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Numerically solving polynomial systems with Bertini, by Daniel J. Bates, Jonathan D. Hauenstein, Andrew J. Sommese, and Charles W. Wampler, Software, Environments, and Tools, Vol. 25, SIAM, Philadelphia, PA, 2013, xii+352 pp., ISBN 978-1-611972-69-6, US.00 $95.00

1. Introduction

Systems of linear equations are ubiquitous in mathematics and applied mathematics. With the advent of computers, it became possible to solve such systems on a massive scale. For example, Google represents the web by a directed graph, with edges corresponding to links and vertices corresponding to websites. The page rank algorithm essentially boils down to finding the largest eigenvalue of the transition matrix for a Markov process, where the associated $n \times n$ transition matrix has n in the billions. The development and implementation of algorithms to handle problems like this is the field of numerical linear algebra.

What happens for more general systems of equations? A linear equation is the simplest type of polynomial equation, so a reasonable next step is to study systems of polynomial equations. The investigation of such systems is the realm of algebraic geometry, a deep and classical subject. The increase in complexity when passing from linear to higher degree polynomials can already be seen in the case of one variable. Such a system is trivial in the linear case, while for a system $\{f_1(x), \ldots, f_m(x)\} \subseteq \mathbb{C}[x]$ of higher degree polynomials, finding the common zeros is synonymous with finding the greatest common divisor of the f_i, which can be done using the Euclidean algorithm. This relies on the fact that the polynomial ring $\mathbb{C}[x]$ is a *principal ideal domain*: the *ideal* $I = \langle f_1(x), \ldots, f_m(x)\rangle$ consisting of all polynomial combinations $\sum_{i=1}^{m} g_i(x) \cdot f_i(x)$ is generated by a single polynomial $h(x)$, which is the greatest common divisor of the f_i.

For systems of polynomials in two or more variables, the corresponding ring $\mathbb{C}[x_1, \ldots, x_k]$ is no longer a principal ideal domain, and finding the *variety* $V(I)$ consisting of common solutions for all $f \in I$ becomes much more complicated. However, a polynomial ring over a field is still quite special: it is *Noetherian*, which means that there are no infinite ascending chains of ideals. As a consequence there is a well understood structure theory for $V(I)$. `Bertini` is a software system for finding the solutions to a system of polynomial equations; the book under review is a combined `Bertini` users manual and quick reference guide to theory and algorithms. We begin with some general background on solving polynomial systems.

1.1. Ideals and varieties.

An ideal I in a ring R is *prime* if $f \cdot g \in I$ implies either f or g is in I. In a Noetherian ring, every ideal I has a decomposition

$$I = \bigcap_{i=1}^{n} Q_i, \text{ where } \sqrt{Q_i} = P_i \text{ for distinct primes } P_i,$$

with $\sqrt{I} = \{f \mid f^m \in I\}$ and the Q_i *primary* ideals. Primary ideals are close cousins of prime ideals, defined by the property that if $f \cdot g \in Q_i$, then f or g^m is in Q_i, for some m. While the Q_i which appear in a primary decomposition are not

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unique, the P_i are, and the $V(P_i)$ are the closed and irreducible sets of the *Zariski topology*. If I has a primary decomposition as above, then

$$V(I) = \bigcup_{i=1}^{n} V(Q_i) = \bigcup_{i=1}^{n} V(P_i),$$

so the main step in analyzing the variety $V(I)$ is to find the *associated primes* P_i which appear in the primary decomposition.

Example 1.1. To understand the difference between Q_i and P_i, consider the ideal $I = \langle x^2, y^2 \rangle \subseteq \mathbb{C}[x, y]$. It is easy to see that $V(I)$ is the origin and $\sqrt{I} = \langle x, y \rangle$. An intuitive way to think about I versus $\sqrt{I}$ is that x^2 and y^2 may be viewed as "doubled" lines; the difference between I and $\sqrt{I}$ encodes the fact that as $\epsilon \mapsto 0$, the four solutions to the equations $\{x(x-\epsilon), y(y-\epsilon)\}$ coalesce into a single point. An isolated solution with $Q_i = P_i$ is *nonsingular*; if not, the solution is singular and has a multiplicity $m \in \mathbb{Z}_{>1}$. In this example the multiplicity is four, matching the intuitive description.

