BOOK REVIEWS

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 53, Number 2, April 2016, Pages 313–324
http://dx.doi.org/10.1090/bull/1510
Article electronically published on July 6, 2015

Twisted Teichmüller curves, by Christian Weiß, Lecture Notes in Mathematics, Vol. 2104, Springer, Cham, Switzerland, 2014, xvi+166 pp., ISBN 978-3-319-04074-5

> *Teichmüller theory is an amazing subject, richly connected to geometry, topology, dynamics, analysis and algebra.*

These words of Thurston, from his preface to [33], certainly apply to the study of Teichmüller curves.

1. Translation surfaces

Topologically, the surface of a coffee cup is the same as that of a bagel; it is also the same as a square with opposite sides identified by translating one to another. This representation of a torus even inherits a complex structure from the Euclidean z-plane, making it a Riemann surface of genus 1. Furthermore, dz induces a holomorphic 1-form (also called an Abelian differential).

There is an inverse process, even in the general genus case. Given a Riemann surface X and a holomorphic 1-form ω with set of zeros Σ, integration defines local coordinates on $X \setminus \Sigma$. Transition functions originate from change of basepoint and are thus translations. This then allows the Euclidean structure of the plane to be induced onto $X \setminus \Sigma$. The Euclidean structure can be extended to all of X, but at the cost of introducing singularities; these are cone singularities with angles that are found to be integral multiples of 2π. The result is a *translation surface*, (X, ω).

For explicit examples in genus 2, take two regular pentagons, glue them along one edge, and then identify opposite sides by translation. Both this and identifying a regular decagon's opposite sides by translation give genus 2 surfaces. However, the first has a single cone singularity of angle 6π, whereas the second has two singular points, each of angle 4π. These translation surfaces correspond to the (smooth compact) surface of complex equation $y^2 = x^5 - 1$ with respective 1-forms dx/y and xdx/y [1, 13, 75].

The affine diffeomorphisms of a translation surface (X, ω) are those self-homeomorphisms of X sending Σ to itself that are diffeomorphisms on $X \setminus \Sigma$; equivalently, these are locally affine maps whose linear parts are constant. The linear parts form the *Veech group* $\mathrm{SL}(X, \omega) \subset \mathrm{SL}_2(\mathbb{R})$, which Veech [75] showed to be a (non-cocompact) Fuchsian group.

2010 *Mathematics Subject Classification.* 14G35, 20H10, 11R11, 37D40.

2. Teichmüller curves

In the case of genus $g = 1$, the integration of ω defines a global map from X to $\mathbb{C}$ modulo the lattice that is the image of first integral homology. This is in fact a biholomorphic map. We can normalize the lattice so that it is generated by 1 and $\tau \in \mathbb{H}$, where $\mathbb{H}$ is the upper half-plane. The normalization depends on a choice of generators for the lattice. Since generating pairs differ by elements of $\mathrm{SL}_2(\mathbb{Z})$, one finds that $\mathcal{M}_1 = \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ has its points in one-to-one correspondence with the biholomorphic equivalence classes of complex tori. That is, $\mathcal{M}_1$ is a (coarse) moduli space for Riemann surfaces of genus 1. (For technical reasons, one should mark a point on each torus, and speak of $\mathcal{M}_{1,1}$; as well, a full introduction would necessarily include a discussion of quadratic holomorphic forms.)

For general genus, the Riemann moduli space $\mathcal{M}_g$ is a singular complex space which is similarly a coarse moduli space of compact Riemann surfaces of genus g. In the 1940s, Teichmüller introduced the use of quasi-conformal maps to study a simply connected (ramified) covering space of $\mathcal{M}_g$, which now carries his name and is denoted $\mathcal{T}_g$. Biholomorphic maps are conformal; they send local small circles to circles. Any Riemann surface of fixed genus g can be mapped to any other by means of quasi-conformal maps; these send circles to ellipses. Teichmüller sketched how any isotopy class of quasi-conformal maps has the appropriate measurement of local eccentricity minimized by a map that is locally affine with respect to flat structures on the respective surfaces, and used this to define a distance on $\mathcal{T}_g$. Royden [70] showed that $\mathcal{M}_g$ is the quotient of $\mathcal{T}_g$ by the group of Teichmüller isometries; in particular the Teichmüller metric descends to $\mathcal{M}_g$. A *Teichmüller curve*, defined by Veech [75], is an algebraic curve in $\mathcal{M}_g$ that is geodesic with respect to the Teichmüller metric. In genus 1 there is exactly one Teichmüller curve—$\mathcal{M}_1$ itself.

3. Teichmüller curves exist, but are rare

Post-composition with any element of $\mathrm{SL}_2(\mathbb{R})$ of the coordinate functions of a translation surface gives again a translation surface, with genus and singularity type preserved. There results an $\mathrm{SL}_2(\mathbb{R})$-action on the bundle $\Omega\mathcal{M}_g$ of holomorphic 1-forms over moduli space. Veech [75] showed that the $\mathrm{SL}_2(\mathbb{R})$-orbit of (X, ω) projects to a Teichmüller curve exactly when $\mathrm{SL}(X, \omega)$ is as large as possible; that is, when it is a lattice. One then says that (X, ω) is a *Veech surface*. The *Veech dichotomy* [75] states that if (X, ω) is a Veech surface then its flat dynamics are in a sense optimal (the straight line flow in any direction is either periodic or it is uniquely ergodic). Smillie (see [76]) showed that the $\mathrm{SL}_2(\mathbb{R})$-orbit of (X, ω) is closed if and only if (X, ω) is a Veech surface. For a further list of equivalent properties to (X, ω) being a Veech surface, see [72, 79].

Veech [75] gave examples showing that there is at least one (algebraically primitive) Teichmüller curve in $\Omega\mathcal{M}_g$ for each g. Bouw and Möller [7] showed that each non-cocompact triangle Fuchsian group is realized, up to finite index, as a Veech group; their construction recovers Veech's examples as well as those of his student Ward [80].

