

BOOK REVIEWS

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Mumford–Tate groups and domains: their geometry and arithmetic, by Mark Green, Phillip A. Griffiths, and Matt Kerr, Princeton University Press, Princeton, NJ, 2012, viii+289 pp., ISBN 978-0-691-154251

In simplest form, Hodge theory is the study of period integrals, i.e., the integrals of algebraic differential forms on algebraic manifolds [47]. Its difficulty and richness stem in part from the non-algebraicity of these integrals. The essential content of the Hodge conjecture and its motivic kin is that what algebraic structure such period integrals do possess arises from algebraic cycles.

The book under review is a masterful study of the interplay between the geometry, arithmetic, and representation theory arising from the period integrals. The first four chapters should be accessible to a reader with a working knowledge of complex algebraic geometry and the finite dimensional representation theory of Lie groups. The remaining chapters concern the distribution of subdomains in Mumford–Tate domains and their implications for the arithmetic of period domains.

To recall the classical roots of the the subject, we begin with the Abel–Jacobi map: Given a compact Riemann surface X of genus g , let $\omega_1, \dots, \omega_g$ be a basis for the space of holomorphic 1-forms on X , and let $\Lambda \subset \mathbb{C}^g$ denote the lattice defined by integration

$$(0.1) \quad \left(\int_{\gamma} \omega_1, \dots, \int_{\gamma} \omega_g \right)$$

over all homology classes $\gamma \in H_1(X, \mathbb{Z})$. The quotient $\text{Jac}(X) = \mathbb{C}^g / \Lambda$ is a compact, complex torus of dimension g called the Jacobian of X . In particular, if γ is any smooth path in X from p to q , then the corresponding integral (0.1) gives a well defined point $\text{AJ}(q - p) \in \text{Jac}(X)$ since the difference of any two paths from p to q determines an element of $H_1(X, \mathbb{Z})$. For a fixed choice of basepoint $p \in X$, the map $x \mapsto \text{AJ}(x - p)$ gives an embedding of X into $\text{Jac}(X)$. The Torelli theorem asserts that $\text{Jac}(X)$ determines X up to isomorphism.

Alternatively, instead of considering $\text{Jac}(X)$, one can package the period integrals of X into a period matrix. For a curve of genus g , this gives a complex $g \times g$ -symmetric matrix with positive definite imaginary part. The set of all such matrices is the Siegel upper half-space \mathcal{H}_g . Suitable quotients of \mathcal{H}_g by arithmetic groups $\Gamma \subset Sp_{2g}(\mathbb{Z})$ are the prototypical examples of Shimura varieties [50]. A family of compact Riemann surfaces $\pi : X \rightarrow S$ of genus g determines a period map $\varphi : S \rightarrow \Gamma \backslash \mathcal{H}_g$ where Γ is the global monodromy group of the family [14].

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For families of higher dimensional projective manifolds, one can define period maps $S \rightarrow \Gamma \backslash D$ and Abel–Jacobi maps from homologically trivial algebraic cycles into a Hodge theoretically defined intermediate Jacobian. For families of varieties with extra symmetries or algebraic cycles, the period map can be refined to a map into the quotient of a Mumford–Tate domain D_M , with Mumford–Tate group M encoding the symmetry of the family. The resulting objects $\Gamma \backslash D_M$ can be thought of as a simultaneous generalization of Shimura varieties and period domain quotients $\Gamma \backslash D$.

The objective of the remaining sections of this review is to weave a summary of the contents of the book into a broader survey of the theory of period maps. Given the amount of material to cover, I shall say relatively little about the connections with representation theory beyond the fact that Mumford–Tate groups and domains have a natural connection with discrete series as outlined below. All references of the form (Thm. III.A.1) etc. refer to the monograph under review.

1. DISCRETE SERIES

Let $G_{\mathbb{C}}$ be a connected, semisimple complex Lie group. Then, the finite dimensional irreducible representations of $G_{\mathbb{C}}$ are classified by their highest weight vectors. By the Borel–Weil theorem, this description can be geometrized as follows: Let B be a Borel subgroup of $G_{\mathbb{C}}$. Then, $G_{\mathbb{C}}/B$ is a smooth complex projective variety, and each integral weight λ of G gives rise to a line bundle L_{λ} over $G_{\mathbb{C}}/B$. If λ is dominant integral, then $H^0(G_{\mathbb{C}}/B, L_{-\lambda})$ is an irreducible representation of $G_{\mathbb{C}}$ of highest weight λ [28].

In the case where $G_{\mathbb{C}} = SL_2(\mathbb{C})$, the homogeneous space $G_{\mathbb{C}}/B$ is the Riemann sphere \mathbb{P}^1 with homogeneous coordinates $[x, y]$ and the space of global sections associated to L_{-d} is the set of homogeneous polynomials of degree d in x and y .

