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The joys of Haar measure, by Joe Diestel and Angela Spalsbury, Graduate Studies in Mathematics, Vol. 150, American Mathematical Society, Providence, RI, xiv+320 pp., ISBN 978-1-4704-0935-7, US$65

The origins of Haar measure are deep in the history of both group theory and real analysis. When groups of matrices were first studied, it was clear that there was a density function that could be used to make the Riemann integral invariant under the action of the group, given a suitable parametric representation of the group.

At the same time, so to speak, the nature of the sizes of things was being considered by many mathematicians. The origins of these questions are ancient, but in the modern era I think we see this first in real depth in the work that Georg Cantor did in introducing cardinality, initially as a result of his interests in the behavior of trigonometric series and the structure of sets. In some sense, the next step toward Haar measure was Lebesgue's introduction of Lebesgue measure in Euclidean spaces, and the growth of real analysis and probability that was facilitated by both the concrete and abstract development of measure theory.

These historical developments are what led mathematicians to know that it was critical to have an invariant countably additive positive measure on a nondiscrete infinite group if one wanted in full generality to pursue the study of functions on groups and to understand the structure of the group's (unitary/irreducible) representations. While some of this was already done for matrix groups and for Lebesgue spaces, it was clear that being able to construct such a measure on an abstract Hausdorff locally compact group (*group*) was not going to be a particularly easy task. With the construction of Haar measure, the development of abstract harmonic analysis on groups could begin.

I do not want to take the reader on a grand tour of everything that can be said about the topic of invariant measures, invariant integrals, and invariant linear forms. This would not be easy, if it were even feasible. Indeed, at least in part, that is what this well-written book does. For example, it gives the background information needed to understand the early constructions of Haar measure in compact groups. Then it moves on to the general case using Weil's proof of the existence of Haar measure. This is not a textbook with extensive sets of exercises to train an eager reader. In some ways, I regret this. However, the absence of these types of features makes the telling of the story of Haar measure, its properties, and some of its applications easier.

But at least let me take the liberty of writing about **three topics** that I have always associated with invariant measures. The **first topic** concerns the idea that sets of large "size" should have even larger sets of differences. Actually, it is not even clear what we need to know about a set $E \subset \mathbb{R}$ to guarantee that $E - E = \{x - y : x, y \in E\}$ contains an interval. If E is of second category with the property of Baire, then this is true (see Piccard [17] and Oxtoby [16]). If E is of positive Lebesgue measure, then this is true. This is Steinhaus's Theorem [22]. See the articles by Bingham and Ostaszewski [1] and Ostaszewski [15] for information and

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references on this phenomenon and its generalizations. I wonder: can the class of such sets (or a large portion of them) be suitably modeled and shown to be a complete analytic set? In any case, Haar measure allows us to give a fairly simple proof of this type of result for a large class of sets, with differences in the line replaced by difference in the group. Indeed, if K is a compact set of positive Haar measure $\mu_G(K) > 0$ in a group G, then the function $C : g \to \mu_G(gK \cap K)$ is a continuous function of compact support. Since $C(e) = \mu_G(K) > 0$, there is a neighborhood of the identity where $C(g) > 0$. For every g in this neighborhood, $gK \cap K$ is certainly not empty. So KK^{-1} contains this neighborhood. So by inner regularity of the Haar measure, for any set E of positive Haar measure, the difference set EE^{-1} contains a nonempty open set.

This can of course be extended by using many terms instead of just two. If $g_1, \ldots, g_n$ are all close enough to e, then $\mu_G(g_1E \cap \cdots \cap g_nE \cap E) > 0$, and so the intersection is not empty. This is how one can show that if (g_k) converges to e, then for any n, there is a large enough N, so that $g_k \in EE^{-1}$ for all $k = N+1, \ldots, N+n$. However, you cannot generally achieve this for all $k \geq N+1$. What is clear is that there would be a subsequence $(g_{k_l} : l \geq 1)$ and some $g \in E$ such that $g_{k_l}g \in E$ for all $l \geq 1$. This is a weak statement, but it is strong enough for some interesting consequences. For example, one can use it to show that given a compact set $K \subset G$ with $\mu_G(K) > 0$, and a function $f : G \to \mathbb{C}$, if f is continuous on gK for all g, then f is continuous on G. See Ciesielski and Rosenblatt [2] for results of this type.

