

## THREE THEMES OF SYZYGIES

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ABSTRACT. We present three exciting themes of syzygies, where major progress has been made recently: Boij-Söderberg theory, Stillman’s question, and syzygies over complete intersections.

Free resolutions are both central objects and fruitful tools in commutative algebra. They have many applications in algebraic geometry, computational algebra, invariant theory, hyperplane arrangements, mathematical physics, number theory, and other fields. We introduce and motivate free resolutions and their invariants in Sections 1 and 3. The other sections focus on three hot topics, where major progress has been made recently:

- Syzygies over complete intersections (see Section 2),
- Stillman’s question (see Section 4),
- Boij-Söderberg theory (see Sections 5 and 6).

Of course, there are a number of other interesting aspects of syzygies. The expository papers [MP, PS] contain many open problems and conjectures.

The first two sections are a slightly expanded version of the AMS Invited Address given by Irena Peeva at the Joint Mathematics Meetings in January 2015.

### 1. INTRODUCTION TO SYZYGIES

Linear algebra studies the properties of vector spaces and their maps. The basic concept there is vector space, defined over some ground field (e.g.,  $\mathbb{C}$ ). In the finite-dimensional case, bases are very fruitful tools. These concepts generalize in Algebra as follows: The generalization of the concept of field is the concept of ring, where not every non-zero element is invertible (for example, the ring of polynomials  $\mathbb{C}[x_1, \dots, x_n]$ ). The generalization of the concept of vector space is the concept of module. The condition that “a vector space is finite dimensional” generalizes to the condition that “a module is finitely generated”. What about bases? A *basis* of a module is a generating set that is linearly independent over the ring. Unfortunately, such sets rarely exist: only free modules have bases. Usually, we have to consider a (minimal) *system of generators* instead of a basis (where “minimal” means that no proper subset generates the module). This is illustrated in the following simple example.

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**Example 1.1.** Consider the ideal  $N = (xy, xz)$  in the polynomial ring  $\mathbb{C}[x, y, z]$ . It is generated by  $f := xy$  and  $g := xz$ . Since we have the relation

$$zf - yg = z(xy) - y(xz) = 0,$$

$\{f, g\}$  is not a basis. It can be shown that  $N$  does not have a basis.

Generators give very little information about the structure of a module because usually there are relations on the generators, and relations on these relations, etc. Thus, we face the following basic problem:

**Basic Question 1.2.** *How can we describe the structure of a module?*

Hilbert’s approach is to use free resolutions. Motivated by applications in invariant theory, he introduced the idea of associating a free resolution to a finitely generated module in a famous paper in 1890 [Hi]; the idea can be also found in the work of Cayley [Ca]. We will first introduce the definition and then explain it.

**Definition 1.3.** Let  $R$  be a commutative noetherian ring (e.g.,  $\mathbb{C}[x_1, \dots, x_n]$ ,  $\mathbb{C}[[x_1, \dots, x_n]]$ , or one of their quotients). A sequence

$$(1.4) \quad \mathbf{F} : \dots \rightarrow F_{i+1} \xrightarrow{\partial_{i+1}} F_i \xrightarrow{\partial_i} F_{i-1} \rightarrow \dots \rightarrow F_1 \xrightarrow{\partial_1} F_0$$

of homomorphisms of finitely generated free  $R$ -modules is a *free resolution* of a finitely generated  $R$ -module  $N$  if

- (1)  $\mathbf{F}$  is an exact complex, that is,  $\text{Ker}(\partial_i) = \text{Im}(\partial_{i+1})$  for  $i \geq 1$ ;
- (2)  $N \cong \text{Coker}(\partial_1)$ , that is,

$$(1.5) \quad \dots \rightarrow F_{i+1} \xrightarrow{\partial_{i+1}} F_i \rightarrow \dots \rightarrow F_1 \xrightarrow{\partial_1} F_0 \rightarrow N \rightarrow 0$$

is exact.

A free resolution is usually written in the form (1.4), but it could be also written in the form (1.5) in order to emphasize which module is resolved.

The collection of maps  $\partial = \{\partial_i\}$  is called the *differential* of  $\mathbf{F}$ .

