BOOK REVIEWS

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 53, Number 3, July 2016, Pages 495–506
http://dx.doi.org/10.1090/bull/1523
Article electronically published on February 2, 2016

Metric embeddings: bilipschitz and coarse embedddings into Banach spaces, by M. I. Ostrovskii, de Gruyter Studies in Mathematics, Vol. 49, de Gruyter, Berlin, 2013, xii+372 pp., ISBN 978-3-11-026401-2, US$154.00.

1. Geometry of metric and Banach spaces: an intertwined story

In his 1906 PhD thesis [18], Fréchet introduced the axiomatic definition of a metric that we use today. The simplicity of the axioms defining a metric space have two obvious consequences that have excited and frustrated mathematicians from the beginning. The good news is that metric spaces are ubiquitous. The bad news is that, due to the generality and abstraction of the concept, a general structural theory of metric spaces is as out of reach today as it was to Fréchet. So mathematicians investigated specific classes of metric spaces with additional and more palpable structures or with additional constraints. In particular, Banach introduced the axioms for complete normed spaces in his 1920 PhD dissertation. Fundamental results about complete normed spaces, termed Banach spaces by Fréchet in his 1928 monograph [20], are contained in Banach's famous 1932 book [7]. It is not much of a stretch to say that metric spaces and Banach spaces now provide the conceptual framework in which virtually all mathematical analysts work.

Since the appearance of Banach's book, researchers have built a rich and elegant theory of the linear structure of Banach spaces and linear operators between them. While the investigation of linear properties has received the most attention from specialists, another direction developed around the desire to understand how much of the linear structure could be recovered just by knowing the metric (or merely the topological) structure of a Banach space. Already in 1932 Mazur and Ulam [47] gave a strong positive result in this direction. They proved that if there were a bijective map from a real Banach space onto another Banach space that preserves distances, then the Banach spaces would be the same up to isometric (linear) isomorphism. That is, in the extreme case where complete information on the pairwise distances is known, one can recover the linear structure! Allowing some type of distortion is the center of research in what is now commonly called the nonlinear geometry of Banach spaces. Challenging rigidity problems were raised by Bessaga and Pełczyński in 1961 [11]. If two Banach spaces are isomorphic in some nonlinear category, are they isomorphic in the Banach category, where the objects are Banach spaces and the morphisms are bounded linear maps? The Mazur–Ulam theorem implies that if two Banach spaces are isomorphic in the isometric category in which the objects

2010 *Mathematics Subject Classification.* Primary 46B85; Secondary 05C12, 30L05, 46B20.

are metric spaces and the morphisms are (nonlinear) contractions, then they are isomorphic in the linear isometric category in which the morphisms are contractive *linear* mappings. It took more time to settle the question that lies at the other end of the spectrum; namely, when are two Banach spaces that are isomorphic in the topological category isomorphic as Banach spaces? The answer is strongly negative due to a beautiful result of Kadets from 1967 [29]: any two separable infinite-dimensional Banach spaces are actually homeomorphic, and therefore indiscernible in the topological category! (Later Toruńczyk [61] extended this to the nonseparable setting.) The rigidity problem for two other natural categories, the uniform category and the Lipschitz category, was ignited by a young J. Lindenstrauss in 1964 [39]. The linearization of uniformly continuous projections and of uniformly continuous liftings allowed Lindenstrauss to prove, among many other things, that $C(K)$ is not uniformly homeomorphic to a reflexive Banach space, and that $L_q(0,1)$ is not uniformly homeomorphic to a Hilbert space when $q > 2$. This began a line of research that continues to the present day. See the monographs [10], [40] for a detailed account of this theory.

