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SELECTED MATHEMATICAL REVIEWS

related to the paper in the previous section by
ANDREI OKOUNKOV

MR0016574 (8,37f) 60.0X

Pólya, George

Sur une généralisation d'un problème élémentaire classique, importante dans l'inspection des produits industriels. (French)

C. R. Acad. Sci. Paris **222** (1946), 1422–1424.

The one-dimensional random walk problem with two absorbing barriers is reformulated by a slight change of notation in the following way. A moving point starts at the origin of the (x,y)-plane and moves in unit steps which are parallel either to the x-axis or the y-axis. The corresponding probabilities are p and $q = 1 - p$. The process stops when the point leaves the domain D defined by $-h_1 + s(x+y) < x < h_2 + s(x+y)$; here $h_1, h_2 > 0$ and $s/(s-1) = a/b$, where a and b are positive integers without common divisor. Clearly there are two "boundaries" F_p and F_q consisting of points (x,y) outside of D for which, respectively, $(x-1,y)$ or $(x,y-1)$ is within D. For any $(x,y) \in D$ let $K(x,y)$ denote the number of possible paths from the origin to (x,y); for the points on the boundaries put $K(x,y) = K(x-1,y)$ or $K(x,y) = K(x,y-1)$. Clearly D is composed of subdomains D_k $(k = 0, 1, \cdots)$ such that, when (x,y) runs through D_k, then $(x+a, y+b)$ runs through D_{k+1}. It follows that there exist integers r, $c_1, \cdots, c_r$ $(c_0 = 1)$ such that

$$c_rK(x,y) + c_{r-1}K(x+a,y+b) + c_{r-2}K(x+2a,y+2b) + \cdots \\ + K(x+ra,y+rb) = 0,$$

provided (x,y) is in D but not in D_0. Now the probability that the process ends at a point of F_p is $\sum K(x,y)p^xq^y$, the summation extending over all points of F_p. It follows, therefore, that both this probability and the expectation of the duration of the process are rational functions of p. These results can be applied to A. Wald's recent related investigations connected with sequential analysis [Ann. Math. Statistics **15,** 283–296 (1944); **16,** 117–186 (1945); MR0010927 (6,88g); **7,** 131]. Numerical applications are to follow.

W. Feller

From MathSciNet, December 2015

MR1293681 (95m:81202a) 81T60; 11Z05, 14H52, 81T13

Seiberg, N.; Witten, E.

Electric-magnetic duality, monopole condensation, and confinement in $N = 2$ supersymmetric Yang-Mills theory. (English summary)

Nuclear Physics. B **426** (1994), *no.* 1, 19–52.

MR1303306 (95m:81202b) 81T60; 11Z05, 14H52, 81T13

Seiberg, N.; Witten, E.

Erratum: "Electric-magnetic duality, monopole condensation, and confinement in $N = 2$ supersymmetric Yang-Mills theory".

Nuclear Physics. B **430** (1994), *no.* 2, 485–486.

In this paper, the usual duality between electric and magnetic charges is realized between a strongly interacting regime of a nonabelian theory with fundamental charged particles and an equivalent system of elementary magnetic monopoles interacting weakly with photons. This leads to an important new result, which is a description of confinement in terms of magnetic monopole condensation.

The paper presents a study of the vacuum structure and dyon (a particle with both electric and magnetic charges) spectrum of the $N = 2$ supersymmetric gauge theory in four dimensions, with gauge group SU(2).

While reviewing the $N = 2$ theory, the authors note that its quantum version also possesses a quantum moduli space of vacua inherited from the classical one, as the quantum corrections do not lift the vacuum degeneracy. Then a low-energy effective theory is constructed for the light fields on the quantum moduli space, and a Kähler metric is written locally in terms of a holomorphic function via the Kähler potential. The local geometric structure of the low-energy Lagrangian is explored by constructing a special coordinate system in which the spin-zero component of the $N = 2$ multiplet has a Kähler metric of a very special kind; the structure of this is related by $N = 2$ supersymmetry to the linear structure of the gauge field. It is also found that, for the spin-zero component, this parametrization is only defined up to an $\mathrm{SL}(2, \mathbf{Z})$ transformation, which is physically interpreted as the electric-magnetic charge duality, constrained by the dyon mass formula found by coupling light fields to the massive dyons. For BPS-saturated states, for which this formula (actually an inequality) becomes an equality, the spectrum is shown not to vary continuously in the strong coupling. This mass equality includes the quantum corrections and is dual in the Olive-Montonen sense. While this duality is quite plausible in an $N = 4$ theory, the $N = 2$ theory does not possess it because electrons and monopoles have different Lorentz quantum numbers. This, together with the fact that the BPS-saturated spectrum depends on the coupling and does not vary continuously, yields an essential difference between $N = 2$ and $N = 4$.