The finite dimensional representation theory of real semisimple Lie groups roughly mirrors the complex case. In infinite dimensions, one of the basic classes of representations of interest are the discrete series, i.e., irreducible unitary representations ρ of $G_{\mathbb{R}}$ on a Hilbert space V such that the “matrix coefficients”

$$\rho_{uv}(g) = \langle \rho(g)u, v \rangle$$

defined by a choice of $u, v \in V$ are square integrable on $G_{\mathbb{R}}$. In [36], Harish-Chandra proved that a connected semisimple real group has a discrete series if and only if it has a compact Cartan subgroup.

In analogy with the Borel–Weil theory for $SL_2(\mathbb{C})$, the discrete series for the underlying real form $G_{\mathbb{R}} = SU(1, 1)$ can be described as holomorphic functions on the unit disk $D \subset \mathbb{C}$. Specifically, for each $n \geq 2$ the set of holomorphic functions on D which are L^2 with respect to the measure $(1 - |z|^2)^{n-2} dx dy$ give a discrete series representation with respect to the action

$$\left(\begin{bmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{bmatrix} . f \right) (z) = (-\bar{\beta}z + \alpha)^{-n} f \left(\frac{\alpha z - \beta}{-\bar{\beta}z + \alpha} \right)$$

(Chapter VI, [44]). Other examples of groups with discrete series representations are $SU(p, q)$, $SO(p, q)$ for p even and $Sp(n, \mathbb{R})$. A complex Lie group viewed as a real Lie group has no discrete series representations. Among the groups $SL_n(\mathbb{R})$, only $SL_2(\mathbb{R})$ has discrete series representations.

In analogy with the Borel–Weil theorem, Kostant [46] and Langlands [48] conjectured that the discrete series should arise as the L^2 cohomology of a homogeneous

line bundle over a flag domain, i.e., a complex manifold D arising as an open $G_{\mathbb{R}}$ -orbit in a generalized flag variety for the complexification of $G_{\mathbb{R}}$. This was established by W. Schmid in [58, 60].

The points of D parametrize certain sequences of subspaces

$$(1.1) \quad 0 \subseteq \dots \subseteq F^p \subseteq F^{p-1} \subseteq \dots \subseteq V_{\mathbb{C}}$$

of a fixed complex vector space $V_{\mathbb{C}}$. In Chapter IV, the authors classify the semisimple \mathbb{Q} -algebraic adjoint groups G (Mumford–Tate groups) and corresponding flag domains D (Mumford–Tate domains) which arise in this setting (cf. [61] and [34]).

Remark 1.2. It is expected that the coherent cohomology of the quotients $\Gamma \backslash D_M$ capture the degenerate limits of discrete series. For the case of totally degenerate limits of discrete series, see Chapter 9, [34]. The case of $U(2, 1)$ was studied extensively by Carayol [10]. By a result of Mirković [52], degenerate limits of discrete series do not appear in the cohomology of a Shimura variety.

2. HODGE STRUCTURES AND MUMFORD–TATE GROUPS

Switching now to algebraic geometry, we review a bit of Hodge theory and the history of Mumford–Tate groups. To begin, recall that an abelian variety is an algebraic group which is also a smooth projective variety. Over the complex numbers, an abelian variety can be presented as a quotient V/Λ of a finite dimensional complex vector space by a lattice Λ which satisfies the following condition: There exists a positive definite hermitian form h on V such that the imaginary part of h assumes integral values on Λ . The fact that V/Λ can be embedded in projective space then follows from the theory of theta functions (Lefschetz embedding theorem). The Jacobian of a compact Riemann surface is an abelian variety.

Let A be an abelian variety defined over a number field $k \subset \mathbb{C}$, and let ℓ be a prime. Let $T_{\ell}(A)$ be the Tate module [51] defined using inverse systems of torsion points (a_1, a_2, \dots) such that $\ell a_j = a_{j-1}$ and $\ell a_1 = 0$. Then, via the natural action of $\text{Gal}(\bar{k}/k)$ on the torsion points of A , the vector space $V_{\ell}(A) = T_{\ell}(A) \otimes \mathbb{Q}_{\ell}$ becomes a representation ρ_{ℓ} of $\text{Gal}(\bar{k}/k)$. The Zariski closure of the image of ρ_{ℓ} is an algebraic subgroup of $\text{Aut}(V_{\ell})$ with Lie algebra $\mathfrak{g}_{\ell}(A)$. The Mumford–Tate conjecture [53] asserts that the Lie algebras $\mathfrak{g}_{\ell}(A)$ are all of the form $\mathfrak{m} \otimes \mathbb{Q}_{\ell}$ where \mathfrak{m} is the Mumford–Tate Lie algebra of A .

