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Sets of finite perimeter and geometric variational problems. An introduction to geometric measure theory, by Francesco Maggi, Cambridge Studies in Advanced Mathematics, Vol. 135, Cambridge University Press, Cambridge, 2012, xvii+454 pp., ISBN 978-1-107-02103-7, US $89.99

Geometric measure theory is a broad and beautiful area of mathematics with deep and persistent connections with geometry, analysis, number theory, combinatorics, and beyond. It is also an area that defies simplistic descriptions as it continually evolves and reinvents its *raison d'etre*. Any attempt to write a book containing a comprehensive treatment of all the areas that currently fall under the umbrella of geometric measure theory would be a futile and thankless task. Instead, Francesco Maggi chose a very coherent and interesting set of problems pertaining to sets of finite perimeter and geometric variational problems and produced an excellent, timely, and thoroughly readable text, accessible to a wide mathematical audience.

One of the most intuitive and best known results of modern mathematics is the iso-perimetric inequality. In its simplest form it says that if L is the length of a closed curve and A is the enclosed area, then

$$4\pi A \leq L^2,$$

and the equality holds if and only if the curve is a circle. This problem goes back to ancient times, and its solution was well known in ancient Greece. A variant of this problem, where one of the sides of the enclosed domain is bounded by a line, is often attributed to Queen Dido of Carthage [10]. A rigorous proof, however, was not discovered until the 19th century. See, for example, [13] for a detailed description of the iso-perimetric and related problems.

In higher dimensions, one can formulate the iso-perimetric inequality in the following way (see, e.g., Federer [8]). For any $E \subset \mathbb{R}^n$, whose closure has finite Lebesgue measure

$$n\omega_n^{\frac{1}{n}} \mathcal{L}^n \left(\overline{E}\right)^{\frac{n-1}{n}} \leq M_*^{n-1}(\partial E),$$

where M_*^{n-1} is $(n-1)$-dimensional; Minkowski content, ω_n is the volume of the unit ball in $\mathbb{R}^n$ and $\mathcal{L}^n$ is the n-dimensional Lebesgue measure. If the boundary of E is also rectifiable, then $M_*^{n-1}(E)$ is simply the $(n-1)$-dimensional Hausdorff measure of E.

In the case of domains with sufficiently smooth boundaries, the n-dimensional iso-perimetric inequality is equivalent to the Sobolev inequality

$$\left(\int_{\mathbb{R}^n} |f(x)|^{\frac{n}{n-1}} dx\right)^{\frac{n-1}{n}} \leq n^{-1}\omega_n^{-1} \int_{\mathbb{R}^n} |\nabla f(x)| dx,$$

which holds for $f \in W^{1,1}(\mathbb{R}^n)$, the space of real valued L^1 functions on $\mathbb{R}^n$ with one weak derivative in $L^1(\mathbb{R}^n)$.

The study of iso-perimetric inequalities is closely tied to the question of what it means for a subset of $\mathbb{R}^n$ to have a finite perimeter. The notion was invented by Renato Caccioppoli in the 1920s and described in his celebrated 1928 paper

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[3]. This topic and related issues were subsequently thoroughly investigated by DiGiorgi in [4–7]. We say that a Lebesgue measurable set E in $\mathbb{R}^n$ is a set of a locally finite perimeter if there exists a $\mathbb{R}^n$-valued Radon measure μ_E on $\mathbb{R}^n$, called the Gauss–Green measure of E, such that the generalized Gauss–Green formula

$$\int_E \nabla\phi = \int_{\mathbb{R}^n} \phi d\mu_E \tag{0.1}$$

holds for all $\phi \in C_c^1(\mathbb{R}^n)$, the space of differentiable function with compact support. The total variation $|\mu_E|$ of μ_E induces the notion of both the relative perimeter of E with respect to $F \subset \mathbb{R}^n$, defined by

$$P(E,F) = |\mu_E|(F),$$

and the total perimeter, defined by

$$P(E) = |\mu_E|(\mathbb{R}^n).$$

These definitions are quite natural in view of the fact that if, for instance, E has a C^1 boundary, then

$$P(E,F) = \mathcal{H}^{n-1}(F \cap \partial E), \text{ and } P(E) = \mathcal{H}^{n-1}(\partial E),$$

where $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff content. Indeed, these definitions lead to a natural generalization of the concept of an open set with a C^1 boundary. This, in turn, results in a variety of interesting applications to the study of geometric variational problems.