In fact, it is shown that the discontinuity of the BPS-saturated spectrum (in the strong coupling) arises where a magnetic monopole or a dyon becomes massless. Allowing for monodromies associated with these singular points, the authors find a good Kähler metric on the whole quantum moduli space. More explicitly, by assuming the singularities as coming from massless monopoles and dyons and by making use of certain ingredients from algebraic geometry, such as the fundamental group of a thrice punctured Riemann surface, the authors solve the model by giving a global metric on the moduli space and unique formulas for particle masses satisfying the required conditions. By further perturbation about the point at which monopoles become massless, it is shown that monopoles condense by developing a vacuum expectation value, and thus lead to confinement of charges.

There remains an open question as to whether this result can also be established in nonsupersymmetric theories or is inherent to supersymmetry?

Hamid Kachkachi

From MathSciNet, December 2015

MR1861442 (2003c:82018) 82B20; 05A16, 05A17, 05E10, 58E50, 82B05

Shlosman, S. B.

The Wulff construction in statistical mechanics and combinatorics.

Russian Mathematical Surveys **56** (2001), *no.* 4, 709–738.

It is always exciting to find out that a mathematical model for a certain physical phenomenon answers questions arising in quite different fields. The present comprehensive survey is devoted to demonstrating that Wulff's theory of the shape of a crystal at equilibrium solves some problems in statistical mechanics and combinatorics.

Wulff's shape is defined as a solution of the following variational problem:

$$W_\tau(M_d) = \int_{M_d} \tau(\mathbf{n}_x) ds_x = \inf,$$

where M_d is a closed and bounded manifold in $\mathbf{R}^{d+1}$, τ is a nonnegative function on M_d, $\mathbf{n}_x$ is the unit normal vector to M_d at a point $x \in M_d$ and the infimum is taken over the set of manifolds M_d, with $\mathrm{vol}(M_d) = q$. It is known that the solution of the problem exists and that it is unique (up to a shift transformation). The physical meaning of the solution is that it gives the shape of a drop (= manifold M_d) having the minimal surface tension W_τ, enclosing a given volume q.

Statistical mechanics: In the context of a 2-dimensional ferromagnetic Ising model, the solution of the Wulff variational problem gives the thermodynamical limit of the shape of the border of one of the two $\pm$ spin phases inside the other. Here it is assumed that the random contours of phases are induced by the Gibbs measure on the set of configurations. For dimensions ≥ 2 a weak analog of the above result holds. The Wulff construction is also related to the theory of metastability.

Combinatorics: Define a uniform distribution on the set of all planar Young diagrams of size N. The seminal result of A.Vershik and S. Kerov asserts that the limiting distribution, as $N \to \infty$ of such diagrams scaled by $\sqrt{N}$, is concentrated on a curve in the plane given by

$$\exp\big(-\frac{\pi x}{\sqrt{6}}\big) + \exp\big(-\frac{\pi y}{\sqrt{6}}\big) = 1.$$

This curve can be viewed as a solution to a variational problem similar to the Wulff problem. Recently, an extension of this theory to the multidimensional case was obtained.

B. L. Granovsky

From MathSciNet, December 2015

MR2045303 (2005i:53109) 53D20; 81T13, 81T45, 81T60

Nekrasov, Nikita A.

Seiberg-Witten prepotential from instanton counting. (English summary)

Advances in Theoretical and Mathematical Physics **7** (2003), *no.* 5, 831–864.

In 1994 N. Seiberg and E. Witten [Nuclear Phys. B **426** (1994), no. 1, 19–52; MR1293681 (95m:81202a); Nuclear Phys. B **430** (1994), no. 2, 485–486; MR1303306 (95m:81202b) Nuclear Phys. B **431** (1994), no. 3, 484–550; MR1306869 (95m:81203)] obtained an ansatz for the exact prepotential of $\mathcal{N} = 2$ Yang-Mills theory in four dimensions with gauge group SU(2). This celebrated solution has been extended to all gauge groups and to theories with matter, it has been rederived in the context of string theory, and has become one of the key results in theoretical physics produced in the last few years. One obvious challenge has been to find a field theory derivation of this result, or at least to verify it with usual quantum field theory methods. The Seiberg-Witten prepotential can be computed with instanton calculus, and there has been much work devoted to developing this calculus in order to test their ansatz (see [N. Dorey et al. [Phys. Rep. **371** (2002), no. 4-5, 231–459; MR1941102 (2004c:81163)] for a detailed review of these developments). Unfortunately, it has been difficult to obtain explicit results beyond instanton number two due to the complexity of the ADHM construction.