To define Mumford–Tate groups, we recall that given a compact complex manifold X , a choice of hermitian metric h allows us to describe the cohomology groups of X in terms of harmonic differential forms, that is C^{∞} complex differential forms which are annihilated by the exterior differential d and its adjoint d^* with respect to h . When (X, h) is compact Kähler, this representation is compatible with the complex structure, and we obtain a Hodge decomposition [67]

$$(2.1) \quad H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X)$$

such that $\overline{H^{p,q}(X)} = H^{q,p}(X)$. Let C be the Weil operator which acts as multiplication by i^{p-q} on $H^{p,q}(X)$. Using the hard Lefschetz theorem, integration over X determines a bilinear form $Q : H^k(X, \mathbb{R}) \otimes H^k(X, \mathbb{R}) \rightarrow \mathbb{R}$ which is symmetric for k even and skew symmetric for k odd such that

$$(2.2) \quad \langle u, v \rangle = Q(Cu, \bar{v})$$

is a positive definite hermitian inner product which makes (2.1) orthogonal [67]. In the case where X is a complex projective manifold embedded in projective space \mathbb{P}^m and h is obtained by restriction of the Fubini–Study metric of \mathbb{P}^m , the bilinear form Q can be taken to assume rational values on $H^k(X, \mathbb{Q})$.

Remark 2.3. $H^{p,q}(X)$ is isomorphic to the Dolbeault cohomology group $H_{\bar{\partial}}^q(X, \Omega^p)$.

Analogously [30], one defines a pure (rational) Hodge structure H of weight k to consist of a finite dimensional complex vector space $H_{\mathbb{C}}$ equipped with a rational form $H_{\mathbb{Q}}$ and a Hodge decomposition $H_{\mathbb{C}} = \bigoplus_{p+q=k} H^{p,q}$ such that $\overline{H}^{p,q} = H^{q,p}$. A polarization of such a Hodge structure is a bilinear form $Q : H_{\mathbb{Q}} \otimes H_{\mathbb{Q}} \rightarrow \mathbb{Q}$ of parity $(-1)^k$ which defines a positive definite hermitian form on $H_{\mathbb{C}}$ via (2.2). The following two alternative descriptions of a pure Hodge structure of weight k are fundamental:

A pure Hodge structure determines a decreasing Hodge filtration

$$(2.4) \quad F^p H_{\mathbb{C}} = \bigoplus_{a \geq p} H^{a, k-a}$$

of $H_{\mathbb{C}}$ such that $H_{\mathbb{C}} = F^p \oplus \overline{F}^{k-p+1}$. Conversely, a decreasing filtration which satisfies this condition determines a pure Hodge structure of weight k by setting $H^{p, k-p} = F^p \cap \overline{F}^{k-p}$. Replacement of the Hodge decomposition by the Hodge filtration provides the link to the flag varieties (1.1) appearing in the representation theory considered above.

A Hodge structure H also determines a representation

$$(2.5) \quad \varphi : U(1) \rightarrow GL(H_{\mathbb{R}})$$

by the requirement that $z \in \mathbb{C}^*$ acts on $H^{p,q}$ as multiplication by $z^p \bar{z}^q$. This representation is compatible with the underlying real structure and preserves the polarization Q . More formally, a pure Hodge structure of weight k is a representation of the Deligne torus $\mathbb{S} = \text{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m)$ such that $r \in \mathbb{R}^*$ acts as r^k . Given such a representation, the corresponding pure Hodge structure H is determined by the action of $U(1)$ as above. This is the link that allows the authors to classify semisimple Mumford–Tate groups of adjoint type in Chapter IV of the monograph using the machinery of finite dimensional representation theory.

Remark 2.6. A morphism of Hodge structures $V \rightarrow W$ is a linear map $V_{\mathbb{Q}} \rightarrow W_{\mathbb{Q}}$ which maps $F^p V_{\mathbb{C}}$ to $F^p W_{\mathbb{C}}$. The category of pure, polarized Hodge structures is a semisimple abelian category [24] closed under the operations of taking tensor products and duals. Inclusion $V \rightarrow W$ and orthogonal projection $W \rightarrow V$ with respect to the polarization are examples of morphisms of Hodge structures.

Definition 2.7. The (special) Mumford–Tate group $M = M(H) = M_{\varphi}$ of a Hodge structure H is the smallest \mathbb{Q} -algebraic subgroup of $GL(H)$ such that $M(\mathbb{R})$ contains the image of (2.5).

To motivate this definition, we recall that the Hodge conjecture [49] asserts that if X is a smooth complex projective variety, then every element of the Hodge group

$$H^{p,p}(X, \mathbb{Q}) = F^p H^{2p}(X, \mathbb{C}) \cap H^{2p}(X, \mathbb{Q})$$

arises as the cohomology class of a codimension p algebraic cycle on X . Via the Künneth formula, the Hodge conjecture implies that a morphism $f : V \rightarrow W$

between sub-Hodge structures of a pair of smooth projective varieties can be represented by an algebraic cycle on the product.

In particular, starting from a pure, polarized Hodge structure H , we can consider the tensor algebra $T = \bigoplus_{a,b \geq 0} H^{\otimes a} \otimes \check{H}^{\otimes b}$ generated by H and its dual \check{H} . An element of T is a Hodge tensor if it is rational and of Hodge type (p, p) for some integer p . The Mumford–Tate group M of H is then exactly the subgroup of $\text{Aut}(Q)$ which fixes all Hodge tensors of T (Prop. I.B.1). By the previous paragraphs, if $H = H^k(X)$, the Hodge conjecture links M to algebraic cycles on self products of X . Chapter I of the book reviews the basic properties of Hodge structures and Mumford–Tate groups.