We are going to present three representative results (Theorems 1–3 below) that capture the essence of Francesco Maggi's text. These results should be viewed in the context of reconciling the modern mathematical literature on variational problems with the classical literature expressing the viewpoint of physics and engineering (see, e.g., [12]), where a variety of symmetry assumptions are made, frequently without rigorous justification. In the past century, geometric measure theory has made tremendous strides in the direction of putting many of those results on a solid mathematical footing. Along with the symmetry considerations, regularity is also frequently assumed and this issue is addressed during the discussion of Theorem 3 below.

It is difficult to discuss variational problems without mentioning the Plateau problem, which asks for the minimal area of surfaces passing through a given curve (see the picture above). The problem was first raised by Joseph-Louis Lagrange in 1760, but it is named after Plateau who added a practical dimension to the question by experimenting with soap films. If two soap bubbles meet, they merge and a thin film is created between them. In this way, foams are composed of a network of films connected by Plateau borders, which serve as models for the surfaces with minimal area passing through a given curve. A thoroughly accessible description of these concepts can be found in Frank Morgan's highly accessible text on geometric measure theory [11].

For the formulation of a rough version of the Plateau problem, let $A \subset \mathbb{R}^n$ and E_0 be a finite perimeter subset of $\mathbb{R}^n$. We define

$$\gamma(A, E_0) = \inf\{P(E), E\backslash A = E_0\backslash A\}, \tag{0.2}$$

where $P(E)$ is defined as above.

Theorem 1. *Let $A \subset \mathbb{R}^n$, and let E_0 be a finite perimeter subset in $\mathbb{R}^n$. Then there exists a set of finite perimeter $E \subset \mathbb{R}^n$ such that*

$$E \backslash A = E_0 \backslash A$$

and $P(E) \leq P(F)$ for every F such that

$$F \backslash A = F \backslash E_0.$$

In particular, E is the minimizer of the variational problem given by (0.2).

Another important area of variational research is centered around equilibrium shapes of a liquid confined in a given container. A related well-known example in science is hydrostatic equilibrium, which occurs when the flow velocity of a liquid at each point is constant as a function of time. This happens when external forces are balanced by a pressure gradient. See [14] for further details.

The study of problems of this type was originated by Gauss who introduced the free energy functional. Suppose that the liquid occupies a region E (of finite perimeter) inside a container A (an open set with a sufficiently smooth boundary). The free energy of the liquid is given by

$$\sigma(P(E;A) - \beta P(E;\partial A)) + \int_E g(x)dx \equiv \mathcal{F}_\beta(E;A) - \mathcal{G}(E), \tag{0.3}$$

where $\sigma > 0$ denotes the surface tension at the interface between the liquid and the other medium filling A. The coefficient β is called the relative adhesion coefficient between the fluid and bounding solid walls of the recipient. The integral in (0.3) denotes the potential energy acting on the liquid.

One of the most impressive results pertaining to this problem is the symmetrization principle for liquid drops in strips. Let S_T denote the strip $\{x \in \mathbb{R}^n : 0 < x_n < T\}$, and let E^* denote the Schwartz symmetrization of E, where intersections of E with hyperplanes of the form $x_n = const$ are replaced by centered balls of the same volume. By the celebrated Schwartz inequality, $P(E) \geq P(E^*)$, which is one of the fundamental ideas behind iso-perimetric inequalities.

Theorem 2. *If $\beta \in \mathbb{R}$, $g \in L^1(\mathbb{R}^n)$, $E \subset S_T$ are a set of finite perimeter with $0 < |E| < \infty$ and*

$$\mathcal{F}_\beta(E;S_T) + \mathcal{G}(E) \leq \mathcal{F}_\beta(F) + \mathcal{G}(F)$$

for ever $F \subset S_T$ with $|E| = |F|$, then there exists $z \in \mathbb{R}^{n-1}$ such that E is equivalent to $z + E^*$.