Example 1.2. Consider the ideal $I = \langle xy + 2xz, y^2z + 2yz^2 \rangle \subseteq \mathbb{C}[x, y, z]$. Here the first polynomial factors as $x(y + 2z)$ and the second factors as $yz(y + 2z)$, so $I = (y + 2z) \cap (x, y) \cap (x, z)$ and the irreducible components are a line $V(y + 2z)$ and two points $V(x, y) = (0, 0, 1)$ and $V(x, z) = (0, 1, 0)$.

Example 1.3. A four-bar linkage is a planar hinged quadrilateral, and a problem of mechanical design is to move the linkage to trace a desired path if one bar is fixed. Points on the two bars adjacent to the fixed bar trace out circles as the linkage moves. A *coupler point* is a point on the bar opposite the fixed bar; a coupler point traces out a sextic curve. A dimension count shows there are nine parameters for the problem: the lengths of the four bars, two translations and a rotation to fix one bar, and two coordinates for a coupler point.

In 1923 Alt [2] asked how many four-bar linkages there are whose coupler curve passes through nine general points. Because the number of parameters equals the number of constraints, the answer will be finite. Using isotropic coordinates allows exploitation a multihomogeneous structure: write the nine points in isotropic coordinates $(p_i, \overline{p_i}), i \in \{1, \ldots, 9\}$. Fixing p_1 to be the origin also drops one parameter from the system, and in a watershed 1992 paper, Morgan, Sommese, and Wampler [23] used homotopy methods to solve the resulting system of eight equations in eight unknowns. Letting $(a_1, \overline{a_1}, a_2, \overline{a_2}, b_1, \overline{b_1}, b_2, \overline{b_2})$ be the parameters for the system, they obtain equations

$$f(p_i, \overline{p_i}, q) = |v, \overline{u}||v, u| + |u, \overline{u}|^2 = 0, \tag{1}$$

where

$$v = \begin{bmatrix} (p_i - a_1)(\overline{p}_i - \overline{a}_1) + b_1\overline{b}_1 - (b_1 - a_1)(\overline{b}_1 - \overline{a}_1) \\ (p_i - a_2)(\overline{p}_i - \overline{a}_2) + b_1\overline{b}_2 - (b_2 - a_2)(\overline{b}_2 - \overline{a}_2) \end{bmatrix}$$

and

$$u = \begin{bmatrix} b_1(p_i - \overline{a}_1) \\ b_2(p_i - \overline{a}_2 \end{bmatrix}, \qquad \overline{u} = \begin{bmatrix} \overline{b}_1(p_i - a_1) \\ \overline{b}_2(p_i - a_2) \end{bmatrix}.$$

These equations have degree eight, but the lead terms cancel. Hence, if all solutions are isolated, we expect $7^8 = 5,764,801$ solutions (counting multiplicity). However,

many are singular, and others diverge to infinity. `Bertini` returns an answer of

```
                  nonsing     total       singular    infinite    higher
|level| paths    | endpoints | discarded | endpoints | endpoints | dim'l  |
-------------------------------------------------------------------------
| 0   |   8      | 8         | 0         | 0         | 0         |  0     |
| 1   |   64     | 61        | 3         | 0         | 1         |  2     |
| 2   |   488    | 352       | 136       | 28        | 104       |  4     |
| 3   |   2816   | 1526      | 1290      | 346       | 938       |  6     |
| 4   |   11824  | 4687      | 7137      | 2099      | 5038      |  0     |
| 5   |   33056  | 9804      | 23252     | 7575      | 15677     |  0     |
| 6   |   55240  | 12176     | 43064     | 13304     | 29760     |  0     |
| 7   |   48704  | 8652      | 40052     | 11148     | 28904     |  0     |
-------------------------------------------------------------------------
|total| 15220

Non-singular Finite Solution Summary
Nonsingular vs singular is based on rank deficiency and identical endpoints

             | Number of real solns | Number of non-real solns |  Total  |
-------------------------------------------------------------------------
Non-singular | 0                    | 8652                     |  8652   |
-------------------------------------------------------------------------
Total        | 0                    | 8652                     |  8652   |
```

It turns out that there is a symmetry group of order six acting on the solutions, so the end result is that up to symmetry, there are exactly 1442 distinct coupler curves passing thru nine general points. The input file to run this example is available at: `https://bertini.nd.edu/book.html`; for more details see §5.5.2 of [4].