Remark 2.8. For abelian varieties, the Mumford–Tate conjecture links the Hodge conjecture and the Tate conjecture.

From the viewpoint of algebraic cycles, the Mumford–Tate groups are clearly the fundamental symmetry groups of Hodge theory. The corresponding geometric objects are the Mumford–Tate domains: If H is a pure Hodge structure with Mumford–Tate group M , the corresponding Mumford–Tate domain D_M is the orbit of the Hodge flag of H (2.4) under the real points of M . The corresponding compact dual \bar{D}_M is the orbit of any point of D_M under $M(\mathbb{C})$. The first geometric realization [10] of the simple exceptional Lie group G_2 was via a Mumford–Tate domain (see Chapter IV).

Chapter II discusses the basic properties of Mumford–Tate domains. In particular, D_M is a complex manifold upon which $M(\mathbb{R})$ acts via biholomorphisms. The two extreme cases are when D_M is as large as possible or as small as possible. In the first case $M = \text{Aut}(Q)$ and $D = D_M$ is a standard period domain of the type introduced by Griffiths in [30]; see (3.1) below.

In the second case, if M is a torus and hence D_M consists of a single point, we say H is of CM type. By [1], every (effective) Hodge structure of CM type occurs in the cohomology of a CM abelian variety. In the case of an elliptic curve $E = \mathbb{C}/\Lambda$, the property of being CM means that there is a complex number α which is not an integer such that $\alpha(\Lambda) \subset \Lambda$, and hence multiplication by α induces a map $E \rightarrow E$. For example, multiplication by $\sqrt{-1}$ induces a self map of the elliptic curve $\mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\sqrt{-1})$.

Example 2.9. Any two dimensional irreducible Hodge structure of even weight is of CM type. A simple example of this form is $H^{2,0}(X) \oplus H^{0,2}(X)$, where X is a K3 surface of maximal Picard rank. See [5] for additional examples. In [65] Totaro combined the ideas of Shimura and the monograph under review to classify the endomorphism algebras of Hodge structures of the form $H^{m,0} \oplus H^{0,m}$.

The Mumford–Tate conjecture for abelian varieties of CM type was proven by Pohlmann [56]. In [25], Deligne proved that $\mathfrak{g}_\ell(A)$ is always contained in the tensor product of the Lie algebra of the Mumford–Tate group of $H^1(A)$ with \mathbb{Q}_ℓ . Deligne’s proof [25] that Hodge classes on an abelian variety are absolute Hodge also depends on Mumford–Tate theory. Chapters V and VI of the monograph are devoted to CM Hodge structures and other arithmetic aspects of Mumford–Tate domains.

3. PERIOD MAPS

As the authors discuss in the introduction, most of the applications of Mumford–Tate theory to algebraic geometry to date involve moduli of curves, abelian varieties, and other cases where the infinitesimal period relation (IPR) of Griffiths is trivial. To explain this, we now introduce the notions of period maps and variations of Hodge structure. This topic is the focus of Chapter III.

Let $X \rightarrow S$ be a smooth family of complex projective manifolds with fiber X_s over $s \in S$. Fix a basepoint $b \in S$ and a polarization Q of $H^k(X_b)$. Then, over an open polydisk $\Delta^r \subset S$ containing b , we can write down the Hodge filtration $F^\bullet H^k(X_s, \mathbb{C})$ relative to a fixed basis of $H^k(X_b, \mathbb{Q})$ [27]. In this way, we obtain a map $\Delta^r \rightarrow D$ where D is the set of all Hodge structures on $H_{\mathbb{C}} = H^k(X_b, \mathbb{C})$ with the same Hodge numbers $\{h^{p,k-p}\}$ as X_b which are polarized by Q . The set D is a complex manifold which is a homogeneous space for the action of $G(\mathbb{R}) = \text{Aut}_{\mathbb{R}}(Q)$. Applying this construction globally, we obtain a holomorphic map

$$(3.1) \quad \phi : S \rightarrow \Gamma \backslash D$$

called the period map [30] of $X \rightarrow S$, where Γ is the image of the monodromy representation of $\pi_1(S, b)$.

The infinitesimal period relation asserts that every local lifting $F(s)$ of (3.1) satisfies

$$(3.2) \quad \frac{\partial F^p}{\partial s_j} \subseteq F^{p-1}(s).$$

As a flag manifold, the tangent space to D at F can be identified with a subspace of $\bigoplus_p \text{Hom}(F^p, V/F^p)$, and hence (3.2) can be viewed as saying that the period map is tangent to the horizontal distribution defined by $\bigoplus_p \text{Hom}(F^p, F^{p-1}/F^p)$. The classifying space D is an open subset of the compact dual \check{D} given by the orbit of any point of D under of $G(\mathbb{C}) = \text{Aut}_{\mathbb{C}}(Q)$. The horizontal distribution extends to a $G(\mathbb{C})$ -equivariant subbundle of $T(\check{D})$.