A new branch of Banach space theory, called local theory, started in the late 1960s. Local theory is devoted to the study of quantitative parameters of finite-dimensional normed spaces and how the asymptotics of these parameters for a suitable paving of an infinite-dimensional space affect the structure of the infinite-dimensional space. For example, ℓ_p and $L_p(0,1)$ are both paved by $(\ell_p^n)_{n=1}^{\infty}$, so from the local point of view they are very similar even though when $p \neq 2$ the spaces are very different as Banach spaces. More generally, a property of a Banach space that depends only on its finite-dimensional subspaces and not on how these spaces are put together is called a local property. Dvoretzky's theorem, proved in the late 1950s [16], which says that for every $\varepsilon > 0$, every infinite-dimensional Banach space contains $(1+\varepsilon)$-isomorphic copies of Euclidean spaces of every (finite) dimension, is a cornerstone in the local theory, as well as the precursor of contemporary convex geometry. The local properties of linear type and linear cotype, developed under the lead of Maurey and Pisier [46] in the 1970s, led to many new results and also gave a new look to the proofs of fundamental results in the linear theory that were previously based on infinite-dimensional considerations. In the late 1960s and mid-1970s, works of Enflo and Ribe established a link between the metric and the local structures of Banach spaces. Enflo [17] showed in 1969 that $L_p(0,1)$ and $L_q(0,1)$ are not uniformly homeomorphic when $1 \leq p \neq q \leq 2$. Enflo exhibited invariants, e.g., the roundness of a metric space, that served as obstructions to the existence of uniform homeomorphisms. Enflo's metric/geometric approach does not reduce the problem in the uniform category to a problem in the linear category via a linearization technique, so it is less dependent on the level of maturity of the linear theory and has a potentially wider field of applicability. Enflo's proof also said something important about the structure of certain families of finite metric spaces by producing the first family of finite metric spaces with unbounded cardinality that could not be embedded into a Hilbert space with uniformly bounded distortion. Ribe's rigidity theorem told us that Enflo's idea would not be an isolated act but merely the ancestor of a profound research program. Ribe [59], using the metric/geometric approach, proved that if two Banach spaces are uniformly homeomorphic, then they have essentially the same finite-dimensional subspaces, i.e., they are locally the same. (Later Heinrich and Mankiewicz [25] gave a linearization proof of this.)

The fact that the local structure of a Banach space is determined by its uniform structure suggested that one could discover metric invariants characterizing local properties. In the mid-1980s there was an important transformation in the geometric study of Banach spaces and metric spaces. At least three seminal articles were published between 1984 and 1986. The first one [26] investigated the Lipschitz extension problem for maps defined on finite metric spaces with values in Hilbert spaces, and it attempted to study nonlinear analogues of classical extension results for linear maps. To achieve their goal the authors needed a lemma, now termed the Johnson–Lindenstrauss Lemma, that says that every finite subset of a Hilbert space can be represented in a slightly distorted way in a Euclidean space whose dimension is significantly smaller than the number of points. In 1985 Bourgain [12] proved that every finite metric space can be represented inside a Hilbert space while controlling the distortion to a reasonable level. Both of these theorems are now used, e.g., by computer scientists who are interested in the design of algorithms. In 1986, Bourgain [13] struck again by showing that superreflexivity (a local property of Banach spaces that is equivalent to being isomorphic to a uniformly convex space) has a purely metric characterization.

A graph is a metric space when it is endowed with its canonical graph metric, where the distance between two vertices is the number of edges of the shortest path connecting them. In 1995 [42] the computer scientists Linial and Rabinovich and the mathematician London established a clear and beautiful connection between the faithful representability of metric spaces of a certain type, especially graphs, into well behaved Banach spaces and the design of efficient algorithms. For instance, they showed how their new geometric approach can provide near-tight cuts for multicommodity flow problems in deterministic polynomial time. Given a finite connected graph, finding a cut, i.e., a partition, of the graph into two pieces that minimizes in a certain sense the number of edges connecting the two pieces is a difficult problem from the complexity point of view. However, the sparsest cut problem can be relaxed and an approximate cut can be find in a reasonable time. Linial, London, and Rabinovich explained that the quality of the approximate solution is related to the possibility of representing without too much distortion the graph, considered with its graph metric, into finite-dimensional Banach spaces. The lower the distortion and the dimension are, the better the approximation. Luckily they had everything at their disposal since the Johnson–Lindenstrauss Lemma took care of the low-dimensionality while Bourgain's embedding theorem took care of the low-distortion issue. This result spurred an explosion of interest in the theory of low-distortion embeddability into low-dimensional spaces. Chapter 15 of Matoušek's monograph [44] on discrete geometry contains a beautiful account of the theory. It is also enlightening to consult [41] and [52] to get a sense of what the fast developing field of finite quantitative metric geometry is about.