One of the results of the research on instanton calculus has been, however, that the coefficients of the Seiberg-Witten prepotential can be computed as the integral of a differential form on the moduli space of instantons, and it was suggested that localization techniques could be used to evaluate these integrals in an efficient way [see, e.g., T. J. Hollowood, J. High Energy Phys. **2002**, no. 3, No. 38, 24 pp.; MR1900823 (2003h:81221)]. In a tour de force presented in the paper under review, Nekrasov was able to pursue this program and finally produce explicit formulae for the Seiberg-Witten prepotential for gauge group $\mathrm{SU}(N)$ and general matter content.

For localization to work, one needs first a smooth and compact space, and secondly a clever group action that simplifies the computation as much as possible, with isolated fixed points. Nekrasov uses the resolution of singularities of the moduli space of instantons of $\mathbf{R}^4$, studied for example in [H. Nakajima, *Lectures on Hilbert schemes of points on surfaces*, Amer. Math. Soc., Providence, RI, 1999; MR1711344 (2001b:14007)] and related to noncommutative instantons [N. A. Nekrasov and A. S. Schwarz, Comm. Math. Phys. **198** (1998), no. 3, 689–703; MR1670037 (2000e:81198)]. In order to obtain isolated fixed points, he considers the action on the instanton moduli space of rotations in spacetime, with parameters $\epsilon_{1,2}$ in equivariant cohomology. He then obtains explicit formulae for a function $\mathcal{F}(a, \epsilon_{1,2})$ analytic in $\epsilon_{1,2}$. The zeroth order term of this function in the expansion around $\epsilon_{1,2} = 0$ is the usual Seiberg-Witten prepotential. Nekrasov also formulates the computation of this function in terms of the twisted $\mathcal{N} = 2$ theory in $\mathbf{R}^4$, equivariantly extended to take into account the action of the rotation group. The function $\mathcal{F}(a, \epsilon_{1,2})$ appears naturally when computing a natural observable of the equivariantly extended theory, and one can easily relate it to the standard prepotential by using the underlying $\mathcal{N} = 2$ structure.

It is clear that Nekrasov's function $\mathcal{F}(\epsilon_{1,2})$ contains much more information than the usual Seiberg-Witten prepotential. What is the meaning of such information? The answer proposed by Nekrasov is the following: the $\mathrm{SU}(N)$ Seiberg-Witten prepotential for pure Yang-Mills can be obtained by taking a particular limit of the prepotential of topological strings on a noncompact Calabi-Yau geometry given by an A_N fibration over $\mathbf{S}^2$. This procedure, introduced in [S. H. Katz, A. Klemm and C. Vafa, Nuclear Phys. B **497** (1997), no. 1-2, 173–195; MR1467889 (98h:81097)] is known as geometric engineering of the gauge theory, and can be extended to theories with matter as well. Therefore, the prepotential of Seiberg-Witten theory can be computed by considering spherical worldsheet instantons in this Calabi-Yau. However, one can "count" genus g instantons with the free energies of topological strings at genus g, denoted by F_g. Nekrasov proposes that, when $\epsilon_1 = -\epsilon_2 = \hbar$, the higher order terms $\mathcal{F}_g$ in the expansion

$$\mathcal{F}(\hbar) = \sum_{g=0}^{\infty} \hbar^{2g} \mathcal{F}_g$$

can be obtained from the F_g by taking the geometric engineering limit. Therefore, the equivariant extension considered by Nekrasov contains information about the embedding of the gauge theory in string theory.