In general, the infinitesimal period relation is highly non-trivial. Indeed, in [13] it is shown that if there are no gaps in the Hodge numbers of D , then the horizontal distribution is bracket generating, i.e., Lie brackets of horizontal vector fields generate the tangent bundle.

The theory outlined in the previous three paragraphs corresponds to well developed theory of period maps and variations of Hodge structure. For families of varieties for which the generic member has extra algebraic cycles or automorphisms, the corresponding period map takes values in the quotient of a Mumford–Tate domain $D_M \subseteq D$ parametrizing Hodge structures with generic Mumford–Tate group $M \subseteq \text{Aut}(Q)$. The basic objective of the monograph is the systematic study of period maps in this context.

A simple illustration of the value of this perspective is the following result of Deligne and André [4]: Let $\Pi \subset \text{Aut}(Q)$ denote the identity connected component of the Zariski closure of the monodromy group Γ appearing in (3.1). Then, Π is a normal subgroup of the derived subgroup of M . Here are two examples where the period map takes values in a quotient of a proper Mumford–Tate subdomain D_M of the ambient Griffiths period domain D :

Example 3.3. The family of Picard curves [55] $y^3 = x(x-1)(x-s_1)(x-s_2)$ has a natural action of $\mathbb{Z}/3\mathbb{Z}$. The Mumford–Tate group for $H^1(C)$ of a generic

member of this family is $U(2, 1)$, corresponding to the fact that $H^{1,0}(C)$ carries a hermitian form of signature $(2, 1)$. The corresponding Mumford–Tate domain is the unit ball in \mathbb{C}^2 , realized as the subset of \mathbb{P}^2 defined by $|w_1|^2 + |w_2|^2 < |w_0|^2$. In contrast, the period map for a generic curve of genus 3 takes values in the space of 3×3 symmetric matrices with positive definite imaginary part. For further work on embeddings of geometric moduli spaces into ball quotients via period maps, see [2, 3, 17, 45, 57].

Example 3.4. Using middle convolutions, Dettweiler and Reiter [26] constructed a family of singular varieties of dimension 6 over $\mathbb{P}^1 - \{0, 1, \infty\}$ such that a period map derived from H^6 has Mumford–Tate group G_2 as opposed to the expected $SO(3, 4)$. See [23] for discussion of the arithmetic of limiting periods of this variation. The group G_2 also furnishes an example of a Mumford–Tate domain which has twelve different homogeneous complex structures (see Chapter IV).

The Baily–Borel theorem [7] asserts that suitable quotients $\Gamma \backslash D$ of a Mumford–Tate domain with trivial IPR are algebraic. More precisely, $\Gamma \backslash D$ has an embedding into \mathbb{P}^m by automorphic forms as a Zariski open subset of an algebraic variety. In contrast, whenever the IPR is non-trivial, the generic point of a Mumford–Tate domain cannot be of geometric origin since periods of families of smooth projective varieties satisfy the IPR. Moreover, the quotients $\Gamma \backslash D$ are not algebraic: modulo some technical conditions, it is shown in [35] that if D_M is a Mumford–Tate domain for M which does not fiber holomorphically or anti-holomorphically over a hermitian symmetric domain, then the quotients of D_M by infinite, finitely generated discrete subgroups of M are not algebraic.

Remark 3.5. The non-algebraicity of $X = \Gamma \backslash D_M$ in the compact case was established by Carlson and Toledo [15] using the theory of harmonic maps. In general [35], one first reduces to the case where Γ is torsion free and proves that X is rationally chain connected. Kollar’s Shafarevich map then implies that X , if algebraic, has finite fundamental group.

4. STRUCTURE THEOREMS

The basic result of Chapter III is a structure theorem for period maps which gives a factorization of a period map $\phi : S \rightarrow \Gamma \backslash D_M$ according to factors of the Mumford–Tate group M of a generic Hodge structure F_o in the image of ϕ . More precisely, over \mathbb{Q} , the Lie algebra \mathfrak{m} of M decomposes as a direct sum of simple factors $\mathfrak{m}_1, \dots, \mathfrak{m}_\ell$ and an abelian subalgebra \mathfrak{a} . Accordingly, at the level of Lie groups, we have

$$M(\mathbb{R}) = M_1(\mathbb{R}) \times \cdots \times M_\ell(\mathbb{R}) \times A(\mathbb{R}).$$

Moreover (Thm. III.A.1), after passage to a finite cover $S' \rightarrow S$, we can assume that this decomposition is compatible with the monodromy group, i.e.,

$$\Gamma = \Gamma_1 \times \cdots \times \Gamma_k$$

with the \mathbb{Q} -closure of Γ_i equal to M_i for $i = 1, \dots, k$. Let D_i denote the orbit of F_o under $M_i(\mathbb{R})$. Then (Cor. III.A.2),

$$\phi : S' \rightarrow \Gamma_1 \backslash D_1 \times \cdots \times \Gamma_k \backslash D_k \times D',$$

where $D' = \Pi_{j>k} D_j$ (i.e., Γ acts trivially on D'). Moreover, D' and each D_i are complex submanifolds of D . In particular, as a consequence of this result it

follows that even though Γ need not be an arithmetic group, it has the same tensor invariants as $M_1(\mathbb{Z}) \times \cdots \times M_k(\mathbb{Z})$.