The trace field of (X, ω) is the field extension of $\mathbb{Q}$ given by adjoining the trace of every element in $\mathrm{SL}(X, \omega)$. Call (X, ω) *arithmetic* if it is a Veech surface whose trace field is $\mathbb{Q}$ itself. Gutkin and Judge [27] showed that every arithmetic surface is a cover of a torus, with possible ramification above one point. These so-called square-tiled surfaces (also known as origami) are dense in $\Omega\mathcal{M}_g$, and provide key specific examples in the theory [25, 29]. Schmithüsen [71] gave an algorithm for computing the Veech group in this setting. In general, computing $\mathrm{SL}(X,\omega)$ given (X,ω) is difficult; results and algorithms in this direction are in [77], [6], [8], [66]. The determination of the $\mathrm{SL}_2(\mathbb{R})$-orbits of square-tiled surfaces remains open in general; for $g = 2$, see [36] and [50].

Ellenberg and McReynolds [14] adapted algebro-geometric arguments about certain Hurwitz spaces (moduli spaces of ramified covers) to show that every algebraic curve defined over $\overline{\mathbb{Q}}$ is birationally equivalent to a Teichmüller curve; the isomorphism classification of Teichmüller curves remains open. They also showed that a large collection of lattice subgroups of $\mathrm{SL}(2,\mathbb{Z})$ are realized as Veech groups. Rigidity arguments show that every Teichmüller curve is defined over $\overline{\mathbb{Q}}$, [56] and [57, 63].

Non-arithmetic Veech surfaces are rare. Aside from the $g = 1$ setting, almost every (X,ω) has trivial Veech group [61]. The passage from Euclidean billiard table to translation surface [39] is one motivation for the study of Teichmüller dynamics—see [42], and [12] for a recent overview; Kenyon and Smillie [40] and Puchta [68] show that only three non-isosceles acute rational angled triangles give Veech surfaces. The space $\Omega\mathcal{M}_g$ is partitioned into strata, identified by multiplicity of zeros of 1-forms, hence by the partitions of $2g-2$ into positive integers. For example, the double pentagon and decagon surface give points in $\Omega\mathcal{M}_2(2)$ and $\Omega\mathcal{M}_2(1,1)$, respectively. McMullen [53] proved that there is only one non-arithmetic Teichmüller curve from $\Omega\mathcal{M}_2(1,1)$; it arises from the decagon surface. Thereafter, Möller [59] showed that for all g there are only finitely many appropriately primitive Teichmüller curves from the so-called hyperelliptic component of $\Omega\mathcal{M}_g(g-1,g-1)$. Bainbridge and Möller [5] show that this is also true for the stratum $\Omega\mathcal{M}_g(3,1)$, and they conjecture that finitude holds for all of $\Omega\mathcal{M}_3$. Most experts now seem to expect that the finitude of primitive Teichmüller curves holds for all strata other than $\Omega\mathcal{M}_2(2)$.

If a Veech group has a hyperbolic element (equivalently, (X,ω) has an affine *pseudo-Anosov* diffeomorphism [73]), then [40] showed that adjoining to $\mathbb{Q}$ the trace of this single element already gives the full trace field of the surface. This can be viewed as an obstruction to realizability as a Veech group [32]. Hubert and Lanneau [34] (see also [10]) showed that the existence of (appropriately distinct) parabolic elements implies that the trace field is *totally real*, thus that all of the field embeddings into $\mathbb{C}$ actually lie in $\mathbb{R}$. It remains unknown which number fields are realized as trace fields. However, Calta and Smillie [10] introduced the notion of periodic direction field, which they proved is equal to the trace field of (X,ω) given the presence of a hyperbolic element in $\mathrm{SL}(X,\omega)$, while also showing that every totally real number field is realized as a periodic direction field. Infinitely generated Veech groups exist [37, 49]. A long standing question is if there exists any (X,ω) whose Veech group is cyclic hyperbolic; Hubert, Lanneau, and Möller [35] ruled out a particularly promising candidate.

4. Genus 2 and Jacobians

A breakthrough in the study of Teichmüller curves in genus 2 occurred when, independently, Calta [9] and McMullen [48] showed that there are infinitely many non-arithmetic Teichmüller curves in $\mathcal{M}_2$. Calta used so-called period coordinates to describe the corresponding Veech surfaces; see [81] for a revisiting of this approach in the setting of linear submanifolds. McMullen's approach relied on showing that the Jacobian variety of X has extra structure whenever (X,ω) of genus 2 has an affine pseudo-Anosov diffeomorphism.

For X of genus g, the holomorphic 1-forms form a g-dimensional vector space $\Omega(X)$. Given $\omega \in \Omega(X)$, each of the real and imaginary parts of ω is a real harmonic 1-form and thus uniquely defines a real first cohomology class and in fact, $\omega \mapsto [\mathrm{Re}(\omega)]$ identifies $\Omega(X)$ with $H^1(X,\mathbb{R})$. Using the duality of homology and cohomology, one then defines the *Jacobian variety* $\mathscr{J}(X)$ of X as the complex torus given by $\Omega^*(X)/H_1(X,\mathbb{Z})$. The intersection form on $H_1(X,\mathbb{Z})$ arising from the intersection of curves on X induces a symplectic form on $\Omega^*(X)$ and thus on (the tangent space at the zero point of) $\mathscr{J}(X)$. The existence of this form is tantamount to an embedding into projective space; that is, $\mathscr{J}(X)$ is a principally polarized Abelian variety.

Any affine diffeomorphism acts on $H^1(X,\mathbb{R})$ so as to preserve Stable(ω), the $\mathbb{R}$-span of the classes defined by the real and imaginary parts of ω. It also gives rise to a linear endomorphism of $H_1(X,\mathbb{Z})$ whose real extension to $\Omega^*(X)$ is self-adjoint; McMullen [48] showed that when $g = 2$, this extension is complex linear and an endomorphism of $\mathscr{J}(X)$ results. The trace field of (X,ω) then has a subring that acts as endomorphisms on the Jacobian; since the field is totally real, one says that $\mathscr{J}(X)$ has *real multiplication* by this field. This key insight allowed McMullen to deeply explore the genus 2 setting in a series of papers, including [48]–[54]. In particular, he identified all Teichmüller curves in genus 2, showing that each lies on some Hilbert modular surface, and he also gave a full determination of the ergodic $\mathrm{SL}_2(\mathbb{R})$-invariant probability measures on $\Omega\mathcal{M}_2$.