This, therefore, is a very important paper with many implications. First of all, it goes halfway toward deriving the Seiberg-Witten ansatz from standard semiclassical methods in field theory. To complete the derivation, one just has to prove that Nekrasov's expression for the prepotential indeed agrees at all orders with the proposal of Seiberg and Witten, which is formulated in terms of computing periods of a meromorphic differential over a hyperelliptic curve. Since this paper appeared, three papers have been written proving this with different methods [N. Nekrasov and A. Okounkov, "Seiberg-Witten theory and random partitions", preprint, `arxiv.org/abs/hep-th/0306238`; H. Nakajima and K. Yoshioka, "Instanton counting on blowup. I. 4-dimensional pure gauge theory", preprint, `arxiv.org/abs/math.AG/0306198`; A. Braverman and P. Etingof, "Instanton counting via affine Lie algebras. II. From Whittaker vectors to the Seiberg-Witten prepotential", preprint, `arxiv.org/abs/math.AG/0409441`]. Also, the connection to string theory has triggered many results. The fact that Nekrasov's function is a limit of the all-genus free energy of topological strings in certain backgrounds was first verified in [A. Klemm, M. Mariño and S. J. Theisen, J. High Energy Phys. **2003**, no. 3, 051, 23 pp.; MR1975679 (2004e:81187)] and then derived using large-N dualities in [A. Iqbal and A.-K. Kashani-Poor, Adv. Theor. Math. Phys. **7** (2003), no. 3, 457–497; MR2030057 (2004m:81154); "SU(N) geometries and topological string amplitudes", preprint, `arxiv.org/abs/hep-th/0306032`; T. Eguchi and H. Kanno, J. High Energy Phys. **2003**, no. 12, 006, 30 pp. (electronic); MR2041169 (2005b:81178)]. Extensions of Nekrasov's results to the other classical gauge groups have been worked out in [M. Mariño and N. Wyllard, J. High Energy Phys. **2004**, no. 5, 021, 24 pp. (electronic); MR2085030] and [N. A. Nekrasov and S. V. Shadchin, Comm. Math. Phys. **252** (2004), no. 1-3, 359–391; MR2104883].

It should be mentioned that this paper is sometimes sketchy, and the interested reader may find useful other papers that clarify and extend Nekrasov's ideas on the computation of the function $\mathcal{F}(\epsilon_{1,2})$, like for example [U. Bruzzo et al., J. High Energy Phys. **2003**, no. 5, 054, 24 pp. (electronic); MR1994160 (2005a:81203); R. Flume and R. H. Poghossian, Internat. J. Modern Phys. A **18** (2003), no. 14, 2541–2563; MR1982714 (2004e:81180)] (from a physical point of view) or [H. Nakajima and K. Yoshioka, in *Algebraic structures and moduli spaces*, 31–101, Amer. Math. Soc., Providence, RI, 2004; MR2095899] (from a mathematical point of view).

Marcos Mariño

From MathSciNet, December 2015

MR2215138 (2007f:60014) 60D05; 82B26, 82B41

Kenyon, Richard; Okounkov, Andrei; Sheffield, Scott

Dimers and amoebae. (English summary)

Annals of Mathematics. Second Series **163** (2006), *no.* 3, 1019–1056.

The authors study random height functions which can be assigned to dimer configurations (perfect matchings) on a weighted planar doubly periodic bipartite graph (e.g., the square lattice). On an integer multiple of the fundamental domain with periodic boundary conditions the probability of a dimer configuration is defined to be proportional to the product of dimer weights.

In the thermodynamic (infinite volume) limit there is a one-to-one correspondence between ergodic Gibbs measures and accessible macroscopic slopes. The authors derive explicit formulas for the surface tension (free energy per two-dimensional volume) and the local structure of the Gibbs measure. The key to these quantities is the characteristic polynomial (in two variables) of the Kasteleyn matrix given by the piece of the graph in the fundamental domain alone. Its Ronkin function is the Legendre transform of the free energy. The Ronkin function can also be interpreted as an equilibrium crystal shape (ECS), i.e., the shape which minimizes the surface free energy of the height function for a given crystal volume with suitable boundary conditions. Flat pieces of the Ronkin function, therefore, correspond to facets of the ECS. The boundaries of these facets form the amoeba of the spectral curve, i.e., the set of zeros of the characteristic polynomial.

The authors classify the possible ergodic Gibbs measures as frozen, liquid, and gaseous:

A frozen phase corresponds to a crystal facet at temperature zero. Since there are no localized defects the facet is microscopically flat. In the ECS these facets are unbounded.

A gaseous phase corresponds to a facet at finite temperature, but below the roughening transition. Localized excitations form a gas of defects on the otherwise perfectly regular facet. In the ECS these facets have a bounded domain.

In between the facets the ECS consists of a rounded component. Each of its points corresponds to a liquid phase. The surface is microscopically rough, with logarithmically diverging height-height correlations. The Gibbs measure is expected to have a Gaussian limit.

It is shown that the spectral curves have a certain maximality property, which identifies them as so-called Harnack curves. This implies qualitative and quantitative properties of the models in question. For example the number of bounded facets can be related to the genus of the spectral curve. Another consequence of maximality is a universality of height fluctuations in the liquid phases. The logarithmic divergence of height-height correlations always has the prefactor $1/\pi$.