Remark 4.1. The key ingredients of the proof of the structure theorem are as follows:

- (i) The subalgebras \mathfrak{m}_i and \mathfrak{a} are sub-Hodge structures of \mathfrak{m} .
- (ii) The image of the period map is contained in the Mumford–Tate domain D_M .
- (iii) The \mathbb{Q} -closure of Γ is a normal subgroup of the derived group of M by the work of Deligne and Andre cited above.

Chapter III closes with an estimate for the codimension of the Noether–Lefschetz loci and a discussion of Schoen’s criterion for when a complex projective manifold X can be dominated by a product of lower dimensional varieties $Y = X_1 \times \cdots \times X_k$.

Chapter IV addresses the question of which groups are Mumford–Tate groups (Thm. IV.E.1) and classifies the possible representations (2.5). Namely,

Theorem 4.2. *A \mathbb{Q} -simple adjoint group M is a Mumford–Tate group if and only if $M(\mathbb{R})$ contains a compact maximal torus T .*

Remark 4.3. See [54] for the generalization of this result to the case where M is reductive.

Regarding the possible representations (2.5), let $\rho : M \rightarrow GL(H)$ be a representation of M over \mathbb{Q} and $\sigma : U(1) \rightarrow T$ be a cocharacter. Then, the composite

$$\varphi = \rho \circ \sigma : U(1) \rightarrow GL(H_{\mathbb{R}})$$

potentially arises from a polarized Hodge structure. In the case where ρ is the adjoint representation of M on \mathfrak{m} , we have the following description (Prop. IV.B.3) of which cocharacters yield Hodge representations: Let $K \subseteq M(\mathbb{R})$ be the maximal compact subgroup containing T . The associated Cartan decomposition of $\mathfrak{m}_{\mathbb{R}}$ gives a notion of compact and non-compact roots of T . Likewise, if we write $T = \mathfrak{t}/\Lambda$ via the exponential map, then we have standard identification of cocharacters of T with elements $\lambda \in \Lambda$:

$$e^{2\pi i u} \mapsto \exp(u\lambda).$$

Theorem 4.4. *A cocharacter σ gives rise to a Hodge representation on \mathfrak{m} via the adjoint representation if and only if*

$$\begin{aligned} \langle \sigma, \alpha \rangle &\equiv 0 \pmod{4} && \text{for every compact root } \alpha, \\ \langle \sigma, \beta \rangle &\equiv 2 \pmod{4} && \text{for every non-compact root } \beta. \end{aligned}$$

Remark 4.5. The Hodge structure on \mathfrak{m} has weight 0, and hence the Weil operator C acts on $\mathfrak{m}^{p,-p}$ as i^{2p} . The requirement for $\langle \sigma, \bullet \rangle$ to be congruent to 0 or 2 mod 4 is related to the Hodge–Riemann bilinear relations.

Intertwined with the development of the abstract machinery to characterize Mumford–Tate groups and their Hodge representations, Chapter IV also analyzes a large number of examples, including the classical groups in (IV.C) and the exceptional groups G_2 , F_4 , and E_6 in (IV.D).

5. PARTIAL COMPACTIFICATIONS

Returning now to the Baily–Borel theorem, for suitable quotients $X = \Gamma \backslash D_M$ of Mumford–Tate domains with trivial IPR, we have a system of partial compactifications [29]

$$\bar{X}^{BS} \rightarrow \bar{X}^{RBS} \rightarrow \bar{X}^{BB} \leftarrow \bar{X}_\Sigma,$$

where \bar{X}^{BS} is the Borel–Serre compactification (manifold with corners), \bar{X}^{RBS} is the reductive Borel–Serre compactification (stratified topological space), \bar{X}^{BB} is the Baily–Borel compactification (projective variety), and X_Σ is a toroidal compactification of [6] which depends on a choice of fan Σ . In §9 of [31], Griffiths posed the problem of constructing Hodge theoretic partial compactifications of period domains $\Gamma \backslash D$ to which the period map of a family of algebraic varieties would extend.