5. Teichmüller dynamics and the Hodge bundle

Let $\Omega_1\mathcal{M}_g \subset \Omega\mathcal{M}_g$ correspond to translation surfaces of area one, $\mathrm{SL}_2(\mathbb{R})$ also acts here. The *Teichmüller flow* is given by the action of the diagonal matrices $g_t = \mathrm{diag}(e^t, e^{-t})$. On translation surfaces, this contracts the vertical direction while expanding the horizontal. Independently, Masur [46] and Veech [74] showed that there is a natural finite measure on each stratum of $\Omega_1\mathcal{M}_g$, and that the Teichmüller flow is ergodic on each component with respect to this measure. Eskin and Okounkov [19] and Eskin, Okounkov, and Pandharipande [20] determined the precise measure of each stratum by a Hurwitz space approach and the fact that arithmetic translation surfaces are appropriately dense.

Especially due to applications to interval exchange transformations, a main goal of Teichmüller dynamics is to fully understand the quality of mixing of Teichmüller flow for all $\mathrm{SL}_2(\mathbb{R})$-invariant probability measures on $\Omega\mathcal{M}_g$. Despite the fact that Teichmüller space is in a sense completely inhomogeneous [70], analogies to the dynamics of homogeneous spaces provide insight that has allowed for astounding progress. A decade after McMullen's [54] treatment of the genus 2 case, Eskin and Mirzakhani [17] show that finite ergodic $\mathrm{SL}_2(\mathbb{R})$-invariant measures are of Lebesgue class and supported on affine varieties; Eskin, Mirzakhani, and Mohammadi [18] show that the closures of $\mathrm{SL}_2(\mathbb{R})$-orbits are affine manifolds; and Filip [21,22] shows that these closures are algebraic varieties defined over number fields, thus generalizing the aforementioned results for Teichmüller curves. For more on these results and the flurry of work they have inspired see [82]. Finiteness results for primitive

Teichmüller curves in certain strata based on these results include those of [47] and [45].

Almost contemporaneously, Eskin, Kontsevich, and Zorich [15] culminated a 15-year project showing the rationality of the sum of the Lyapunov exponents for Teichmüller flow on the Hodge bundle over any connected component of a stratum. The (real) *Hodge* (normed vector) *bundle* over $\Omega_1\mathcal{M}_g$ is the C^∞-bundle whose fibers are $H^1(X,\mathbb{R})$ with the Hodge norm: $||v||^2 = \frac{i}{2}\int_X \omega\wedge\overline{\omega}$, where $\omega\in\Omega(X)$ has real part whose class is v. The Teichmüller flow can be lifted by parallel transport to give a flow on the Hodge bundle $v\mapsto G_t^{\mathrm{KZ}}(v), t\in\mathbb{R}$. Writing this flow in terms of transition matrices shows it to be a cocycle, called the *Kontsevich–Zorich cocycle*. By a theorem of Oseledets, the ergodicity of the Teichmüller flow with respect to the standard measure implies that there is a measurable decomposition $H^1(X,\mathbb{R}) = \bigoplus_{i=1}^k \mathcal{E}_i(\omega)$ depending on the point (X,ω) such that for non-zero $v\in\mathcal{E}_i(\omega)$, we have $||G_t^{\mathrm{KZ}}(v)|| = \exp(\lambda_i t + o(t))$. Relabelling and listing these Lyapunov exponents with multiplicity, the symplectic structure gives that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{2g}$, with $\lambda_{2g-i+1} = -\lambda_i$ and the first g values being all non-negative. (And thus, the rationality result is for the sum of these first g exponents!) The subspace comprised of the Stable(ω) is indeed stable under the flow: it accounts for $1=\lambda_1=-\lambda_{2g}$. This project of Eskin, Kontsevich, and Zorich has been central to the developments in the field; significant steps toward the result of [15] were made by Forni [23] showing positivity of λ_g, and Avila and Viana [3] proving the *Zorich–Kontsevich conjecture* that all of the Lyapunov exponents are non-zero and distinct. The long list of related works includes those of Eskin, Masur, and Zorich [16], Rafi [69], and Avila, Matheus, and Yoccoz [2].

6. Hodge bundle over a Teichmüller curve

Möller [57] characterized Teichmüller curves in terms of the variation of Hodge structure. In naive terms, above each point b in the Teichmüller curve $B\subset\mathcal{M}_g$ is a curve X, and we can form a bundle with fiber over b being $H^1(X,\mathbb{R})$. Möller showed that there is a decomposition of $H^1(X,\mathbb{R})$, preserved under the monodromy action of $\pi_1(B,b)$, of the form $M\oplus\bigoplus_{j=1}^r L_j$ with the L_j vector spaces of dimension 2 that are appropriately Galois conjugates of Stable(ω) over the associated trace field K. One of the implications is that the Jacobians of the curves fibering over B each has an r-dimensional Abelian subvariety that has real multiplication by K, a direct generalization of McMullen's genus 2 result.

This sophisticated use of modern algebraic geometry is key to many later developments. In particular, it allowed Möller [58] to prove a result underpinning many finiteness results: If (X,ω) is a Veech surface, then the formal difference of two zeros of ω defines a torsion element (that is, an element of finite order with respect to the group structure) in $\mathscr{J}(X)$. The characterization is a central ingredient in the aforementioned Bouw and Möller realization result [7]. It also leads to ways to evaluate the sum of Lyapunov exponents for Teichmüller flow restricted to a Teichmüller curve (or rather its canonical lift to $\Omega_1\mathcal{M}_g$) in terms of invariants of the L_j [7], and in terms of intersection data in the Deligne–Mumford compatification $\overline{\mathcal{M}}_g$ ([11], see also [15]). Using this latter approach, Chen and Möller [11] show that for small genus, all primitive Teichmüller curves arising from the same stratum $\Omega\mathcal{M}_g$ have the same Lyapunov exponents.