Michael Prähofer

From MathSciNet, December 2015

MR2181816 (2008a:81227) 81T60; 05E10, 11Z05, 14D21, 60C05, 81T45

Nekrasov, Nikita A.; Okounkov, Andrei

Seiberg-Witten theory and random partitions. (English summary)

The unity of mathematics, 525–596, *Progr. Math., 244*, *Birkhäuser Boston, Boston, MA*, 2006.

In the article under review, Nekrasov and Okounkov give a proof of a conjecture of the first author stated in [Adv. Theor. Math. Phys. **7** (2003), no. 5, 831–864; MR2045303 (2005i:53109)]. In that paper, Nekrasov gave a field-theoretic derivation of a celebrated result of N. Seiberg and E. Witten [Nuclear Phys. B **426** (1994), no. 1, 19–52; MR1293681 (95m:81202a); erratum, Nuclear Phys. B **430** (1994), no. 2, 485–486; MR1303306 (95m:81202b)], computing the prepotential that encodes the low-energy behavior of $\mathcal{N}=2$ supersymmetric SU(N) Yang-Mills theory. Nekrasov introduced a partition function $Z(a,\epsilon_1,\epsilon_2;q)=\sum_{k\le 0} q^k \int_{\mathcal{M}_{k,N}} 1$, where $\mathcal{M}_{k,N}$ are the moduli-spaces of framed SU(N) instantons on $\mathbb{R}^4$ (or desingularizations thereof), and the integral is formally defined in the equivariant cohomology with respect to a torus action on the moduli spaces. The prepotential $\mathcal{F}^{\rm inst}$ is then defined by Nekrasov through the relation $Z(a,\epsilon_1,\epsilon_2;q)=\exp(\frac{\mathcal{F}^{\rm inst}(a,\epsilon_1,\epsilon_2;q)}{\epsilon_1\epsilon_2})$. Seiberg and Witten earlier introduced a prepotential $\mathcal{F}^{\rm SW}(a;q)$ expressed in terms of period integrals on a family of curves. From physical motivations Nekrasov then claimed that $\mathcal{F}^{\rm SW}(a;q)=\mathcal{F}^{\rm inst}(a,\epsilon_1,\epsilon_2;q)|_{\epsilon_1,\epsilon_2=0}$. As each side of this equation has a rigorous mathematical definition, one can interpret this as a conjecture in geometry, and it is this conjecture that the paper under review sets out to prove. Two other proofs of this conjecture have also appeared in the literature [H. Nakajima and K. Yoshioka, Invent. Math. **162** (2005), no. 2, 313–355; MR2199008 (2007b:14027a); A. Braverman and P. I. Etingof, in *Studies in Lie theory*, 61–78, Progr. Math., 243, Birkhäuser Boston, Boston, MA, 2006; MR2214246 (2007b:14026)], using very different techniques.

The proof given here is statistical in nature, and relies heavily on the combinatorics of the moduli spaces. The equivariant integrals over the moduli-spaces $\mathcal{M}_{k,N}$ localize onto the fixed points for the torus action. For the suitable desingularizations, which have a mathematical interpretation as moduli spaces of framed torsion-free sheaves on $\mathbb{CP}^2$, these fixed points are enumerated by N-tuples of partitions. In the case of $N=1$, the weights of the corresponding fixed points determine a deformation of the Plancherel measure on partitions. It is a classical theorem of B. F. Logan, Jr. and L. A. Shepp [Advances in Math. **26** (1977), no. 2, 206–222; MR1417317 (98e:05108)] and A. M. Vershik and S. V. Kerov [A. M. Veršik and S. V. Kerov Dokl. Akad. Nauk SSSR **233** (1977), no. 6, 1024–1027; MR0480398 (58 #562)] that as k goes to infinity the scaled profiles of the partitions of k converge onto some universal function in the Plancherel measure. The proof given here can be seen as a generalization of this result. The authors argue that in the limit $\epsilon_1,\epsilon_2\to 0$ the partition function Z becomes an integral over a certain function space that contains the profiles of the tuples of partitions (being the sum of the shifted profiles of the constituents' partitions), and that this integral is dominated by the saddle point f_* of a functional on this space. Furthermore, each such critical point corresponds to the induced function on the boundaries of a conformal map from the upper-half plane to a half-strip. An explicit construction is then given for such a map as a sequence of conformal maps and in terms of this construction the relationship with the Seiberg-Witten prepotential is obtained. The limit shape looks like a piecewise analytic function, which on alternating intervals is either linear (on the gaps) or has positive second derivative (on the bands). The authors relate these gaps and bands to those in the spectrum of the Lax operator for the corresponding periodic Toda chain whose spectral curves are the Seiberg-Witten curves.