The thinking above is what leads to the following type of problem: given a nondiscrete locally compact group G, is there an infinite *pattern* that appears in every set of positive Haar measure $E \subset G$? There is a great deal that can be said about this depending on what one means by pattern. We take here the simplest version: is there a nondiscrete group and a sequence $(g_k : k \geq 1)$ of distinct terms tending to e, such that for all sets E of positive Haar measure, there is some g such that $g_kg \in E$ for all (sufficiently large) values of k? This question asked in $G = \mathbb{R}$ becomes this: is there a sequence of nonzero numbers (x_n) tending to 0 such that every set of positive Lebesgue measure contains an *isometric copy* of the (tail) of the sequence? We know the answer to this question is negative because of this: for any sequence of nonzero numbers tending to 0, there is a set E of positive Lebesgue measure such that for any x infinitely many $x + x_k$ are not in E. This is related to *sweeping out* properties that have a role in almost everywhere convergence in ergodic theory. Here is the elementary fact in real analysis that is relevant: given an infinite set $\{x_k : k \geq 1\}$ in $[0,1]$, and $\epsilon > 0$, there is a Borel measurable set E with Lebesgue measure $\mu(E) \leq \epsilon$, such that $[0,1] \subset \bigcup_{k=1}^{\infty} x_k + E$. If one consults Ellis [6], one sees a way to prove that sweeping out can be used in general groups and show that for any nondiscrete group, there cannot be a sequence such that up to translation the sequence (or even its tail) appears in every set of positive Haar measure.

But what if one allows more generally a *similar copy*: that is, a pattern $sx_k + x$ for some x and some $s > 0$? This problem is in Erdős [7]. In Mauldin [14, p. 38], Erdős says, "I don't think the problem is difficult, but perhaps it is not quite trivial." He offered \$100 for a solution. We still do not know the answer. It seems unlikely that there would be such a sequence. However, at this time, we do not know if the following holds: given any set E of positive Lebesgue measure, there is some x and some $s > 0$ such that some tail of the sequence $(s\frac{1}{2^n} + x : n \geq 1)$

is in E. See Humke and Laczkovich [10] for more on this question, as well as a formulation of the problem in terms of infinite combinatorics.

A **second topic** with a long history is the Banach–Tarski paradox and its variations. This construction showed that when trying to extend the idea of volume to a large class of sets, you really need to restrict the class of sets, at least in the presence of the Axiom of Choice (AC), if you want natural ideas from the physical world to hold. We know now that you can use the AC to show there is a partition of the unit ball B_3 in $\mathbb{R}^3$ into nine subsets $E_1, \dots, E_9$, and there are nine isometries $\sigma_1, \dots, \sigma_9$ of $\mathbb{R}^3$ such that $\sigma_1 E_1, \dots, \sigma_9 E_9$ are pairwise disjoint and

$$\bigcup_{i=1}^{4} \sigma_i E_i = B_3 = \bigcup_{i=5}^{9} \sigma_i E_i.$$

This is showing that we cannot extend Lebesgue measure to all sets, even if we only expect finite additivity, if we also want to have invariance under isometries. It is generally thought that the reason for this paradox is the existence of nonabelian free groups in the group of rotations of $\mathbb{R}^3$. This was considered to be perhaps also enough to solve the Banach–Ruziewicz problem: the only rotation-invariant, finitely additive, normalized measure on the Lebesgue measurable sets in the unit ball in $\mathbb{R}^3$ is Lebesgue measure itself. This problem was solved by Margulis [13] and Sullivan [21] for $d \geq 5$ and by Drinfel'd [3] in dimensions $d = 3, 4$. But it is not the existence of nonabelian free groups, but rather groups with Khazdan's property T (for $d \geq 5$) and essentially Khazdan's property T *in situs* (for $d = 2, 3$) that makes this happen. Hence, a deep property of the representations of certain groups comes into play to solve the Banach–Ruziewicz problem. See Wagon [24].