1.2. **Other methods.** Much early work on computing solutions of polynomial systems involved symbolic algebra. The main tool made its appearance in work of Hironaka [19] and Gröbner [12]. The Buchberger algorithm [8] provides a useful and elegant way to compute a *Gröbner basis* for an ideal, which is a generating set with very good properties. Roughly speaking, Gröbner bases give a way to run Gaussian elimination for nonlinear polynomials.

Gaussian elimination implicitly orders the variables, and Gröbner bases order not just the variables, but all monomials; the Buchberger algorithm yields a generating set (the Gröbner basis) where the division algorithm works. Once a Gröbner basis has been found, it is easy to compute numerical invariants of $V(I)$ (e.g., dimension and degree). The drawback is that the algorithm is doubly exponential in the number of variables [21], so while in situations involving few equations or few variables Gröbner bases are useful, symbolic methods are often not well suited to large systems. For more on symbolic methods and the material in §1.1, see [9].

2. Numerical algebraic geometry

In the 1953 paper "On a new method of numerical solution of systems of nonlinear equations" [10], Davidenko described a homotopy continuation method to solve polynomial systems. Many researchers contributed to the method for finding isolated solutions: good surveys of this work are [1] and [24]. The foundational articles for applying the method to possibly positive dimensional solutions sets were

[32], [27], [29], [30], [31], [5]. Even with advances in computing power, homotopy continuation and numerical methods [3] are still the only approach able to answer Example 1.3. Homotopy and numerical methods (henceforth, *numerical algebraic geometry*) have contributed to recent advances in a wide range of settings, including

(1) Biology: model selection in cell signaling and epidemiology [13].
(2) Statistics: maximum likelihood degree [18], [25].
(3) Physics: vacuum moduli space of supersymmetric field theory [16].
(4) Numerical PDE: hyperbolic conservation laws, [15].
(5) Applied geometry: decomposing tensors [17], genus of a curve [6].
(6) Chemistry: biochemical reaction networks [14].
(7) Robotics: robot motion planning [35].

Numerical algebraic geometry is now a standard element in the computational toolkit; for more on symbolic versus numerical methods, see [3].

Roughly speaking, homotopy continuation functions like a race to the summits of a mountain range, where the homotopy paths are the routes taken by the runners. Since the goal is to win, each runner chooses a good starting position (of necessity, distinct). Next, to make the ascent as easy as possible, each runner chooses a smooth route to the top. Finally, to prevent conflict between runners, each must ascend along a separate path, so if m runners end up on a particular summit, they originated from m distinct start points. From a mathematical standpoint, there are two hurdles to overcome. First, how do we choose good start positions? Second, how do we follow the path? Of course, runners can get lost, so there is a third problem: paths can diverge.

2.1. **Homotopy continuation.** To describe the mathematics of homotopy continuation, consider a system $\mathbf{f}(\mathbf{x}) = \{f_1(x_1, \ldots, x_k), \ldots, f_m(x_1, \ldots, x_k)\}$ of polynomials having a finite number of solutions (this implies that $m \geq k$). The strategy of numerical algebraic geometry is to take a simple system $\mathbf{g}(\mathbf{x})$ having at least as many solutions as the original system, and then find a path from the solutions to the simple system to the solutions of the original system. Consider the homotopy

$$\mathbf{h}(\mathbf{x}, t) = (1 - t) \cdot \mathbf{f}(\mathbf{x}) + t \cdot \mathbf{g}(\mathbf{x}).$$

