says "I will give you this powerful machine, and it will answer any question you like. All you need to do is give me your soul; give up geometry and you will have this marvellous machine."
>
> Atiyah (2001), p. 659.

But Atiyah is giving his view of the situation at the research level. At high school level, algebra is by no means dominant. There have been calls for its total abolition. Algebraically minded mathematicians also lament the state of high school algebra. The algebraic geometer Igor Shafarevich wrote:

> Algebra courses in schools comprise a strange mixture of useful rules, logical judgments, and exercises in using aids such as tables of logarithms and pocket calculators. Such a course is closer in spirit to the brand of mathematics developed in ancient Egypt and Babylon than to the line of development that appeared in ancient Greece and then continued from the Renaissance in western Europe. Nevertheless, algebra is just as fundamental, just as deep, and just as beautiful as geometry.
>
> Preface to Shafarevich (2003).

We see here not only two views of algebra, but two views of geometry. How these views developed, and how they intertwined, is an interesting story.

In Atiyah's view, the difference between geometry and algebra is the difference between *space* and *time*. Geometry is powerful because humans have a highly developed visual system, enabling us to perceive a complicated spatial situation at a glance; algebra is powerful because it enables us to process a sequence of

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operations occurring in time. In Shafarevich's view both are powerful because they have a *logical* structure, first exemplified by Euclid's *Elements*.

Whichever view one takes, it seems important to understand the historical development of both algebra and geometry.

2. Algebra and geometry

For most of its history, algebra had the goal of solving equations—finding an "unknown" quantity in terms of known quantities. Simple examples occur in Egyptian mathematics from around 1850 BCE, mostly equivalent to linear equations but on one occasion a quadratic equation with integer roots, solved partly by trial and error. In Mesopotamia, around the same time, there was a more sophisticated mathematical culture that studied problems relating length to area. This of course leads to quadratic equations, and the Mesopotamians found a general algorithm for their solution, essentially equivalent to our quadratic formula.

So, the ancients understood "algebra" up to the level of quadratic equations, but was it algebra as we know it? Not exactly, because *mathematics* as we know it (theorems and proofs) began much later. The first systematic account of theorems and proofs that we know of is Euclid's *Elements*, from around 300 BCE. It contains well-developed treatments of geometry and number theory, but little, if any, algebra. Yet when algebra began to develop, 1000 years later, the *Elements* provided the model for its first proofs.

To understand how geometry could be a model for proofs in algebra, consider two of Euclid's theorems that seem to have algebraic content: Propositions 1 and 4 of the *Elements*, Book II. These propositions involve the "rectangle" of two line segments a and b, which is literally the rectangle with adjacent sides a and b, but which *behaves*, in some ways, like the product of (numerical) lengths a and b. (The Greeks avoided identifying line segments with numbers because they knew irrational line segments, which were not their idea of numbers.) When the "rectangle" is understood as a product, Proposition 1 is essentially the distributive law,

$$a(b+c+\cdots) = ab + ac + \cdots ,$$

in the geometric form shown in Figure 1.

Proposition 4 is essentially the formula

$$(a+b)^2 = a^2 + 2ab + b^2,$$

in the geometric form shown in Figure 2.

Debate once raged over the question whether this was really algebra; see Unguru (1975) versus van der Waerden (1975) and Weil (1978). Today it seems to be agreed that the Greeks were not really doing algebra—because they did not grant lengths all the properties of numbers—but that geometry played a supporting role in the development of algebra and eventually merged with it.

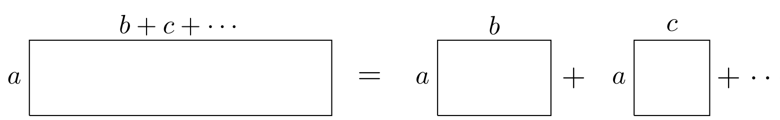

Figure 1. Geometric version of the distributive law

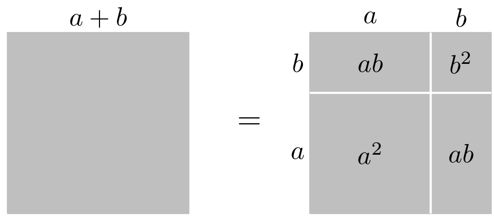

FIGURE 2. Geometric version of $(a+b)^2 = a^2 + 2ab + b^2$

Algebra became a cohesive discipline in the Islamic world between 700 and 1000 CE, with the goals of solving equations and explaining algorithms for their solution. The name "algebra" comes from the Arabic word *al-jabr*, meaning something like "restoring". in the title of an influential book written around 825 by al-Khwarizmi (whose name also gave us the word "algorithm"). Equations were still written in words, but they were *manipulated* according to explicit rules—such as the distributive law—and the rules were justified by appeal to Euclidean propositions like those above. In particular, al-Khwarizmi solved quadratic equations by "completing the square" geometrically.