Almost simultaneously, another emerging field was about to open new opportunities for Banach space geometers. Gromov laid down in his monograph [23] a program that relates the algebraic structure of a finitely generated group to the large-scale structure of its Cayley graph considered with its word metric. One striking piece of Gromov's program [22], which actually predates his monograph, is his theorem about polynomial growth. A group is said to have polynomial growth if the number of group elements in a ball is polynomial in the radius as the radius tends to infinity. Gromov characterized finitely generated groups of polynomial growth (a geometric property) as those groups that have nilpotent subgroups of finite index (an algebraic property). Connections between geometric group theory and topology were deepening. Since the early 1980s, Connes and others had been developing the subject of noncommutative geometry, a natural generalization of Riemannian geometry. Part of Connes' noncommutative program [15] bridges classical geometry and topology. An important feature of the program is the Baum–Connes conjecture, which suggests that two objects (e.g., an analytic one and a topological one) that one can associate to a group can be identified. The Baum–Connes conjecture is closely related to an older conjecture of Novikov on the homotopy invariance of higher signatures that were in turn motivated by the Borel conjecture in topology, which asserts that an aspherical closed manifold is determined, up to homeomorphism, by its fundamental group. In 1995 Gromov hinted ([24], p. 67) that understanding the groups whose large-scale geometry is compatible in a certain sense with the geometry of a Hilbert space or a superreflexive (i.e., admits an equivalent uniformly convex norm) Banach space should be interesting regarding the Novikov conjecture. Gromov's intuition has since been shown to be accurate (see [62], [36], and [37] for particularly striking examples). This research has drawn the attention of Banach space geometers as well as geometric group theorists to the coarse geometry of Banach spaces.

2. Faithful embeddings of metric spaces into Banach spaces

Similar to how the advances in the theory of linear operators between Banach spaces deepened our understanding of the Banach spaces themselves and vice-versa, it has been productive to investigate the structure of metric spaces by studying maps between them. Two moduli encode and quantify the properties of a map f (called a metric embedding in the sequel) from a metric space (X, d_X) into another metric space (Y, d_Y). The expansion modulus

$$\omega_f(t) = \sup\{d_Y(f(x), f(y))\colon d_X(x,y) \le t\}$$

quantifies how much the map expands distances, while the compression modulus

$$\rho_f(t) = \inf\{d_Y(f(x), f(y))\colon d_X(x,y) \ge t\}$$

witnesses the contraction of distances. In particular, for every $x, y \in X$,

$$\rho_f(d_X(x,y)) \le d_Y(f(x), f(y)) \le \omega_f(d_X(x,y)).$$

Fréchet [19] considered isometric embeddings, i.e., mappings for which $\rho_f(t) = \omega_f(t) = t$ for every $t \in (0,\infty)$:

Theorem. *Every separable metric space embeds isometrically into ℓ_∞.*

To prove this, take a dense sequence $(x_n)_{n\ge1}$ in X. Then the map $x \mapsto (d_X(x,x_n) - d_X(x_n,x_1))_{n\ge1}$ does the job. Somewhat less expected is the fact that Fréchet-type embeddings continue to play a prominent role in the construction of metric embeddings; in fact, Bourgain's proof of his embedding theorem starts with a variation of Fréchet's embedding. Fréchet's embedding theorem says that the nonseparable Banach space ℓ_∞ is isometrically universal for the class of separable metric spaces. The fact that the separable Banach space $C([0,1])$ is also isometrically universal is a simple consequence of the Banach–Mazur theorem. It was thus known since the early ages of metric space theory that every separable metric space can be represented without distortion into "large" Banach spaces. However, demanding a flawless representation of every finite metric space into a universal host space precludes the host space from enjoying nice properties. For example, it is a simple exercise to show that a tripod with its graph metric does not embed isometrically into any uniformly convex Banach space. It is therefore natural to consider various relaxations of the isometric embeddability condition in order to obtain universality results where the universal space possesses desirable geometric properties. The tradeoff, between faithfulness of the embedding versus geometric regularity of the host space is a battle that geometers of all kinds have fought. Ostrovskii discusses in his monograph two natural and important relaxations, namely bi-Lipschitz embeddings and coarse embeddings.

2.1. **Rudiments of bi-Lipschitz geometry.** A *bi-Lipschitz embedding* from (X, d_X) to (Y, d_Y) is a map $f\colon X \to Y$ such that there exist a scaling factor $s \in (0, \infty)$ and a constant $D \in [1, \infty)$, and for all $x, y \in X$,

$$s \cdot d_X(x, y) \le d_Y(f(x), f(y)) \le D \cdot s \cdot d_X(x, y).$$

In this case we use the convenient notation $X \underset{D-\mathrm{Lip}}{\hookrightarrow} Y$. The Y-distortion of X, denoted $c_Y(X)$, is defined as $c_Y(X) = \inf\{D\colon X \underset{D-\mathrm{Lip}}{\hookrightarrow} Y\}$.

The Lipschitz embedding problem for finite metric spaces is quantitative in nature since obviously every n-point metric space admits a bi-Lipschitz embedding into any metric space containing at least n points, and an obvious variant of Fréchet's embedding yields that every finite metric space on n points embeds isometrically into ℓ_∞^{n-1}. The Hamming cubes form the first family of finite metric spaces whose embeddings into a Hilbert space was shown to incur large distortion. The n-dimensional Hamming cube H_n is the set $\{0,1\}^n$ equipped with the Hamming distance: that is, the restriction of the ℓ_1-distance to the elements of the Hamming cube considered as elements in ℓ_1^n. Enflo proved [17]:

Theorem. *For every $n \in \mathbb{N}$, one has $c_{\ell_2}(H_n) \ge \sqrt{n}$.*

It is easily seen that Enflo's lower bound is tight for the Hamming cubes, and since H_n has 2^n points it shows that $c_{\ell_2}(H_n) \ge \sqrt{\log |H_n|}$. Bourgain showed the following upper bound for general metric spaces [12].

Theorem. *Let $p \in [1, \infty)$. There exists $\alpha \in (0, \infty)$ such that for every finite metric space X one has $c_{\ell_p}(X) \le \alpha \log(|X|)$.*

The gap between Enflo's lower bound and Bourgain's upper bound was closed by Linial, London, and Rabinovich [42] as they realized that a sequence of expander graphs could not be embedded with a better distortion than the one achievable via Bourgain's embedding.

Theorem. *Let $p \in [1, \infty)$. Let $(\Gamma_n)_{n\in\mathbb{N}}$ be a sequence of expander graphs. One has $c_{\ell_p}(\Gamma_n) \ge \beta \log(|\Gamma_n|)$, for some positive constant β independent of n.*

The geometric insight in [42] offers a way to understand the approximation ratios achieved by linear programming or semidefinite programming relaxations for cut problems. While the linear programming relaxation involves general metrics and Bourgain's upper bound, the semidefinite programming relaxation involves metrics of negative type. A metric space (X, d_X) is said to be of negative type if the metric space $(X, \sqrt{d_X})$ embeds isometrically into a Hilbert space. It turns out that for this restricted class of metrics Bourgain's general upper bound can be significantly sharpened. Arora, Lee, and Naor [2] proved the following embedding theorem and derived some important algorithmic applications.