2.1.1. *Start state.* How do we choose the start state $\mathbf{g}(\mathbf{x})$? One natural choice is a *total degree homotopy*; if the initial polynomials are of degrees $\{d_1, \ldots, d_n\}$, then choose polynomials g_i of degree d_i with nice solutions, for example $g_i = x_i^{d_i} - 1$. Depending on the problem, there are a number of other useful options for start state: a solution set with product structure leads to multihomogeneous homotopy [22]; another common option is a linear product homotopy [34], where the g_i are products of linear forms. `Bertini` does not include the polyhedral homotopy [20], which is based on D. Bernstein's work [7] on Newton polygons. It does, however, allow user defined homotopies, so if data is available which can simplify the path tracking, `Bertini` can take advantage of this.

2.1.2. *Path tracking.* Suppose we have a system of two equations $(f_1(x, y), f_2(x, y))$, and choose a system $\mathbf{g}$ whose solutions are easy to find:

$$\begin{aligned} \mathbf{f} &= (f_1(x, y), f_2(x, y)), \\ \mathbf{g} &= (g_1(x, y), g_2(x, y)). \end{aligned}$$

For technical reasons it is better to track from easy solutions at $t = 1$ to our desired solutions at $t = 0$, so our homotopy is

$$\begin{array}{rcccl} h_1(x,y,t) & = & (1-t)\cdot f_1 + t\cdot g_1 & = & 0, \\ h_2(x,y,t) & = & (1-t)\cdot f_2 + t\cdot g_2 & = & 0. \end{array}$$

Because we are tracking a path, $(x, y) = (x(t), y(t))$; in particular, if $h_i(x(t), y(t), t) = 0$, then $h_i(x(0), y(0), 0) = f_i(x(0), y(0)) = 0$, and we have our desired solutions. Differentiate the system above, obtaining for $i = 1, 2$

$$\frac{\partial h_i}{\partial x}\cdot dx + \frac{\partial h_i}{\partial y}\cdot dy + \frac{\partial h_i}{\partial t}\cdot dt = 0.$$

Rewriting this as the system

$$\begin{array}{rcccl} \frac{\partial h_1}{\partial x}\cdot\frac{dx}{dt} & + & \frac{\partial h_1}{\partial y}\cdot\frac{dy}{dt} & = & -\frac{\partial h_1}{\partial t}, \\ \frac{\partial h_2}{\partial x}\cdot\frac{dx}{dt} & + & \frac{\partial h_2}{\partial y}\cdot\frac{dy}{dt} & = & -\frac{\partial h_2}{\partial t}, \end{array}$$

and letting $\frac{dx}{dt} = \Delta x$, $\frac{dy}{dt} = \Delta y$ yields

$$J_h \cdot \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} -\frac{\partial h_1}{\partial t} \\ -\frac{\partial h_1}{\partial t} \end{bmatrix}, \text{ where } J_h = \begin{bmatrix} \frac{\partial h_1}{\partial x} & \frac{\partial h_1}{\partial y} \\ \frac{\partial h_2}{\partial x} & \frac{\partial h_2}{\partial y} \end{bmatrix}. \tag{2}$$

As long as the determinant of the Jacobian matrix J_h above is nonzero, we can solve this equation for the increment $[\Delta x, \Delta y]$ and use this to move closer to the solutions for the original system. What if the determinant of J_h vanishes? It turns out that this problem has an easy fix: introducing an extra complex parameter γ into the homotopy so that $(1 - t)$ is replaced by $\gamma(1 - t)$ causes roots of the discriminant system

$$\begin{array}{rcl} \det(J_h) & = & 0, \\ \mathbf{h}(x,y,t) & = & 0 \end{array}$$

to lie outside the interval $(0, 1]$. So there are no problems, save possibly the "collision" of roots at $t = 0$, which is discussed in §2.1.4.