Further on, a dual partition function Z^D is defined from Z, and it is shown that Z^D can be expressed as a matrix element for a representation for the corresponding affine Lie algebra. As this formula naturally generalizes it leads to a conjecture for the (dual) partition functions of other gauge groups.

In the latter parts of the article a further modification of the ideas above is given to encompass the cases beyond pure gauge theory, including matter.

The paper draws upon an impressively broad body of knowledge in theoretical physics, algebraic geometry, integrable systems, representation theory, analysis, combinatorics and probability, and beautifully uses their often very intricate interactions. Another exposition of the central part of the article was given by Okounkov [in *International Congress of Mathematicians. Vol. III*, 687–711, Eur. Math. Soc., Zürich, 2006; MR2275703 (2008h:81167)], providing some further details and comments.

{For the entire collection see MR2182597 (2006e:00016)}.

Johan A. Martens

From MathSciNet, December 2015

MR2566160 (2011d:14012) 14D21; 14M25, 14N35, 81T13, 81T60

Gasparim, Elizabeth; Liu, Chiu-Chu Melissa

The Nekrasov conjecture for toric surfaces.

Communications in Mathematical Physics **293** (2010), *no.* 3, 661–700.

The Nekrasov conjecture [N. A. Nekrasov, Adv. Theor. Math. Phys. **7** (2003), no. 5, 831–864; MR2045303 (2005i:53109)], based on the comparison of the infrared and the ultraviolet limit along a renormalization-group flow in the Wilson theory space of the SUSY gauge theory, relates the partition function of a $d = 4, N = 2$ super Yang-Mills theory to a related Seiberg-Witten prepotential ([N. Seiberg and E. Witten, Nuclear Phys. B **426** (1994), no. 1, 19–52; MR1293681 (95m:81202a); erratum, Nuclear Phys. B **430** (1994), no. 2, 485–486; MR1303306 (95m:81202b)]; see also [J. Terning, *Modern supersymmetry*, Oxford Univ. Press, Oxford, 2006; MR2292490 (2008f:81002)] for an updated review of SUSY theory). For the case of super Yang-Mills theory on $\mathbb{R}^4$, the conjecture was proved independently by Nekrasov and A. Okounkov [in *The unity of mathematics*, 525–596, Progr. Math., 244, Birkhäuser Boston, Boston, MA, 2006; MR2181816 (2008a:81227)] and by H. Nakajima and K. Yoshioka [Invent. Math. **162** (2005), no. 2, 313–355; MR2199008 (2007b:14027a); Transform. Groups **10** (2005), no. 3-4, 489–519; MR2183121 (2007b:14027b)]. The current work extends the line to the case of (complex) noncompact toric surfaces.

Let $X_0 = X - l_\infty$ be an open toric surface that can be compactified, as toric variety, to a smooth projective toric surface X by adding a line $l_\infty \simeq \mathbb{P}^1$ at infinity with the self-intersection number $l_\infty \cdot l_\infty > 0$, and let $\mathfrak{M}_{r,d,n}(X, l_\infty)$ be the moduli space of torsion-free sheaves of rank r over X, with Chern classes $c_1 = d$ and $c_2 = n$, that is framed over l_∞. The built-in $T_t := (\mathbb{C}^*)^2$ on X and X_0 and the built-in maximal torus $T_e := (\mathbb{C}^*)^r \subset \mathrm{GL}(r, \mathbb{C})$-action on the framing of sheaves induces a $\tilde{T} := T_t \times T_e$-action on the smooth variety $\mathfrak{M}_{r,d,n}(X, l_\infty)$ with isolated fixed points. Let $T_{\mathfrak{M}}$ be the tangent bundle of $\mathfrak{M}_{r,d,n}(X, l_\infty)$, and $V_{\mathfrak{M}} := (R^1 p_2)_*(\mathcal{E} \otimes p_1^*(\mathcal{O}_X(-l_\infty)))$ be the natural bundle of $\mathfrak{M}_{r,d,n}(X, l_\infty)$, where $\mathcal{E}$ is the universal sheaf over $X \times \mathfrak{M}_{r,d,n}(X, l_\infty)$, and $p_1\colon X \times \mathfrak{M}_{r,d,n}(X, l_\infty) \to X$ and $p_2\colon X \times \mathfrak{M}_{r,d,n}(X, l_\infty) \to \mathfrak{M}_{r,d,n}(X, l_\infty)$ are the projection maps. Both $T_{\mathfrak{M}}$ and $V_{\mathfrak{M}}$ are $\tilde{T}$-equivariant. Fix presentations $H^*_{T_t}(\mathrm{pt}; \mathbb{Q}) \simeq \mathbb{Q}[\epsilon_1, \epsilon_2]$ and $H^*_{T_e}(\mathrm{pt}; \mathbb{Q}) \simeq \mathbb{Q}[a_1, \ldots, a_r]$ of the $T_\bullet$-equivariant cohomologies via the canonical generators.