7. Kobayashi curves

The Schwarz–Pick Lemma states that any holomorphic mapping from the unit disk to itself is distance-decreasing with respect to the hyperbolic metric. Should the composition of two such maps be distance-preserving, then of course each must be. Hence, to show that a map is an isometry, it suffices to exhibit a second holomorphic map for which the composition is an isometry. Analogous arguments for holomorphic maps between complex manifolds are possible by using Kobayashi metrics, since the distance-decreasing property holds here as well. (The Kobayashi pseudo-metric on a complex manifold is the largest pseudo-metric such that every holomorphic map from the unit disk is distance-decreasing. When this pseudo-metric is non-degenerate, one has a metric. The unit disk's Kobayashi metric agrees with its hyperbolic metric.) A *Kobayashi curve* is an algebraic curve that is geodesic with respect to the Kobayashi metric on some complex manifold. Royden proved that on $\mathcal{T}_g$ the Kobayashi metric agrees with the Teichmüller metric. Thus, any Teichmüller curve is a Kobayashi curve in $\mathcal{M}_g$.

Fixing a symplectic basis of integral homology and a correspondingly normalized basis of holomorphic 1-forms on a Riemann surface X, integration gives its period matrix, a complex $g \times g$ matrix whose imaginary part is positive definite. The space of all such matrices is the *Siegel upper half-space*, $\mathfrak{H}_g$, whose quotient by the real symplectic group gives the coarse moduli space of principally polarized Abelian varieties, $\mathcal{A}_g$. When $g = 1$, this is the upper half-plane, and just as in that case, there is an expression of the general Siegel upper half-space as a bounded symmetric convex domain in an appropriate $\mathbb{C}^n$, and it follows that it carries a Kobayashi metric (as opposed to this being merely a pseudo-metric). Kra [43] and later McMullen [48] show that the Teichmüller disk of any holomorphic 1-form is sent by the Torelli map (associating period matrix to Riemann surface) isometrically into $\mathfrak{H}_g$ with its Kobayashi metric. It follows that a Teichmüller curve in $\mathcal{M}_g$, arising from some (X, ω), gives a Kobayashi curve in $\mathcal{A}_g$.

Overly simplifying, a *Shimura curve* is an algebraic curve uniformized by a Fuchsian group arising appropriately from a quaternion algebra. Shimura curves in $\mathcal{A}_g$ are totally geodesic with respect to the Bergman metric [65], and they are also Kobayashi curves [78]. Möller and Viehweg [63] characterize all Kobayashi curves in $\mathcal{A}_g$, showing rigidity. Möller [60] shows that for $g \neq 5$ there are exactly two Teichmüller curves (those given in [25, 29]), that are simultaneously Shimura curves.

8. Kobayashi curves on Hilbert modular surfaces

One way to generalize the construction of the modular surface $\mathcal{M}_1$ is to consider the quotient of the n-fold product $\mathbb{H}^n$ by $\mathrm{SL}_2(\mathcal{O}_K)$, where $\mathcal{O}_K$ is the ring of algebraic integers of a totally real number field K with $n = [K : \mathbb{Q}]$. There are thus n distinct embeddings of K into $\mathbb{R}$, each of which induces an injection of $\mathrm{SL}_2(\mathcal{O}_K)$ into $\mathrm{SL}_2(\mathbb{R})$. When $n = 1$, the singularities of the quotient are removable, and one can also compactify by adding a cusp point. For $n = 2$, Hirzebruch showed how to resolve quotient singularities and the singularities introduced when compactifying. This allowed a school about him to deeply study the arithmetic, geometry, and topology of these *Hilbert modular surfaces*; see [26] and [30]. In particular, Hirzebruch and Zagier [31] gave a construction of classical modular forms by taking the intersection numbers of *twisted diagonals*, the projection to $\mathbb{H}^2/\mathrm{SL}_2(\mathcal{O}_K)$ of $z \mapsto (Mz, M^\sigma z)$, where $M \in \mathrm{GL}_2^+(K)$ and M^σ denotes the matrix obtained by letting the non-trivial

Galois group element of $K/\mathbb{Q}$ act on the entries of M. Franke [24] and Hausmann [28] completed a classification of these curves. The Kobayashi geodesics of $\mathbb{H}^2$ are the graphs of holomorphic functions from $\mathbb{H}$ to itself (and the image of such under interchange of coordinates), and it follows that the twisted diagonals are Kobayashi curves on their Hilbert modular surfaces.

McMullen's identification of Teichmüller curves in $\mathcal{M}_2$ shows that for each there is a discriminant D such that each of the corresponding Jacobians has endomorphism ring containing a copy of $\mathcal{O}_D$, the order (that is, the finite index subring of the full ring of algebraic integers of a real quadratic field) of discriminant D. A coarse moduli space for principally polarized Abelian surfaces $A_\tau = \mathbb{C}^2/(\mathcal{O}_D \oplus \mathcal{O}_D^\vee)$ with real multiplication by $\mathcal{O}_D$ is given by $X_D = (\mathbb{H} \times \mathbb{H})/\mathrm{SL}(\mathcal{O}_D \oplus \mathcal{O}_D^\vee)$, where $O_D^\vee = \{\alpha/\sqrt{D} \,|\, \alpha \in \mathcal{O}_D\}$ is the inverse different of $\mathcal{O}_D$, and $\mathrm{SL}(\mathcal{O}_D \oplus \mathcal{O}_D^\vee)$ is the appropriate automorphism group. Given a Teichmüller curve in $\mathcal{M}_2$, after appropriate normalization of the associated Veech groups, there is a *modular embedding* $\phi : \mathbb{H} \to \mathbb{H}$ intertwining the action of $\mathrm{SL}(X, \omega)$ and its Galois conjugate so that the quotient of $\mathbb{H} \times \phi(\mathbb{H})$ descends so as to give a copy of the Teichmüller curve on X_D. Since the universal cover $\mathbb{H}$ of a Teichmüller curve coming from an Abelian differential (X, ω) maps isometrically with respect to Kobayashi metrics by the Torelli map to Siegel space, and this map does factor through $(\mathrm{id}_{\mathbb{H}}, \phi)$, the distance-decreasing property shows that the Teichmüller curve on X_D is Kobayashi geodesic.