Theorem. *There exists $\gamma \in (0,\infty)$ such that for every finite metric space X of negative type one has $c_{\ell_2}(X) \le \gamma\sqrt{\log(|X|)}\log\log(|X|)$.*

That the upper bound in [2] is nearly tight is shown by Enflo's lower bound, since it is well known that every subset of $L_1(0,1)$ is of negative type. For algorithmic applications it is important to keep the dimension of the target space small. This is achieved via the Johnson–Lindenstrauss Lemma [26]:

Theorem. *For every $\varepsilon \in (0,\infty)$, there exists a constant $K := K(\varepsilon) \in (0,\infty)$ such that for every finite subset $X \subset \ell_2$, if $k \ge K\log(|X|)$, then $c_{\ell_2^k}(X) \le 1+\varepsilon$.*

As far as infinite metric spaces are concerned, in 1974 Aharoni [1] showed that in the Lipschitz category there is a "small" separable Banach space that is universal for all separable metric spaces. After several improvements on the distortion bound due to Assouad [3] and Pelant [58], Kalton and Lancien [35] finally showed that the optimal bound in Aharoni's embedding theorem is 2.

Theorem. *If M is a separable metric space, then $M \underset{2-\mathrm{Lip}}{\hookrightarrow} \mathrm{c}_0$.*

The fact that the space c_0 has, for instance, the geometric property of asymptotical flatness, whereas ℓ_∞ and $C([0,1])$ fail this property, is a typical example of the tradeoff alluded to above.

In order to improve the easier direction in Bourgain's characterization of superreflexivity, the barycentric gluing technique was introduced in [8]. This technique, inspired by the work of Ribe [60], was fruitfully implemented in various situations (see for instance [9]) and had some unexpected consequences for the geometry of infinite metric spaces. In particular it is a crucial ingredient in the proof of Ostrovskii's finite determinacy of the bi-Lipschitz embeddability for locally finite metric spaces [56].

Theorem. *The bi-Lipschitz embeddability of a locally finite metric space into an infinite-dimensional Banach space Y is finitely determined.*

2.2. **Rudiments of the Ribe Program.** A Banach space X is said to be crudely finitely representable in Y if for some $\lambda \in [1,\infty)$ and for every finite-dimensional subspace $F \subset X$ there exists a linear map $T\colon F \to Y$ such that for every $x \in F$, $\|x\|_X \le \|T(x)\|_Y \le \lambda\|x\|_X$. Ribe's rigidity theorem below [59] gave birth to the Ribe Program.

Theorem. *Let X and Y be two Banach spaces. If there is a bijection from X onto Y that is uniformly continuous with a uniformly continuous inverse, then X is crudely finitely representable in Y and Y is crudely finitely representable in X.*

One early step in the Ribe Program is Bourgain's characterization of superreflexivity [13] in terms of the sequence of combinatorial binary trees B_n of height n equipped with their graph metrics:

Theorem. *A Banach space X is superreflexive if and only if $\sup_{n\in\mathbb{N}} c_X(B_n) = \infty$.*

Nowadays there are many other characterizations of superreflexivity in terms of various sequences of finite or infinite graphs. The Johnson–Schechtman characterization [28] in terms of the sequence of diamond graphs is presented in the monograph under review.

In 1992 Ball [5] made a striking advance in the Ribe Program. Ball introduced two metric invariants, Markov type and Markov cotype, that allow for an analogue in the metric setting of a celebrated linear extension result of Maurey [45]. Ball's notion of Markov type has applications in, for example, the compression theory of finitely generated groups (see [4] for an enlightening example). The quest for metric analogues of the important notions of Rademacher type and cotype was a central question in the Ribe Program. Candidates for a metric notion of type, which coincides with the linear notion for Banach spaces, appeared in [17] and [14], and the problem was eventually settled by Mendel and Naor in [48] as what can be seen as a byproduct of their breakthrough discovery of a metric analogue of the linear cotype [49]. This line of research is still very active as evidenced by the recent article of Naor and Schechtman about X_p-inequalites [53], and also their article on KS-inequalities [54]. We refer the reader to the excellent surveys by Ball [6] and Naor [51] for a more detailed account of the Ribe Program.