The general methods of solution, being based on plane geometry, could not succeed on equations of degree more than two. However, algebra advanced in other respects. Around 1000, al-Karaji stated the binomial theorem, and a century later its connection with "Pascal's triangle" was explained in *The Shining Book of Calculation* by al-Samaw'al. (Chinese mathematicians made a similar discovery at about the same time and used it to solve polynomial equations numerically.) The *Shining Book* also extended some of al-Karaji's ideas about polynomials to a full-scale arithmetic, allowing addition, subtraction, multiplication, and division of polynomials.

Meanwhile, the cubic equation beckoned. Around 1100, al-Khayyami (known to most westerners as the poet Omar Khayyam) was able to solve cubic equations geometrically—describing their roots in terms of the intersections of two suitably chosen conic sections—but was disappointed not to find an algebraic solution.

Such a solution was first found in Italy, in the early 16th century. It became known as the *Cardano formula* after its publication by Cardano (1545), though it had been discovered earlier by del Ferro and Tartaglia. The formula states that the solution of $x^3 = px + q$ is

$$x = \sqrt[3]{\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^3}} + \sqrt[3]{\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^3}}.$$

Like the familiar quadratic formula, Cardano's formula is a *solution by radicals*. That is, the solution is obtained from the coefficients of the equation by the operations $+, -, \times, \div$ and the "radicals" $\sqrt{\ }$ and $\sqrt[3]{\ }$. Shortly after Cardano learned of the formula, his assistant Ferrari found a solution by radicals of the general equation of fourth degree (the *quartic*).

These dramatic advances in algebra after centuries of stagnation set off an explosion of algebraic research in Italy, France, and England. Its main goal—solution by radicals of the *quintic* (fifth degree) equation—was not reached. But the *nature* of algebra became clearer, and it was finally able to emancipate itself from geometry.

Around 1600, Viète in France and Harriot in England developed an algebraic symbolism quite close to modern notation, so that computation with unknowns became as easy as computation with numbers. At this point, algebra became able to help geometry more than geometry could help algebra, and the result was *La Géométrie* of Descartes (1637), the ancestor of today's algebraic geometry.

3. Algebra and number theory

Until around 1800, algebra was formally the same as the arithmetic of numbers: mostly about the operations $+, -, \times, \div$, and it was subject to the same laws. The radical operations $\sqrt{\ }, \sqrt[3]{\ }, \ldots$ were occasionally invoked to solve polynomial equations, but they too were viewed as operations on numbers. For all intents and purposes, algebra was just "universal arithmetic", as it was called by Newton (1728) and Lagrange (1795). Of course we now know that algebra was about to be transformed by Galois and his analysis of solution by radicals. This part of the story will be continued in the next section. But another change was in the offing, due to the transformation of arithmetic itself.

Since the time of Euclid it has been noticed that the arithmetic of integers presents problems quite different from the arithmetic of rational numbers. The problems with integers are due to the fact that they do not generally divide each other, so the concepts of division with remainder, common divisors, and primes arise. The most basic problem, finding the greatest common divisor of integers a and b, $\gcd(a, b)$, was solved by Euclid in the *Elements*, Book VII, Proposition 1.

Euclid's solution introduces the *Euclidean algorithm*, a process also discovered (apparently independently) by Indian and Chinese mathematicians. As Euclid presented it, the algorithm is to "continually subtract the lesser number from the greater" (and save the difference), but the equivalent process of performing division with remainder turns out to be more suitable for generalization.

The Euclidean algorithm produces ever smaller integers and hence terminates with the gcd, in the form $\gcd(a, b) = ma + nb$ for some integers m and n. This implies the property discovered by Euclid: if a prime p divides cd, then p divides c or p divides d. The latter property implies *unique prime factorization*, the property we single out today as the fundamental theorem of arithmetic. It was first identified as such by Gauss (1801), but was used long before that, sometimes unconsciously. Other applications of the Euclidean algorithm are the solution of linear Diophantine equations by Brahmagupta (628) and the "Chinese remainder problem" solved by Sun Zi in the third century CE.