One can deform a translation surface (X, ω) with multiple zeros by changing the distances between some of these; this change in relative periods when made rigorous is a key ingredient to understanding the topology of strata and their boundaries [16]. In the genus 2 setting, McMullen [55] determines a foliation of X_D whose initial ingredient is given a (X, ω) to fix its absolute periods and allow relative periods to vary. In particular this provided a way for Bainbridge [4] to define appropriate cycles on X_D so as to calculate the Euler characteristics of the Teichmüller curves in $\mathcal{M}_2$, even though the uniformizing groups, the Veech groups, remain unknown in general. (Further topological data is calculated in [67].) Among other results, Bainbridge explicitly showed that the second Lyapunov exponent, λ_2, for any Teichmüller curve is constant within each of the two strata; see [15] for another proof of this.

Using explicit algebraic models of Hilbert modular surfaces, Kumar and Mukamel [44] find explicit equations for a large number of Teichmüller curves, exposing intriguing geometry.

Möller and Zagier [64] develop a theory of *ϕ-twisted modular forms* on Hilbert modular surfaces (where ϕ is a modular embedding), and in particular express the Teichmüller curves on X_D in equations involving derivatives of theta functions. Möller [62] applied the approach of [64] to give equations for the curves on X_D coming from McMullen's [52] Prym variety construction of Teichmüller curves in $\mathcal{M}_g$, with $g = 3, 4$. It is now the Prym varieties of the parametrized curves that have real multiplication, but in general these Abelian surfaces are not principal polarized. However, there is still an appropriate Hilbert modular surface upon which these give Kobayashi curves. Möller poses the challenge to determine all Kobayashi curves on each X_D.

9. Twisted Teichmüller curves and the book under review

In direct analogy with the twisted diagonals, a *twisted Teichmüller curve* is the projection to X_D of $z \mapsto (Mz, M^\sigma \phi(z)\,)$, where $z \mapsto (z, \phi(z)\,)$ projects to a

Teichmüller curve and once again $M \in \mathrm{GL}_2^+(K)$. As Weiß easily shows, any such twist of a Kobayashi geodesic in $\mathbb{H}^2$ is again Kobayashi, thus giving many new examples of Kobayashi curves. One of his most interesting observations is that the second Lyapunov exponent is invariant under twisting. This allows him to apply results about λ_2 of Bainbridge and of Chen and Möller to show that the curves arising on X_D from McMullen's Prym construction are not twists of the Teichmüller curves from $\mathcal{M}_2$.

Much of this book is devoted to the use of Fuchsian group theory and algebraic number theory to achieve, for certain classes of D, results on the number, volume, and Euler characteristic of distinct twists, thus generalizing results of Franke and Hausmann. The author raises but leaves open the highly interesting question of what the intersection numbers of twisted Teichmüller curves may be.

The book is an elaboration of Weiß's Ph.D. dissertation (under the direction of Möller) and can be compared to three other similar publications cited within it. The Bonner Mathematische Schriften publications of Franke [24] and Hausmann [28] each present fundamental results about curves on Hilbert modular surfaces, while tersely presenting background and merely hinting at motivation; they are written in fairly formal German and this, along with their limited distribution, gives the work of Weiß, and the Karlsruhe IT dissertation by Kappes [38], a real advantage as to accessibility. (That said, a price is paid in elegance and formality of language. Indeed, the book under review suffers from a high frequency of clumsily worded English.) Both Kappes and Weiß give more motivation, also giving clear summaries of the necessary background, although Weiß's treatment of the background of Lyapunov exponents is perhaps too terse. In an appendix, Weiß presents a clear proof that Hilbert modular surfaces are indeed coarse moduli spaces for Abelian surfaces with real multiplication by the appropriate ring. This is the sort of service to the profession that one gladly sees in a published thesis—this well-known result did not have an easily found proof in accessible form.

Quibbles arise with almost any book. Oddly enough, Weiß never quite presents a proof that McMullen's Teichmüller curves *are* Kobayashi geodesic on the X_D (cf. p. 44). He overstates Möller's result on the extent of real multiplication of $\mathscr{J}(X)$ above a Teichmüller curve (p. 125). There is the occasional typographical error, such as on p. 24 where the uniformization theorem is attributed a date some half a century too early.

There remain many interesting questions in this area, and one can easily imagine that, just as [24] and [28], these more recent dissertation monographs will be cited for decades to come.

References

[1] E. Aurell and C. Itzykson, *Rational billiards and algebraic curves*, J. Geom. Phys. **5** (1988), no. 2, 191–208, DOI 10.1016/0393-0440(88)90004-6. MR1029427 (91b:58212)

[2] Artur Avila, Carlos Matheus, and Jean-Christophe Yoccoz, *$SL(2,\mathbb{R})$-invariant probability measures on the moduli spaces of translation surfaces are regular*, Geom. Funct. Anal. **23** (2013), no. 6, 1705–1729, DOI 10.1007/s00039-013-0244-5. MR3132901

[3] Artur Avila and Marcelo Viana, *Simplicity of Lyapunov spectra: proof of the Zorich-Kontsevich conjecture*, Acta Math. **198** (2007), no. 1, 1–56, DOI 10.1007/s11511-007-0012-1. MR2316268 (2008m:37010)

[4] Matt Bainbridge, *Euler characteristics of Teichmüller curves in genus two*, Geom. Topol. **11** (2007), 1887–2073, DOI 10.2140/gt.2007.11.1887. MR2350471 (2009c:32025)

[5] Matt Bainbridge and Martin Möller, *The Deligne-Mumford compactification of the real multiplication locus and Teichmüller curves in genus 3*, Acta Math. **208** (2012), no. 1, 1–92, DOI 10.1007/s11511-012-0074-6. MR2910796

[6] Joshua P. Bowman, *Teichmüller geodesics, Delaunay triangulations, and Veech groups*, Teichmüller theory and moduli problem, Ramanujan Math. Soc. Lect. Notes Ser., vol. 10, Ramanujan Math. Soc., Mysore, 2010, pp. 113–129. MR2667552 (2012d:37101)