The Banach–Tarski paradox, that volume is not preserved under finite partitions and isometric motion, is really "our fault". We should have stuck with sets in the plane. In the plane, the group of isometries is not as algebraically rich as it is in higher dimensions, and in particular there is an invariant "volume" that is well defined for all sets. This is because the group of isometries in the plane is amenable as a discrete group. This volume is a finitely additive invariant measure that extends Lebesgue measure, but it is not countably additive like Lebesgue measure. This led to the question in Tarski [23]: are two natural sets (a disc and a square for example) with the same area in the plane equivalent by finite decomposition? In Maudlin [14, p. 39], Erdős says, "This is a very beautiful problem, and rather well known. If it were my problem I would offer \$1000 for it—a very very nice question, possibly very difficult."

The Tarski problem was solved by Laczkovich [11] about a decade after Erdős wrote this. He showed that a square S in the plane of side length $\sqrt{\pi}$ and a disc D of radius 1 are equivalent by finite decomposition using just translations. That is, there is a partition $\{P_i : i = 1, \dots, M\}$ of S and translations $T_i, i = 1, \dots, M$ so that $\{T_i P_i : i = 1, \dots, m\}$ is a partition of D. The construction uses nonmeasurable sets. Recently, Grabowski, Máthé, and Pikhurko [8] have shown that the pieces can be chosen to be Lebesgue measurable; this article is a good source for known results and references to various issues connected with (finite) equidecomposability. There are also some great results in this area that show why you cannot use "nice" pieces (for example, scissor decompositions). See for example Dubins, Hirsch, and Karush [4]. It is not known if one can solve Tarski's problem using translations (or isometries) of Borel pieces. Also, not much is known about the size of M, except for the upper estimate from Laczkovich's work and there it is very large, about 10^{50}. Getting a smaller number of sets (for example, nine) would be really interesting. Actually it is not known to be impossible with four pieces!

The result of Laczkovich is one of the great results of modern analysis. However, we do not know much about how it generalizes to other groups. The Banach–Tarski paradox shows that if the group is algebraically rich (that is, perhaps, contains nonabelian free groups), then this type of result is not generally possible. On the other hand, if the group is amenable as a discrete group, it is possible that there are equidecomposition results such as Laczkovich's which hold in the group, for suitable pairs of sets. Note: there are clearly obstructions to this process even in the simplest of settings. Given a compact set with no interior in $\mathbb{R}$ of measure 1, it cannot be equidecomposable with $[0,1]$ because the pieces in the decomposition would be first category, even using an infinite but countable number of pieces.

A **third topic** that is closely tied with the structure of groups and Haar measure is the linear independence of translations of functions on the group. Consider a function $f : G \to \mathbb{C}$ on a group G. The translate ${}_gf$ is defined by ${}_gf(h) = f(g^{-1}h)$ for all $h \in G$. We ask when would we know that the translates $\{{}_gf : g \in G\}$ are linearly independent? The scope of this is too broad, so we restrict ourselves to functions in the Lebesgue spaces $L_r(G), 1 \leq r \leq \infty$. This question is clearly closely tied to the nature of the (irreducible) unitary representations of G. Indeed, if there is linear independence in the action of the representation, then the same would hold on the group.