2.3. **Rudiments of coarse geometry.** Coarse geometry ignores small distances, as bounded sets are equivalent to points in this geometry. In particular, coarse geometry is meaningless for bounded metric spaces. The intuition behind the coarse geometry is better grasped when we look at the spaces from a large distance. For instance, one would not be able to distinguish $\mathbb{R}$ and $\mathbb{Z}$ when we look at them from a very large distance, and hence in the coarse category, $\mathbb{Z}$ and $\mathbb{R}$ are coarsely equivalent. An (unbounded) metric space (X, d_X) is said to be *coarsely embeddable* into (Y, d_Y) if there exists $f\colon X \to Y$ such that $\lim_{t\to\infty} \rho_f(t) = \infty$ and $\omega_f(t) < \infty$ for all $t > 0$.

In 2000, Yu turned the spotlight on the coarse embeddability of metric spaces into Banach spaces when he proved in [62] that the coarse Baum–Connes conjecture (and in particular the coarse Novikov conjecture) holds true for uniformly locally finite metric spaces admitting a coarse embedding into a Hilbert space. Recall that a metric space is uniformly locally finite, or has bounded geometry, if for any $r > 0$ there exists a finite number $N = N(r) > 0$ such that any closed ball of radius r has at most N points. This deep theorem was subsequently generalized in [36]:

Theorem. *The coarse geometric Novikov conjecture holds for every discrete bounded geometry metric space admitting a coarse embedding into a superreflexive Banach space.*

It is known ([27], [49]) that there are superreflexive Banach spaces, e.g., ℓ_q for $q > 2$, which do not coarsely embed into a Hilbert space. However, whether the Kasparov–Yu superreflexivity theorem is a strict improvement of Yu's Hilbertian theorem (that is, whether there exists an infinite bounded geometry subset of a superreflexive Banach space that does not admit a coarse embedding into a Hilbert space) is unknown.

Gromov observed that a sequence of expander graphs $(\Gamma_n)_{n\in\mathbb{N}}$ does not admit an equi-coarse embedding into a Hilbert space, i.e., that there do not exist functions ρ and ω, and maps $f_n\colon \Gamma_n \to \ell_2$ so that $\rho(t) \le \rho_{f_n}(t)$ with $\lim_{t\to\infty}\rho(t) = \infty$, and $\omega_{f_n}(t) \le \omega(t)$ with $\omega(t) < \infty$ for every $t > 0$. This remark extends to any L_p-space for $p \in [1, 2)$ and also for $p \in (2, \infty)$ as shown by Matoušek [43]. Ozawa [57] gave an unexpected geometric condition that is an obstruction to the containment of expander graphs.

Theorem. *A sequence of expander graphs does not admit an equi-coarse embedding into a Banach space whose unit ball is uniformly embeddable into a Hilbert space.*

A sequence of expander graphs that does not admit an equi-coarse embedding into any superreflexive Banach space is called a sequence of superexpanders. Whether every sequence of expander graphs is a sequence of superexpanders is open, as is whether the unit ball of every superreflexive Banach space is uniformly embeddable into a Hilbert space. There do exist explicit constructions of sequences of superexpanders: an algebraic construction by Lafforgue [38] and a combinatorial construction by Mendel and Naor [50]. The monograph of Nowak and Yu [55] treats many interesting topics involving the large scale geometry of groups and metric spaces.

Fine quantitative embedding results for large classes of metric spaces can be obtained if the superreflexivity property is relaxed. We simply state nonquantitative coarse versions of two theorems of the late Nigel Kalton [30] about stable metrics, i.e., such that $\lim_{n\to\infty}\lim_{k\to\infty} d(x_n, y_k) = \lim_{k\to\infty}\lim_{n\to\infty} d(x_n, y_k)$ whenever the limits exist.