[7] Irene I. Bouw and Martin Möller, *Teichmüller curves, triangle groups, and Lyapunov exponents*, Ann. of Math. (2) **172** (2010), no. 1, 139–185, DOI 10.4007/annals.2010.172.139. MR2680418 (2012b:32020)

[8] S. Allen Broughton and Chris Judge, *Ellipses in translation surfaces*, Geom. Dedicata **157** (2012), 111–151, DOI 10.1007/s10711-011-9602-3. MR2893481

[9] Kariane Calta, *Veech surfaces and complete periodicity in genus two*, J. Amer. Math. Soc. **17** (2004), no. 4, 871–908, DOI 10.1090/S0894-0347-04-00461-8. MR2083470 (2005j:37040)

[10] Kariane Calta and John Smillie, *Algebraically periodic translation surfaces*, J. Mod. Dyn. **2** (2008), no. 2, 209–248, DOI 10.3934/jmd.2008.2.209. MR2383267 (2010d:37075)

[11] Dawei Chen and Martin Möller, *Nonvarying sums of Lyapunov exponents of Abelian differentials in low genus*, Geom. Topol. **16** (2012), no. 4, 2427–2479, DOI 10.2140/gt.2012.16.2427. MR3033521

[12] Laura DeMarco, *The conformal geometry of billiards*, Bull. Amer. Math. Soc. (N.S.) **48** (2011), no. 1, 33–52, DOI 10.1090/S0273-0979-2010-01322-7. MR2738905 (2012c:37068)

[13] Clifford J. Earle and Frederick P. Gardiner, *Teichmüller disks and Veech's $\mathcal{F}$-structures*, Extremal Riemann surfaces (San Francisco, CA, 1995), Contemp. Math., vol. 201, Amer. Math. Soc., Providence, RI, 1997, pp. 165–189, DOI 10.1090/conm/201/02618. MR1429199 (97k:32031)

[14] Jordan S. Ellenberg and D. B. McReynolds, *Arithmetic Veech sublattices of* $\mathrm{SL}(2,\mathbf{Z})$, Duke Math. J. **161** (2012), no. 3, 415–429, DOI 10.1215/00127094-1507412. MR2881227

[15] Alex Eskin, Maxim Kontsevich, and Anton Zorich, *Sum of Lyapunov exponents of the Hodge bundle with respect to the Teichmüller geodesic flow*, Publ. Math. Inst. Hautes Études Sci. **120** (2014), 207–333, DOI 10.1007/s10240-013-0060-3. MR3270590

[16] Alex Eskin, Howard Masur, and Anton Zorich, *Moduli spaces of abelian differentials: the principal boundary, counting problems, and the Siegel-Veech constants*, Publ. Math. Inst. Hautes Études Sci. **97** (2003), 61–179, DOI 10.1007/s10240-003-0015-1. MR2010740 (2005b:32029)

[17] A. Eskin and M. Mirzakhani, *Invariant and stationary measures for the $SL(2;R)$ action on moduli space*, preprint, arXiv 1302.3320 (2013).

[18] A. Eskin, M. Mirzakhani and A. Mohammadi, *Isolation theorems for $SL(2;R)$-invariant submanifolds in moduli space*, preprint, arXiv:1305.3015 (2013).

[19] Alex Eskin and Andrei Okounkov, *Asymptotics of numbers of branched coverings of a torus and volumes of moduli spaces of holomorphic differentials*, Invent. Math. **145** (2001), no. 1, 59–103, DOI 10.1007/s002220100142. MR1839286 (2002g:32018)

[20] Alex Eskin, Andrei Okounkov, and Rahul Pandharipande, *The theta characteristic of a branched covering*, Adv. Math. **217** (2008), no. 3, 873–888, DOI 10.1016/j.aim.2006.08.001. MR2383889 (2008k:14065)

[21] S. Filip, *Semisimplicity and rigidity of the Kontsevich-Zorich cocycle*, preprint, arXiv:1307.7314 (2013).

[22] S. Filip, *Splitting mixed Hodge structures over invariant manifolds*, preprint, arXiv:1311.2350 (2013).

[23] Giovanni Forni, *Deviation of ergodic averages for area-preserving flows on surfaces of higher genus*, Ann. of Math. (2) **155** (2002), no. 1, 1–103, DOI 10.2307/3062150. MR1888794 (2003g:37009)

[24] Hans-Georg Franke, *Kurven in Hilbertschen Modulflächen und Humbertschen Flächen im Siegel-Raum* (German), Bonner Mathematische Schriften [Bonn Mathematical Publications], 104, Universität Bonn, Mathematisches Institut, Bonn, 1977. Dissertation, Rheinische Friedrich-Wilhelms-Universität, Bonn, 1977. MR516810 (81e:14023)

[25] G. Forni and C. Matheus, *An example of a Teichmüller disk in genus 4 with degenerate Kontsevich–Zorich spectrum*, Preprint, 2008, 8 pp., arXiv:0810.0023.

[26] Gerard van der Geer, *Hilbert modular surfaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 16, Springer-Verlag, Berlin, 1988. MR930101 (89c:11073)

[27] Eugene Gutkin and Chris Judge, *Affine mappings of translation surfaces: geometry and arithmetic*, Duke Math. J. **103** (2000), no. 2, 191–213, DOI 10.1215/S0012-7094-00-10321-3. MR1760625 (2001h:37071)

[28] Wilfried Hausmann, *Kurven auf Hilbertschen Modulflächen* (German), Bonner Mathematische Schriften [Bonn Mathematical Publications], 123, Universität Bonn, Mathematisches Institut, Bonn, 1979. Dissertation, Rheinische Friedrich-Wilhelms-Universität, Bonn, 1979. MR598814 (82g:10045)

[29] Frank Herrlich and Gabriela Schmithüsen, *An extraordinary origami curve*, Math. Nachr. **281** (2008), no. 2, 219–237, DOI 10.1002/mana.200510597. MR2387362 (2008k:14059)

[30] Friedrich E. P. Hirzebruch, *Hilbert modular surfaces*, Enseignement Math. (2) **19** (1973), 183–281. MR0393045 (52 #13856)

[31] F. Hirzebruch and D. Zagier, *Intersection numbers of curves on Hilbert modular surfaces and modular forms of Nebentypus*, Invent. Math. **36** (1976), 57–113. MR0453649 (56 #11909)