A new twist in the story of unique prime factorization occurred in Euler (1770), a supposedly elementary book on algebra which broke new ground in number theory. In seeking positive integer solutions of the equation $x^3 + y^3 = z^3$, Euler made use of numbers of the form $a + b\sqrt{-3}$, where $a, b \in \mathbb{Z}$. He saw that they behave rather like integers and effectively assumed that they have unique prime factorization (which is not quite right, though the error is repairable). Without justifying this bold assumption, Euler showed that it yields the conclusion that $x^3 + y^3 = z^3$ has no positive integer solutions—thus proving a case of Fermat's last theorem. Similarly using integers of the form $a + b\sqrt{-2}$ and assuming their unique prime factorization (this time, quite correctly), he was able to prove that the only positive integer solution of $y^3 = x^2 + 2$ is $x = 5, y = 3$, proving another claim of Fermat.

At the time, Euler's belief that the certain numbers could be viewed as "integers" with unique prime factorization was unfounded. It was eventually justified by a theory of *algebraic integers* developed by Dirichlet, Eisenstein, Kummer, and Dedekind between the 1830s and the 1870s. However, this theory did not develop until it was necesary to fix a more serious problem: *failure* of unique prime factorization. Like the failure of solution by radicals, failure of unique prime factorization led to a reconstruction of algebra on new, abstract, foundations.

4. Abstraction

Now we come to the 19th century, when algebra begins to look "modern"—with groups, rings, fields, and vector spaces. The first modern algebraist was Galois, who introduced the concepts of group and field to explain the structure of polynomial equations and solution by radicals. However, his most important works, such as Galois (1831), were not published until 1846, long after his death, and they were not well understood even then. The first course on Galois theory was given by Dedekind in 1856 and the first book on group theory, an exposition of Galois's ideas, was Jordan (1870).

Galois also noticed interesting examples of fields, the *finite* fields, previously only implicit in the arithmetic of congruence mod p. But, like the theory of groups, the theory of fields did not catch on until more applications came to light, mainly in number theory. As mentioned in the previous section, the need for a theory of rings and fields became urgent when unique prime factorization was found to fail for certain algebraic integers. The failure was first noticed in the 1840s by Kummer, in the rather complicated setting of cyclotomic integers. These are the "integers" one meets in the linear factors of $a^n + b^n$ when a and b are ordinary integers.

Dedekind (1877) gave the simpler example of $\mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} : a, b \in \mathbb{Z}\}$, where 6 has two "prime" factorizations, $6 = 2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5})$. When Kummer found his examples, he believed there were "ideal" prime factors dividing the actual factors, which would restore unique prime factorization in the new world of ideal numbers. In the example above, one would hope in particular for an ideal common divisor of 2 and $1 + \sqrt{-5}$. Harking back to the situation in $\mathbb{Z}$, where the set $\{ma + nb : m, n \in \mathbb{Z}\}$ consists of precisely the multiples of $\gcd(a, b)$, one might guess that $\gcd(2, 1 + \sqrt{-5})$ in $\mathbb{Z}[\sqrt{-5}]$ is realized as a set of combinations $2m + (1 + \sqrt{-5})n$. This idea, suitably generalized, was the Dedekind (1871) concept of an *ideal* in the ring of integers of an algebraic number field—later generalized to the concept of an ideal in arbitrary ring.

The concept of an algebraic number field $\mathbb{Q}(\alpha)$, which Dedekind took as the setting for his theory of algebraic integers and ideals, was modelled on the example $\mathbb{Z}/p\mathbb{Z}$ of integers mod p. Instead of the ring $\mathbb{Z}$ one begins with the ring $\mathbb{Q}[x]$ of polynomials with rational coefficients, and instead of congruence mod p one has congruence mod $p(x)$, where $p(x)$ is the minimal polynomial for the algebraic number α. It easily follows that $\mathbb{Q}(\alpha)$ is a *vector space* of dimension n over $\mathbb{Q}$, where n is the degree of $p(x)$. This is just one of the places where the concept of vector space began to emerge towards the end of the 19th century.

Dedekind and Weber (1882) took this idea a giant step further, with remarkable applications to algebraic geometry and even analysis (as mentioned at the beginning of this review). Following Dedekind's approach to algebraic number fields, they began with the field $\mathbb{Q}(x)$ of rational functions of x and took an extension field

$\mathbb{Q}(x, f)$ of finite degree n. Then $\mathbb{Q}(x, f)$ is an *algebraic function field* of degree n, from which they were able to define a *Riemann surface*, and prove the classical theorems of Abel and Riemann–Roch, originally proved by analytic methods.

"Universal arithmetic" had come a long way since the time of Newton!