2.1.3. *Path correction.* Suppose we are at a point $p(t_i) = (x(t_i), y(t_i))$, and solving equation (2) tells us to increment by $(\Delta x, \Delta y)$. The solution obtained after incrementing will typically not lie on $\mathbf{h}(p(t), t)$, so we need to modify the result. This process is known as path correction and can be done in several ways, for example, by running Newton's method on $\mathbf{h}(x(t), y(t), t_{i+1})$ with start value at position $(x(t_{i+1}), y(t_{i+1}))$, and then replacing $p(t_{i+1})$ with the corrected value. `Bertini` chooses step sizes for the prediction of §2.1.2 adaptively, so if an Euler–Newton step does not fall within the desired tolerance, `Bertini` halves Δt and tries again.

2.1.4. *Endgame.* As t approaches zero on a solution path $p(t)$ with $\mathbf{h}(p(t), t) = 0$, special care is needed. If the solution at $t = 0$ is singular, then the numerical linear algebra becomes ill conditioned. There are several ways to deal with this, either by taking a Pusieux expansion for $p(t)$ or using the Cauchy integral formula. `Bertini` offers both of these options to the user. This is perhaps the most challenging part of the computation from a numerical point of view.

2.2. **Numerical irreducible decomposition.** If a polynomial system has fewer equations than variables, then $V(I)$ will have components of positive dimension. How do we find them? The solution is to slice down with general linear polynomials

until the solution set is finite dimensional and use homotopy continuation; the solutions correspond to points on the top dimensional components of $V(I)$. In general, $V(I)$ will not be equidimensional, so once we have the top dimensional components in hand, it is necessary to isolate them from $V(I)$ and study the residual solutions. Formally, this is done via witness sets: let

$$V(I) = \bigcup_{i=0}^{\dim V(I)} \bigcup_{j \in \Lambda_i} Z_{ij},$$

where Λ_i consists of the dimension i components of $V(I)$ and Z_{ij} is an irreducible component of dimension i. Solutions are tracked by witness sets $W_{ij} = \{f, L_{ij}, w_{ij}\}$, where f is a polynomial system containing Z_{ij} as an irreducible component, L_{ij} is a system of i linear forms cutting down Z_{ij} to a zero dimensional set, and w_{ij} consists of $Z_{ij} \cap V(L_{ij})$. The delicate part of the process is separating out distinct irreducible components of a fixed dimension. This is accomplished using two methods, trace testing and monodromy. Trace testing [26], [28] relies on the fact that if X is irreducible of dimension i and L is a set of i general linear forms, then the centroid of the finite set $X \cap V(L)$ moves linearly as $V(L)$ moves parallel to itself. The second method, monodromy, works by moving $V(L)$ in a loop in the parameter space; points on an irreducible component are permuted under this action. `Bertini` uses a combination of the two methods to produce the most efficient search.

3. The book under review

Bates, Hauenstein, Sommese, and Wampler have produced an engaging, hands-on hybrid of a book, combining an introduction to the basics of numerical algebraic geometry with a users manual for the numerical algebraic software system `Bertini`. Most chapters begin with a quick overview of the main mathematical concept, which is then fleshed out with worked examples. During the analysis of these examples, new questions arise, which segué naturally to related definitions and structures. Interspersed with the examples are fragments of `Bertini` code and output, which give the reader a feel for `Bertini`'s capabilities.

The book is divided into four main parts. Part I describes the basics—analyzing systems where the solution set consists of isolated points, focusing on illustrating the techniques and analyzing the four steps described above. Part II tackles systems with positive dimensional solution sets, numerical irreducible decomposition and witness sets. In Part III, the authors focus on several applications of broad interest: polynomial systems arising from discretized differential equations, finding real solutions, and how to perform certain standard algebro-geometric operations (projection, intersection, secant varieties, singular sets). The book closes with Part IV, the `Bertini` Users Manual.

As the authors note in their introduction, this book is not a theorem-proof mathematical text, but rather a users guide to numerical algebraic geometry. Readers interested in delving more deeply into the underlying theory will want to refer to Sommese and Wampler's book [33]. For those with a "code first and ask questions later" mentality, but who still want a quick and dirty description of what makes an algorithm work, this book is ideal. It occupies the favored middle row of my bookshelf, near at hand for a quick reference when needed. It accomplishes admirably its goal of giving a hands-on, user-friendly introduction to numerical algebraic geometry and the `Bertini` software system.

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