[32] W. Patrick Hooper, *Grid graphs and lattice surfaces*, Int. Math. Res. Not. IMRN **12** (2013), 2657–2698. MR3071661

[33] John Hamal Hubbard, *Teichmüller theory and applications to geometry, topology, and dynamics. Vol. 1*, Matrix Editions, Ithaca, NY, 2006. Teichmüller theory; With contributions by Adrien Douady, William Dunbar, Roland Roeder, Sylvain Bonnot, David Brown, Allen Hatcher, Chris Hruska and Sudeb Mitra; With forewords by William Thurston and Clifford Earle. MR2245223 (2008k:30055)

[34] Pascal Hubert and Erwan Lanneau, *Veech groups without parabolic elements*, Duke Math. J. **133** (2006), no. 2, 335–346, DOI 10.1215/S0012-7094-06-13326-4. MR2225696 (2007d:37055)

[35] Pascal Hubert, Erwan Lanneau, and Martin Möller, *The Arnoux-Yoccoz Teichmüller disc*, Geom. Funct. Anal. **18** (2009), no. 6, 1988–2016, DOI 10.1007/s00039-009-0706-y. MR2491696 (2010a:32026)

[36] Pascal Hubert and Samuel Lelièvre, *Prime arithmetic Teichmüller discs in $\mathcal{H}(2)$*, Israel J. Math. **151** (2006), 281–321, DOI 10.1007/BF02777365. MR2214127 (2008f:37073)

[37] Pascal Hubert and Thomas A. Schmidt, *Infinitely generated Veech groups*, Duke Math. J. **123** (2004), no. 1, 49–69, DOI 10.1215/S0012-7094-04-12312-8. MR2060022 (2005c:30042)

[38] A. Kappes, *Monodromy representations and Lyapunov exponents of origamis*, KIT Scientific Publishing 2011.

[39] A. B. Katok and A. N. Zemljakov, *Topological transitivity of billiards in polygons* (Russian), Mat. Zametki **18** (1975), no. 2, 291–300. MR0399423 (53 #3267)

[40] Richard Kenyon and John Smillie, *Billiards on rational-angled triangles*, Comment. Math. Helv. **75** (2000), no. 1, 65–108, DOI 10.1007/s000140050113. MR1760496 (2001e:37046)

[41] Maxim Kontsevich and Anton Zorich, *Connected components of the moduli spaces of Abelian differentials with prescribed singularities*, Invent. Math. **153** (2003), no. 3, 631–678, DOI 10.1007/s00222-003-0303-x. MR2000471 (2005b:32030)

[42] Steven Kerckhoff, Howard Masur, and John Smillie, *Ergodicity of billiard flows and quadratic differentials*, Ann. of Math. (2) **124** (1986), no. 2, 293–311, DOI 10.2307/1971280. MR855297 (88f:58122)

[43] Irwin Kra, *The Carathéodory metric on abelian Teichmüller disks*, J. Analyse Math. **40** (1981), 129–143 (1982), DOI 10.1007/BF02790158. MR659787 (83m:32027)

[44] A. Kumar and R. Mukamel, *Algebraic models and arithmetic geometry of Teichmüller curves in genus two*, preprint, arXiv 1406.7057 (2014).

[45] E. Lanneau, D.-M. Nguyen and A. Wright, *Finiteness of Teichmüller curves in non-arithmetic rank 1 orbit closures*, preprint arXiv:1504.03742

[46] Howard Masur, *Interval exchange transformations and measured foliations*, Ann. of Math. (2) **115** (1982), no. 1, 169–200, DOI 10.2307/1971341. MR644018 (83e:28012)

[47] Carlos Matheus and Alex Wright, *Hodge-Teichmüller planes and finiteness results for Teichmüller curves*, Duke Math. J. **164** (2015), no. 6, 1041–1077, DOI 10.1215/00127094-2885655. MR3336840

[48] Curtis T. McMullen, *Billiards and Teichmüller curves on Hilbert modular surfaces*, J. Amer. Math. Soc. **16** (2003), no. 4, 857–885 (electronic), DOI 10.1090/S0894-0347-03-00432-6. MR1992827 (2004f:32015)

[49] Curtis T. McMullen, *Teichmüller geodesics of infinite complexity*, Acta Math. **191** (2003), no. 2, 191–223, DOI 10.1007/BF02392964. MR2051398 (2005e:32025)

[50] Curtis T. McMullen, *Teichmüller curves in genus two: discriminant and spin*, Math. Ann. **333** (2005), no. 1, 87–130, DOI 10.1007/s00208-005-0666-y. MR2169830 (2006h:32011)

[51] Curtis T. McMullen, *Teichmüller curves in genus two: the decagon and beyond*, J. Reine Angew. Math. **582** (2005), 173–199, DOI 10.1515/crll.2005.2005.582.173. MR2139715 (2006a:32017)

[52] Curtis T. McMullen, *Prym varieties and Teichmüller curves*, Duke Math. J. **133** (2006), no. 3, 569–590, DOI 10.1215/S0012-7094-06-13335-5. MR2228463 (2007a:32018)

[53] Curtis T. McMullen, *Teichmüller curves in genus two: torsion divisors and ratios of sines*, Invent. Math. **165** (2006), no. 3, 651–672, DOI 10.1007/s00222-006-0511-2. MR2242630 (2007f:14023)

[54] Curtis T. McMullen, *Dynamics of* $\mathrm{SL}_2(\mathbb{R})$ *over moduli space in genus two*, Ann. of Math. (2) **165** (2007), no. 2, 397–456, DOI 10.4007/annals.2007.165.397. MR2299738 (2008k:32035)

[55] Curtis T. McMullen, *Foliations of Hilbert modular surfaces*, Amer. J. Math. **129** (2007), no. 1, 183–215, DOI 10.1353/ajm.2007.0002. MR2288740 (2007k:14044)

[56] Curtis T. McMullen, *Rigidity of Teichmüller curves*, Math. Res. Lett. **16** (2009), no. 4, 647–649, DOI 10.4310/MRL.2009.v16.n4.a7. MR2525030 (2010h:32009)