This type of question is one motivation for the study of the asymptotics of period maps. The basic result is Schmid’s nilpotent orbit theorem [59] which shows that (after passage to a finite cover) every period map $\phi : \Delta^{*r} \rightarrow \Gamma \backslash D$ on a product of punctured disks is asymptotic to a nilpotent orbit generated by the limit Hodge filtration F_∞ and the monodromy logarithms N_1, \dots, N_r of ϕ . This suggests adjoining spaces of nilpotent orbits as boundary components to $\Gamma \backslash D$ to form analogs of \bar{X}_Σ . This was done for Hodge structures of weight 1 in [12] and one parameter degenerations of weight 2 Hodge structures in [19, 20]. After the full development of the theory of nilpotent orbits and SL_2 -orbits in several variables in the 1980s, Kato and Usui were able to construct a general partial compactification [40] of $\Gamma \backslash D$ via nilpotent orbits using the theory of logarithmic manifolds.

In [11], Carayol considered a Kato–Usui type compactification of a Mumford–Tate domain quotient arising from an imaginary quadratic field. The analysis of Schmid’s results at the end of IV.A and Prop. VI.B.11 provide the foundation for extending the constructions of Kato and Usui to arbitrary Mumford–Tate domains. This was done by the reviewer and M. Kerr in [42], resulting in a Hausdorff logarithmic manifold $\Gamma \backslash D_{M,\Sigma}$. Moreover, even when $\Gamma \backslash D_M$ is not algebraic, it is possible for some of the boundary components of $\Gamma \backslash D_{M,\Sigma}$ to be arithmetic varieties. This appears in the work of Carayol, and is treated systematically in §7 and §8 of [42].

In this setting, one also has a reduced or naive limit map [34, 43] from the boundary components of $\Gamma \backslash D_{M,\Sigma}$ (spaces of nilpotent orbits) to the topological boundary of D_M in \check{D}_M . When applied to the toroidal compactifications \bar{X}_Σ of effective weight 1 Hodge structures considered by [12], this reduced limit map recovers the map $\bar{X}_\Sigma \rightarrow \bar{X}^{BB}$ on open strata (Theorem 5.21, [43]).

Remark 5.1. A Hodge structure H of odd weight determines a compact, complex torus $J(H)$ called the intermediate Jacobian of H . Topologically, $J(H) \cong H_{\mathbb{R}}/H_{\mathbb{Z}}$. A period map $\phi : S \rightarrow \Gamma \backslash D$ of odd weight therefore determines a holomorphic family $J \rightarrow S$ of complex tori over S , and one can ask for compactifications $\bar{J} \rightarrow \bar{S}$. Initial attempts at such compactifications were given by Zucker [68], Clemens [22], and Saito [62]. The first general compactification of one parameter families $J \rightarrow S$ was given by Green, Griffiths, and Kerr in [32] and Kato, Nakayama, and Usui in [38]. Both of these constructions should work in the setting of Mumford–Tate domains. Alternative compactifications of $J \rightarrow S$ using D -modules were given in [9, 63, 64].

6. ARITHMETIC OF PERIODS AND SUBDOMAINS

Definition 6.1. A CM field is a number field K equipped with a non-trivial involution ι such that for any embedding of $\sigma : K \rightarrow \mathbb{C}$, the action of ι corresponds to complex conjugation, i.e., $\sigma(\iota(k)) = \overline{\sigma(k)}$.

The object of Chapter V is to classify polarized CM Hodge structures H of arbitrary weight n . To give a rough idea of what is involved in such a classification [21], assume H is irreducible. Then, the algebra E of endomorphisms of $H_{\mathbb{Q}}$ which commute with the corresponding representation (2.5) is a CM field.

Let S denote the set of embeddings of E into \mathbb{C} . Then,

$$H \cong \bigoplus_{s \in S} H_s,$$

where H_s is the subspace upon which $e \in E$ acts as multiplication by $s(e)$. The subspaces H_s turn out to have dimension 1 and be of Hodge type $(p, n-p)$. Accordingly, H determines a function

$$(6.2) \quad \pi : S \rightarrow \mathbb{Z}, \quad \pi(s) + \pi(\bar{s}) = n$$

such that $\pi(s) = p$. Conversely, since the cardinality of S is equal to $[E : \mathbb{Q}]$, a function π of the type described in (6.2) determines a Hodge structure of weight n by decomposing $E \otimes_{\mathbb{Q}} \mathbb{C}$ into eigenspaces for the action of E . Explicitly, for $s \in S$ let δ_s be the element of \mathbb{C}^S which vanishes on $S - \{s\}$ and evaluates to 1 on s . Then, we have an isomorphism

$$j : E \otimes_{\mathbb{Q}} \mathbb{C} \rightarrow \mathbb{C}^S, \quad j(e \otimes t) = \sum_{s \in S} s(e)t \delta_s$$

relative to which $\mathbb{C}\delta_s$ is of Hodge type $(\pi(s), \pi(\bar{s}))$.

Chapter VI is about the arithmetic of Mumford–Tate domains, or more precisely the fields of definition of Hodge theoretically defined Noether–Lefschetz loci. To state a quick application which occurs at the end of the chapter, let D_M be a unitary Mumford–Tate domain, i.e., $M(\mathbb{R}) = U(p, q)$. Then, the CM fields of every CM Hodge structure in D_M contain a common factor $\mathbb{Q}(\sqrt{-d})$. This includes the Picard curves $y^3 = x(x-1)(x-s_1)(x-s_2)$ considered above and the Mumford–Tate domains occurring in the work of Carayol.