Theorem. *Every stable metric space embeds coarsely into a reflexive Banach space.*

Theorem. *The space* c_0 *is not coarsely embeddable into any reflexive Banach space.*

Profound insights on the coarse (and uniform) geometry of Banach spaces were obtained by Kalton in an impressive series of articles [31], [32], [33], [34]. None of this material is discussed in the monograph under review, but the interested reader will find it enlightning to consult the survey by Godefroy, Lancien, and Zizler [21].

3. M. Ostrovskii's monograph

Ostrovskii's monograph has eleven chapters, all of them followed by a (very often historical) discussion about the material presented. The first ten chapters also contain a series of exercises together with hints associated to some of them. Open problems are described in the last chapter. The monograph ends with a bibliography, an author index, and a subject index.

Chapter 1 is an introductory chapter containing the necessary definitions, some useful facts, and some background from probability theory. The author also discusses two motivating examples to study embeddings between metric spaces. The fairly detailed first example is the sparsest cut problem in theoretical computer science, and the second one is related to the Novikov conjecture in topology.

Chapter 2 is almost entirely devoted to proving a result of the author and its corollaries. It is shown that the bi-Lipschitz embeddability of an infinite locally finite metric space into a infinite-dimensional Banach space is completely determined by its finite subsets. It also contains some important Banach space theory material (type and cotype of Banach spaces, Khinchine and Kahane inequalities).

In Chapter 3 the technique of using stochastic padded decompositions to produce bi-Lipschitz embeddings of finite metric spaces with low distortion is explained.

In Chapter 4 the author describes various Poincaré-type inequalities, how to obtain them, and the nonembeddability results that can be derived from them.

The topic of Chapter 5 (the longest chapter of the monograph) is the explicit or random construction of graphs with special properties, including families of expander graphs, families of graphs with large girth, and families of expander graphs not coarsely embeddable into each other. Several techniques or tools to construct such graphs are discussed (zig-zag product, Kazhdan's property (T), graph lift, variational techniques, etc.)

In Chapter 6 the author describes certain Banach spaces not admitting coarse embeddings of families of expander graphs. The rest of the chapter studies some classes of Banach spaces for which the problem of embeddability of families of expander graphs into them is still open.

In Chapter 7 structural properties of spaces which are not coarsely embeddable into a Hilbert space are given.

The central discussion in Chapter 8 is the utilization of Markov chains (in particular the notion of Markov-type) regarding embeddability problems. This chapter also contains a significant portion devoted to the renorming of superreflexive spaces—a classical Banach space theory topic.

In Chapter 9 the author starts by proving Ribe's rigidity theorem using Bourgain's discretization theorem. The metric characterization of superreflexivity in terms of the bi-Lipschitz embeddability of the diamond graphs is then explained.

Chapter 10 is an introduction to Lipschitz free spaces. A significant portion of the chapter deals with Lipschitz free spaces over finite graph metrics or some finite metric spaces.

In Chapter 11 the author recalls, raises, and discusses 14 open problems.

Ostrovskii's *Metric embeddings: Bilipchitz and coarse embeddings into Banach spaces* is a very valuable addition to the literature. It contains an impressive amount of material and is recommended to anyone having some interest in these geometric problems. The area is developing at an extremely fast pace and it is difficult to find in a book format the recent developments; the monograph under review contains some very interesting ones. It has a strong graph and Banach space theoretic flavor, and, as mentioned by the author, "The selection of topics for this book is strongly influenced by [his] interests and expertise". The structure of the monograph shares more common features with a survey, whose proofs are not omitted, than with a graduate course textbook that has certain predetermined learning objectives. Another particularity of the monograph resides in the insertion within some chapters, and at the time they are needed, entire subsections dealing exclusively with classical topics in Banach space geometry. The author evidently put a lot of effort into tracking the history behind results and notions; this information can be found in the "Notes and Remarks" subsections. The extensive bibliography (447 items) is another valuable asset of the monograph, and the author gives precise indication on where to find the relevant material that is not covered in the book. Finally, the degree of difficulty of the exercises ranges from "completely straightforward" to "very challenging" to "too difficult for these reviewers".

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