5. Comments on the book

Most of the story sketched above is covered in detail in *Taming the unknown*, with roughly the same allocation of space to the ancient, classical, and modern parts of algebra. The authors are very thorough and avail themselves of the most recent scholarship. Indeed, I learned many things (such as the work of al-Samaw'al) for the first time when reading this book. Readers of the *Bulletin* might wish that more space had been allocated to the modern period, but the history of modern algebra is a huge topic, and there are three excellent chapters of the book (11, 12, and 13) devoted to the 19th-century roots of the modern period.

I have some small quibbles with the coverage of the fundamental theorem of algebra at the end of Chapter 10: the factorization of $z^n - a^n$ is attributed to Euler in 1748 rather than to Cotes in 1714, and the gaps in Gauss's first proof in 1799 are glossed over. The heart of the difficulty with the fundamental theorem—that it depends on general properties of continuous functions, and hence does not entirely belong to algebra—is not made very clear.

The authors explain in their Prelude (p. 3) that they are writing for an audience ranging from high school students and their teachers to college students and their professors, since even the latter should "know the roots of the algebra they study". I wholeheartedly agree, and would add that knowing the roots of algebra is probably the best antidote to Atiyah's fear that algebra is just a machine, and the best way to realize Shafarevich's vision of an algebra as fundamental, deep, and beautiful as geometry.

References

Atiyah, M. (2001). Mathematics in the 20th century. *Amer. Math. Monthly 108*(7), 654–666.

Brahmagupta (628). *Brâhma-sphuṭa-siddhânta*. Partial English translation in (1817).

Cardano, G. (1545). *Ars magna*. 1968 translation *The great art or the rules of algebra* by T. Richard Witmer, with a foreword by Oystein Ore. The M.I.T. Press, Cambridge, MA-London.

Dedekind, R. (1871). Supplement X. In Dirichlet's *Vorlesungen über Zahlentheorie*, 2nd ed., Vieweg 1871.

Dedekind, R. (1877). *Theory of Algebraic Integers*. Cambridge University Press, Cambridge. Translated from the 1877 French original and with an introduction by John Stillwell.

Dedekind, R. and H. Weber (1882). *Theory of Algebraic Functions of One Variable*, Volume 39 of *History of Mathematics*. American Mathematical Society, Providence, RI; London Mathematical Society, London, 2012. Translated from the 1882 German original and with an introduction, bibliography and index by John Stillwell.

Descartes, R. (1637). *The Geometry of René Descartes. (With a facsimile of the first edition, 1637.)*. Dover Publications Inc., New York, NY. Translated by David Eugene Smith and Marcia L. Latham, 1954.

Euler, L. (1770). *Elements of Algebra*. Translated from the German by John Hewlett. Reprint of the 1840 edition, with an introduction by C. Truesdell, Springer-Verlag, New York, 1984.

Galois, E. (1831). Mémoire sur les conditions de résolubilité des équations par radicaux. English translation in Neumann (2011).

Gauss, C. F. (1801). *Disquisitiones arithmeticae.* Translated and with a preface by Arthur A. Clarke. Revised by William C. Waterhouse, Cornelius Greither and A. W. Grootendorst and with a preface by Waterhouse, Springer-Verlag, New York, 1986.

Jordan, C. (1870). *Traité des substitutions et des équations algébriques.* Sceaux: Éditions Jacques Gabay. 1989 Reprint of the 1870 original.

Lagrange, J.-L. (1795). *Leçons élémentaires sur les mathematiques.* English translation by T. J. McCormack, *Lectures on Elementary Mathematics*, Dover 2008.

Neumann, P. M. (2011). *The mathematical writings of Évariste Galois.* Heritage of European Mathematics. European Mathematical Society (EMS), Zürich.

Newton, I. (1728). *Universal Arithmetick.* Senex, Innys, Osborne and Longman, London. Translation of Newton's *Arithmetica universalis* (1707) by Joseph Raphson.

Shafarevich, I. R. (2003). *Discourses on Algebra.* Universitext. Springer-Verlag, Berlin. Translated from the Russian by William B. Everett.

Unguru, S. (1975). On the need to rewrite the history of Greek mathematics. *Arch. History Exact Sci. 15*(1), 67–114.

van der Waerden, B. L. (1975). Defence of a "shocking" point of view. *Arch. History Exact Sci. 15*(3), 199–210.

Weil, A. (1978). Who betrayed Euclid? Extract from a letter to the editor. *Arch. History Exact Sci. 19*(2), 91–93.

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