To set the stage for the remaining chapters of monograph, we recall that some of the deepest evidence in support of the Hodge conjecture is the algebraicity of the Hodge loci [18]: The basic idea here is that one starts with an algebraic family of complex projective manifolds $X \rightarrow S$ and a Hodge class α on some fiber X_s . Let $T \subseteq S$ be the locus where some parallel translate of α is a Hodge class. Then, the Hodge conjecture implies that the germ of T at s is algebraic. The Hodge conjecture further implies that that if $X \rightarrow S$ is defined over an algebraically closed subfield of \mathbb{C} , then so can T . In [66], Voisin obtains partial results on the field of definition of Hodge loci, which unfortunately, are untenable for isolated points of T (because the fundamental group of a point is trivial).

In a similar fashion, a family of homologically trivial algebraic cycles on the fibers of $X \rightarrow S$ gives rise to a normal function $\nu : S \rightarrow J$, where $J \rightarrow S$ is the associated family of intermediate Jacobians. One of the main applications of the compactifications $\bar{J} \rightarrow \bar{S}$ to date has been to show that the zero locus $Z(\nu)$ of ν is an algebraic subvariety of S [8, 39, 63]. The vanishing of ν can also be described

in terms of having an extra Hodge tensor, and the analogous question about the field of definition of $Z(\nu)$ is closely connected to some of the conjectural properties of the Bloch–Beilinson filtration. See section 5 of [41] for further discussion of this topic. As with Voisin’s result, the fundamental obstruction is that there are no results which apply to isolated points of $Z(\nu)$. (For a base of $\dim = 1$, see [32, 37].)

Motivated by the above, given a period domain D with polarization Q and a point $F \in D$, the authors define the Noether–Lefschetz locus NL_F to consist of all points $\tilde{F} \in D$ such that every Hodge tensor of F is a Hodge tensor of \tilde{F} . This has a natural extension to a variety $\check{\text{NL}}_F \subseteq \check{D}$ which is defined over \mathbb{Q} such that $\text{NL}_F = \check{\text{NL}}_F \cap D$. The connected components of NL_F are smooth: If φ is the representation attached to a point of NL_F , then the connected component of NL_F through this point is equal to the orbit of φ under the identity component of $M_\varphi(\mathbb{R})$. In Chapter VI, it is also shown that the irreducible components of $\check{\text{NL}}_F$ are defined over number fields.

Remark 6.3. One can define analogous Noether–Lefschetz loci NL_M and $\check{\text{NL}}_M$ of flags with Mumford–Tate group contained in M .

To continue, let \mathcal{Z} denote the locus of flags $F \in \check{D}$ which do not define a pure Hodge structure, i.e., $H_{\mathbb{C}} \neq F^p \oplus \bar{F}^{k-p+1}$. Let D^- denote the complement of \mathcal{Z} and $\check{\text{NL}}_M^- = \check{\text{NL}}_M \cap D^-$. By Theorem VI.B.1, $\check{\text{NL}}_M$ never has any connected component contained in \mathcal{Z} . Consequently, $\check{\text{NL}}_M$ is just the Zariski closure of $\check{\text{NL}}_M^-$, and hence is the union of finitely many $M(\mathbb{C})$ -orbits.

Theorem VI.A.3 shows that the largest subgroup of $G(\mathbb{R}) = \text{Aut}_{\mathbb{R}}(Q)$ which stabilizes NL_M is the normalizer $N_G(M, \mathbb{R})$ of $M(\mathbb{R})$ in $G(\mathbb{R})$. The analogous statement is true of $\check{\text{NL}}_M$ and $G(\mathbb{C})$. Accordingly, the group of connected components of $N_G(M, \mathbb{R})$ acts transitively on NL_M . Lemma VI.C.1 shows that each component of $\check{\text{NL}}_M^-$ contains a CM point. Using this type of argument, one sees that the upper bound on the degree of the field of definition of any component of $\check{\text{NL}}_M$ is $2^{\lfloor r/2 \rfloor}$ where r is the dimension of the underlying vector space of D .

Chapter VII describes a method to determine the Mumford–Tate subdomains of a given period domain D . The subdomain attached to the family of Picard curves considered earlier provides an example of such phenomena. The basic idea is that every Mumford–Tate domain contains a CM point, and hence one can start with CM points on D and look at the possible infinitesimal deformations to a Mumford–Tate subdomain. In particular, as the first step of this algorithm, one must calculate all possible CM fields of rank up to the dimension of the underlying vector space of D . This is then worked out in the case of the period domain with Hodge numbers $(1, 1, 1, 1)$.

The book concludes with the discussion of a series of conjectures about the arithmetic and transcendence theory of period maps, including the absolute Hodge conjecture and a generalization of the André–Oort conjecture to non-classical domains.

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