Denote $(a_1, \ldots, a_r)$ by $\overrightarrow{a}$. Then, a prototypical statement of the Nekrasov conjecture says that:

(a) $\mathcal{F}^{\cdots}_{X_0,A,B,d}(\epsilon_1, \epsilon_2, \overrightarrow{a}, \overrightarrow{m}; \Lambda)$ is analytic in ϵ_1, ϵ_2 near $\epsilon_1 = \epsilon_2 = 0$.

(b) $\lim_{\epsilon_1,\epsilon_2 \to 0} \mathcal{F}^{\cdots}_{X_0,A,B,d}(\epsilon_1, \epsilon_2, \overrightarrow{a}, \overrightarrow{m}; \Lambda) = k\mathcal{F}^{\cdots}_0(\overrightarrow{a}, \overrightarrow{m}, \Lambda)$.

Here,

(i) A and B are multiplicative characteristic classes, each of which is associated to a formal power series in one variable; denote the corresponding $\tilde{T}$-equivariant cohomology classes for sheaves $(\cdot)$ over $\mathfrak{M}_{r,d,n}(X, l_\infty)$ by $A_{\tilde{A}}(\cdot)$ and $B_{\tilde{A}}(\cdot)$ respectively;

(ii) $$\mathcal{F}^{\cdots}_{X_0,A,B,d}(\epsilon_1, \epsilon_2, \overrightarrow{a}, \overrightarrow{m}; \Lambda) := - uv \log Z^{\cdots}_{X_0,A,B,d}(\epsilon_1, \epsilon_2, \overrightarrow{a}, \overrightarrow{m}; \Lambda) \in \mathbb{Q}((\epsilon_1, \epsilon_2, a_1, \ldots, a_r, m_1, \ldots, m_N))[[\Lambda]]$$

with

- $u, v \in \mathbb{Z}\epsilon_1 \oplus \mathbb{Z}\epsilon_2$ being the weights of the T_t-action on the fiber of the normal bundle $N_{l_\infty/X}$ of l_∞ in X over the two T_t-fixed points on l_∞,
- the partition function

$$Z^{\cdots}_{X_0,A,B,d}(\epsilon_1, \epsilon_2, \overrightarrow{a}, \overrightarrow{m}; \Lambda) := \sum_{n \geq 0} \int_{\mathfrak{M}_{r,d,n}(X,l_\infty)} \Lambda^{\dim_{\mathbb{C}} \mathfrak{M}_{r,d,n}(X,l_\infty)} A_{\tilde{T}}(T_{\mathfrak{M}}) B_{\tilde{T} \times T_N}(V_{\mathfrak{M}} \times M),$$

with the integral defined by formally applying the Atiyah-Bott localization formula (here M is the fundamental representation of $\mathrm{U}(N)$ and T_m is the maximal torus of $\mathrm{U}(N)$ with $H^*_{T_m}(\mathrm{pt}; \mathbb{Q}) \simeq \mathbb{Q}[m_1, \ldots, m_N]$);

and

(iii) $\mathcal{F}^{\cdots}_0(\overrightarrow{a}, \overrightarrow{m}, \Lambda)$ is the related Seiberg-Witten prepotential of type $(A, B, \overrightarrow{m})$ and $k = l_\infty \cdot l_\infty > 0$.

Eight cases of the conjecture are stated and proved in detail. For the instanton part, i.e. with $^{\cdots}$ replaced by $^{\mathrm{inst}}$, the authors state and prove the conjecture in the following four cases:

(1) 4d pure gauge theory: $A = B = 1$, $\overrightarrow{m} = \varnothing$;

(2) 4d gauge theory with N_f fundamental matter multiplet: $A = 1$, $B = E_{\overrightarrow{m}}(V_{\mathfrak{M}})$ the T_m-equivariant Euler class of $V_{\mathfrak{M}} \times M$, $\vec{m} = (m_1, \ldots, m_{N_f})$;

(3) 4d gauge theory with one adjoint matter multiplet: $A = E_m(T_{\mathfrak{M}})$, $B = 1$, $\overrightarrow{m} = (m)$;

(4) 5d gauge theory compactified on a circle of circumference β: $A = \widehat{A}_\beta(T_{\mathfrak{M}})$ the $\widehat{A}_\beta$-genus of $T_{\mathfrak{M}}$, $B = 1$, $\overrightarrow{m} = \varnothing$; $\mathcal{F}$ now depends on the additional parameter β.