[57] Martin Möller, *Variations of Hodge structures of a Teichmüller curve*, J. Amer. Math. Soc. **19** (2006), no. 2, 327–344, DOI 10.1090/S0894-0347-05-00512-6. MR2188128 (2007b:32026)

[58] Martin Möller, *Periodic points on Veech surfaces and the Mordell-Weil group over a Teichmüller curve*, Invent. Math. **165** (2006), no. 3, 633–649, DOI 10.1007/s00222-006-0510-3. MR2242629 (2007e:14012)

[59] Martin Möller, *Finiteness results for Teichmüller curves* (English, with English and French summaries), Ann. Inst. Fourier (Grenoble) **58** (2008), no. 1, 63–83. MR2401216 (2010d:14037)

[60] Martin Möller, *Shimura and Teichmüller curves*, J. Mod. Dyn. **5** (2011), no. 1, 1–32, DOI 10.3934/jmd.2011.5.1. MR2787595 (2012f:14055)

[61] Martin Möller, *Teichmüller curves, mainly from the viewpoint of algebraic geometry*, Moduli spaces of Riemann surfaces, IAS/Park City Math. Ser., vol. 20, Amer. Math. Soc., Providence, RI, 2013, pp. 267–318. MR3114688

[62] Martin Möller, *Prym covers, theta functions and Kobayashi curves in Hilbert modular surfaces*, Amer. J. Math. **136** (2014), no. 4, 995–1021, DOI 10.1353/ajm.2014.0026. MR3245185

[63] Martin Möller and Eckart Viehweg, *Kobayashi geodesics in* $\mathcal{A}_g$, J. Differential Geom. **86** (2010), no. 2, 355–579. MR2772554 (2012f:14016)

[64] M. Möller and D. Zagier, *Theta derivatives and Teichmüller curves*, preprint (2015), arXiv:1503.05690

[65] Ben Moonen, *Linearity properties of Shimura varieties. I*, J. Algebraic Geom. **7** (1998), no. 3, 539–567. MR1618140 (99e:14028)

[66] R. Mukamel, *Fundamental domains and generators for lattice Veech groups*, preprint (2013)

[67] Ronen E. Mukamel, *Orbifold points on Teichmüller curves and Jacobians with complex multiplication*, Geom. Topol. **18** (2014), no. 2, 779–829, DOI 10.2140/gt.2014.18.779. MR3180485

[68] Jan-Christoph Puchta, *On triangular billiards*, Comment. Math. Helv. **76** (2001), no. 3, 501–505, DOI 10.1007/PL00013215. MR1854695 (2002f:37060)

[69] Kasra Rafi, *Thick-thin decomposition for quadratic differentials*, Math. Res. Lett. **14** (2007), no. 2, 333–341, DOI 10.4310/MRL.2007.v14.n2.a14. MR2318629 (2008g:30035)

[70] H. L. Royden, *Automorphisms and isometries of Teichmüller space*, Advances in the Theory of Riemann Surfaces (Proc. Conf., Stony Brook, N.Y., 1969), Ann. of Math. Studies, No. 66. Princeton Univ. Press, Princeton, N.J., 1971, pp. 369–383. MR0288254 (44 #5452)

[71] Gabriela Schmithüsen, *An algorithm for finding the Veech group of an origami*, Experiment. Math. **13** (2004), no. 4, 459–472. MR2118271 (2006b:30080)

[72] John Smillie and Barak Weiss, *Characterizations of lattice surfaces*, Invent. Math. **180** (2010), no. 3, 535–557, DOI 10.1007/s00222-010-0236-0. MR2609249 (2012c:37072)

[73] William P. Thurston, *On the geometry and dynamics of diffeomorphisms of surfaces*, Bull. Amer. Math. Soc. (N.S.) **19** (1988), no. 2, 417–431, DOI 10.1090/S0273-0979-1988-15685-6. MR956596 (89k:57023)

[74] William A. Veech, *Gauss measures for transformations on the space of interval exchange maps*, Ann. of Math. (2) **115** (1982), no. 1, 201–242, DOI 10.2307/1971391. MR644019 (83g:28036b)

[75] W. A. Veech, *Teichmüller curves in moduli space, Eisenstein series and an application to triangular billiards*, Invent. Math. **97** (1989), no. 3, 553–583, DOI 10.1007/BF01388890. MR1005006 (91h:58083a)

[76] William A. Veech, *Geometric realizations of hyperelliptic curves*, Algorithms, fractals, and dynamics (Okayama/Kyoto, 1992), Plenum, New York, 1995, pp. 217–226. MR1402493 (98f:14022)

[77] William A. Veech, *Bicuspid F-structures and Hecke groups*, Proc. Lond. Math. Soc. (3) **103** (2011), no. 4, 710–745, DOI 10.1112/plms/pdq057. MR2837020 (2012h:37097)

[78] Eckart Viehweg and Kang Zuo, *A characterization of certain Shimura curves in the moduli stack of abelian varieties*, J. Differential Geom. **66** (2004), no. 2, 233–287. MR2106125 (2006a:14015)

[79] Ya. B. Vorobets, *Plane structures and billiards in rational polygons: the Veech alternative* (Russian), Uspekhi Mat. Nauk **51** (1996), no. 5(311), 3–42, DOI 10.1070/RM1996v051n05ABEH002993; English transl., Russian Math. Surveys **51** (1996), no. 5, 779–817. MR1436653 (97j:58092)

[80] Clayton C. Ward, *Calculation of Fuchsian groups associated to billiards in a rational triangle*, Ergodic Theory Dynam. Systems **18** (1998), no. 4, 1019–1042, DOI 10.1017/S0143385798117479. MR1645350 (2000b:30065)

[81] Alex Wright, *The field of definition of affine invariant submanifolds of the moduli space of abelian differentials*, Geom. Topol. **18** (2014), no. 3, 1323–1341, DOI 10.2140/gt.2014.18.1323. MR3254934

[82] Alex Wright, *From rational billiards to dynamics on moduli spaces*, Bull. AMS, to appear.

Thomas A. Schmidt
Oregon State University
E-mail address: `toms@math.orst.edu`