We now turn our attention to the issue of regularity of solutions of variational problems. We say that E is a local perimeter minimizer at scale r_0 in some open set A if the support of μ_E is ∂E and $P(E;A) \leq P(F;A)$ whenever $E \triangle F \subset B(x, r_0) \cap A$ and $x \in A$. Here μ_E denote the Gauss–Green measure on E defined in (0.1) above and $E \triangle F$ is the symmetric difference of E and F defined by $(E \backslash F) \cup (F \backslash E)$.

The following deep result shows that local minimizers have a remarkably smooth structure or, in other words, that the smoothness is the consequence of the assumptions inherent in assumptions of the problem. As we we will see in a moment, $A \cap \partial^* E$ is an analytic hypersurface and the singular set is quite small, where

$$\partial^* E = \left\{ x \in support(\mu_E) : \lim_{r \to 0^+} \frac{\mu(B(x,r))}{|\mu_E|(B(x,r))} \text{ exists and belongs to } S^{n-1} \right\}.$$

Theorem 3. *If $n \geq 2$, A is an open set in $\mathbb{R}^n$, and E is a local perimeter minimizer in A, then $A \cap \partial^* E$ is an analytic hypersurface of zero mean curvature which is relatively open in $A \cap \partial E$, while the singular set of E in A,*

$$\Sigma(E; A) = A \cap (\partial E \backslash \partial^* E),$$

satisfies the following:

- *If $2 \leq n \leq 7$, then $\Sigma(E; A)$ is empty.*
- *If $n = 8$, $\Sigma(E; A)$ has no accumulation points in A.*
- *If $n \geq 9$, then $\mathcal{H}^s(\Sigma(E; A)) = 0$ for every $s > n - 8$. (Here $\mathcal{H}^s$ denotes the s-dimensional Hausdorff measure).*

One of the earlier results in this direction was proved by Enrico Bomberi in [2]. See also a seminal paper by Almgren and Lieb [1].

The results presented above give you only a taste of this wonderful book. Francesco Maggi weaves together a precise, self-contained, and deeply insightful narrative of which Theorems 1–3 above are representative examples. A reader interested in a more basic grasp of the underlying concepts with a slightly different emphasis may want to begin by reading the aforementioned interesting text [11] by Frank Morgan. On the other hand, any scientist interested in a deep understanding of the notion of geometric measure theory pertaining to sets of finite perimeter and regularity theory will find this book to be a very valuable asset.

A difficult question every writer of an advanced monograph must answer is how much basic background to put in. The book under review begins with a thorough and clearly written series of sections covering Borel, Radon and Hausdorff measures, Lipschitz functions, the area formula, Gauss–Green formula, rectifiable sets and tangential differentiability. The whole Part I of the book is dedicated to this meticulous review of basic notions of geometric measure theory. A student with a strong background in undergraduate analysis should have little trouble reading this book, though some knowledge of basic theory of Lebesgue integration would certainly speed things along. The first part of the book is extremely useful regardless of what flavor of geometric measure theory one is interested in covering, and I plan to use it extensively the next time I teach a graduate course on the Falconer distance problem or related problems.

Once Part I of the book is carefully mastered, the beautiful world of sets of finite perimeter and regularity theory and analysis of singularities, covered in Parts II and III, respectively, becomes thoroughly accessible. The transitions between the various parts of the book are carefully thought out and designed to make this book widely accessible. This is a truly a first rate text that will serve to bring some of the most fundamental ideas of geometric measure theory to a significant portion of the mathematical community.

Let me conclude my review by noting that I find it interesting and inspiring that Francesco Maggi's book arrived on the mathematical scene within a year of another monograph on geometric measure theory, entitled *Fourier analysis and Hausdorff dimension* by Pertti Mattila ([9]). The emphasis of Mattila's book is on the Fourier analytic aspects of geometric measure theory and on continuous variants of extremal problems in combinatorics, such as the Falconer distance problem. Nevertheless, similarities both on the level of techniques and ideas certainly exist, and experts in both areas would benefit from examining those connections. The resulting symbiosis would serve as a further testament to the universal appeal of the underlying concepts.

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