The easiest example of this phenomenon is for nonzero functions $f \in L_1(\mathbb{R})$. Using the Fourier transform, it is clear that these functions have linearly independent translations. It is an obvious point that the irreducible representations are one dimensional in this case, and one cannot look at the irreducible representations individually to see the linear independence. On the other hand, in groups with infinite-dimensional irreducible representations, a single irreducible representation can play a critical role in the issue of linear independence. A great example of this is the classical Heisenberg group. The infinite-dimensional irreducible representations of this group are the source of an unsolved problem of linear independence. This problem arose in another fashion, through the study of frames. See Heil, Ramanathanm, and Topiwala [9]. The basic problem is this: given a nonzero function $f \in L_2(\mathbb{R})$, are the functions $\{\exp(iat)f(b+t) : (a,b) \in \mathbb{R}^2\}$ linearly independent? The obvious method of taking Fourier transforms does not work here because it just switches the translation and the frequency factor and turns the problem into the same question but with the Fourier transform $\widehat{f}$ in place of f. See Rosenblatt [19] for related examples of very similar groups where there is linear dependence.

We really do not know very much about the questions of linear independence if we vary the group, even among the standard discrete and nondiscrete groups that have been used by many authors in studies of harmonic analysis, probability, and graph theory. Perhaps the issue is that there is just too much variability in the nature of the group, both algebraically and topologically. For example, a special case of this problem is actually the long-standing question of Kaplansky: given that the group has no torsion, are translates of elements of the group ring linearly independent?

Additionally, there is the issue of what happens as one varies the Lebesgue space $L_r(G)$. For instance, see the articles by Puls [18] and Linnell and Puls [12]. When considering $\mathbb{R}^n$, the fact that there is a critical index $r = 2n/(n-1)$ shows also that this phenomenon can be a subtle one. In this case, for $r > 2n/(n-1), n \geq 2$, there are nonzero functions $f \in L_r(\mathbb{R}^n)$ whose translates are not linearly independent. Using classical and distributional Fourier transforms, one can see that for $r \leq 2n/(n-1), n \geq 2$ there is always linear independence. In addition, there is linear independence of translations on $L_r(\mathbb{R}), 1 \leq r < \infty$, and on $C_o(\mathbb{R})$. The classical Fourier transform will do for many of these arguments. However, in some cases, for example $L_r(\mathbb{R}), 2 < r < \infty$, and $C_o(\mathbb{R})$, the arguments seem to require the use of tempered distributions. See Edgar and Rosenblatt [5] and Rosenblatt [20].

In all **three topics** above, Haar measure plays an important role. The book under review by Joe Diestel and Angela Spalsbury is all about Haar measure. The authors start their book with Lebesgue measure in Euclidean spaces and then introduce topological groups, Banach limits, and Haar measure for compact groups. In the process of this there is some discussion of great results, such as the Brunn–Minkowski Theorem, covering theorems of Vitali, the Birkhoff–Kakutani theorem, and the Arzelá–Ascoli Theorem. Then the serious business of constructing Haar measure in general groups is done, giving Weil's existence proof and Cartan's existence and uniqueness proof. The book ends with some material on Steinlage's work on Borel content and the connection with Haar measure and Oxtoby's views on Haar measure.

In summary, I like this book and recommend it for what it can provide. I wouldn't recommend this book for a young reader looking for entertainment. The Phantom Tollbooth by Norton Juster is better for that. I wouldn't recommend this book for someone who really wants to understand the structure of the irreducible representations of the classical groups. There are many other articles and books to look at for that, an enormous and intricate subject. However, if you want to know a little of the history of analysis on groups, this will give you some perspective on that, and introduce the basic tools needed for further work. If you just want to see the scope of what can be done to construct Haar measure easily in special cases, this book does that too. If you want to see how Haar measure can start you into the representation theory of groups and harmonic analysis on groups, this text will provide that. I believe that this is what the authors had in mind: to tell the basic story of Haar measure and along the way capture the imagination of the reader so that they would want to learn more on this subject in one of the many directions that can be pursued.

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Joseph Rosenblatt
Indiana University–Purdue University Indianapolis
E-mail address: joserose@iupui.edu