The statement and the proof of these four cases follow from localization calculation (cf. Sections 4 and 5 of the paper). The necessary ingredients—a basic understanding of the framed moduli space $\mathfrak{M}_{r,d,n}(X, l_\infty)$ and the induced $\tilde{T}$-action on $\mathfrak{M}_{r,d,n}(X, l_\infty)$ and its fixed-point locus—are given in Sections 2 and 3.

The perturbative part, i.e. with $^{\cdots}$ replaced by $^{\mathrm{pert}}$, comes from the difference between framed instantons on X and unframed instantons on X_0. Let $\mathfrak{M}_0 = \mathfrak{M}_{r,d,n}(X_0)$ be the moduli space of torsion-free sheaves on X_0 of type (r, d, n).

Then both the virtual tangent bundle $T^{\text{vir}}_{\mathfrak{M}_0}$ and the virtual natural bundle $V^{\text{vir}}_{\mathfrak{M}_0}$ have to be considered. Furthermore, the evaluation of the required multiplicative classes at such bundles requires a zeta-function regularization. With these taken care of, the authors state and prove the conjecture for the following four cases (see Section 6):

(1) 4d pure gauge theory;
(2) 4d gauge theory with N_f fundamental matter multiplet;
(3) 4d gauge theory with one adjoint matter multiplet;
(4) 5d gauge theory compactified on a circle of circumference β.

To make this article more self-contained, the authors provide in the appendix a very compact overview of the Kobayashi-Hitchin correspondence/Donaldson-Uhlenbeck-Yau theorem relating gauge instantons to stable vector bundles; equivariant cohomology and the Atiyah-Bott localization formula; and the Seiberg-Witten prepotential for gauge groups $\mathrm{SU}(r)$, $r \geq 2$.

Chien-Hao Liu

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Universality in the 2D Ising model and conformal invariance of fermionic observables. (English summary)

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The article provides a self-contained presentation and analysis of specific observables defined on two representations of a broad generalization of the family of 2-d statistical systems known as the Ising model. Both the critical spin representation and the random cluster (FK) representation of the Ising model are defined with concise notation and carefully labeled figures that accompany the development of the two representations. The authors then proceed to establish results concerning the observables (called fermions) defined on these models that put the concept of universality, often stated vaguely in the literature, in precise terms.

Conjectures of invariance (scale invariance, Möbius invariance and full conformal invariance) and universality in 2-d statistical systems date back to the mid 20th century. Until recently, mathematical literature on the 2-d Ising model has been primarily focused on subdomains of a square lattice. Here, Chelkak and Smirnov clearly articulate a rich family of 2-d lattices, known as isoradial graphs, on which the universality and conformal invariance are established for limits of fermionic observables taken as the mesh width of these isoradial graphs becomes small.

Here, the isoradial graphs of interest are lattice approximations of 2-d simply connected domains with neighboring vertices all equidistant from common centers. Of primary interest is the behavior of the Ising models defined on these graphs as the lattice approximations become increasingly fine on some fixed, simply connected domain. One consequence of conformal invariance is that certain "crossing probabilities" associated with the random interfaces of this model can be calculated directly for one marked domain (the half-plane, with 4 boundary points identified) and then calculated for any other marked, simply connected domain via the conformal modulus that relates the two marked domains. This result, stated precisely in the context of the Ising model interface, is exactly the content of Theorem C of this paper.

This is the second of a series of papers dedicated to establishing universality of the critical Ising model with respect to isoradial graphs. Results from the first paper of this series [D. Chelkak and S. K. Smirnov, Adv. Math. **228** (2011), no. 3, 1590–1630; MR2824564 (2012k:60137)] play a crucial role in the derivations here; however, many of these results are already summarized within the current paper and the definitions and statements repeated in both of the papers are more polished here.

In summary, the fermionic observables identified here converge as the mesh width of the isoradial graphs become small irrespective of the exact structure of the isoradial graph. Using any two designated boundary points of the underlying domain, the scaling limit also yields a random interface path defined between the two points; crossing probabilities associated with these interfaces are also shown to converge in the fine mesh limit irrespective of the exact structure of the graph. In this sense, the authors prove universality for a family of (weakly) convergent statistical systems indexed by the simply connected sets in the plane.

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