**Example 1.6.** In Example 1.1,  $N = (xy, xz)$  has a free resolution

$$(1.7) \quad 0 \rightarrow S \xrightarrow{\begin{pmatrix} z \\ -y \end{pmatrix}} S^2 \xrightarrow{\begin{pmatrix} xy & xz \end{pmatrix}} N \rightarrow 0.$$

Note that it can be interpreted as

$$0 \rightarrow F_1 \xrightarrow{\begin{pmatrix} \text{relations} \\ \text{on the} \\ \text{generators} \\ \text{of } N \end{pmatrix}} F_0 \xrightarrow{\begin{pmatrix} \text{generators} \\ \text{of } N \end{pmatrix}} N \rightarrow 0.$$

This interpretation holds in general, except that the resolution can continue for many steps (possibly infinitely many). Hilbert’s key insight was that a free

resolution of  $N$  has the form

$$\cdots \rightarrow F_2 \xrightarrow{\partial_2 = \begin{pmatrix} \text{relations} \\ \text{on the} \\ \text{relations} \\ \text{in } \partial_1 \end{pmatrix}} F_1 \xrightarrow{\partial_1 = \begin{pmatrix} \text{relations} \\ \text{on the} \\ \text{generators} \\ \text{of } N \end{pmatrix}} F_0 \xrightarrow{\begin{pmatrix} \text{generators} \\ \text{of } N \end{pmatrix}} N \rightarrow 0,$$

and so it is a description of the structure of the module  $N$ .

**Example 1.8.** The ideal  $N = (xy, xz)$  considered in Example 1.6 has many free resolutions. For example,

$$0 \rightarrow S \xrightarrow{\begin{pmatrix} -y \\ 1 \end{pmatrix}} S^2 \xrightarrow{\begin{pmatrix} z & yz \\ -y & -y^2 \end{pmatrix}} S^2 \xrightarrow{\begin{pmatrix} xy & xz \end{pmatrix}} N \rightarrow 0$$

is a larger free resolution than (1.7). The first differential contains the relation  $\begin{pmatrix} yz \\ -y^2 \end{pmatrix}$ , which is a multiple of the relation  $\begin{pmatrix} z \\ -y \end{pmatrix}$ , and so it could be omitted in order to produce the smaller resolution (1.7).

One may want to construct a free resolution as efficiently as possible, that is, to pick a *minimal* system of relations at each step. This is captured by the concept of *minimal* free resolution. The concept of minimality relies on Nakayama's lemma and thus makes sense in two main cases:

- the local case, when  $R$  is a local ring (for example,  $R$  is a quotient of  $\mathbb{C}[[x_1, \dots, x_n]]$ );
- the graded case, when  $R$  is a standard graded finitely generated algebra over a field and the module  $N$  is graded (for example,  $R$  is a graded quotient of  $\mathbb{C}[x_1, \dots, x_n]$  and  $\deg(x_i) = 1$  for all  $i$ ). In this case, a free resolution  $\mathbf{F}$  is called *graded* if its differential is a homogeneous map of degree 0, that is, the differential preserves degree.

**Definition 1.9.** A free resolution  $\mathbf{F}$  is *minimal* if

$$\partial_{i+1}(F_{i+1}) \subseteq \mathfrak{m}F_i \quad \text{for all } i \geq 0,$$

where either  $R$  is local with maximal ideal  $\mathfrak{m}$ , or  $\mathbf{F}$  is graded and  $R$  is standard graded with irrelevant maximal ideal  $\mathfrak{m} = R_+$ . This means that no invertible elements appear in the differential matrices.

The word “minimal” refers to the following two properties:

- (1) A free resolution is minimal if and only if at each step we make an optimal choice, that is, we choose a minimal system of generators of the kernel in order to construct the next differential; see [Pe, Theorem 3.4].
- (2) A minimal free resolution is smallest in the sense that it lies (as a direct summand) inside any other free resolution of the module; see [Pe, Theorem 3.5].

The crucial fact about such resolutions is:

**Theorem 1.10** (see [Pe, Theorem 7.5]). *In the local or graded case, every finitely generated (graded) module has a minimal (graded) free resolution, and it is unique up to isomorphism.*

Non-minimal free resolutions are much easier to produce and can have very simple structure, but they yield less information about the module we are resolving. For example, over the polynomial ring  $\mathbb{C}[x_1, \dots, x_n]$ , the well-known Auslander-Buchsbaum formula states that the depth of a finitely generated graded module  $N$  is equal to  $n$  minus the length of its minimal free resolution (see [Pe, 15.3]).

**Definition 1.11.** The ranks of the free modules in the minimal free resolution  $\mathbf{F}$  of a finitely generated  $R$ -module  $N$  are called the *Betti numbers* of  $N$  and denoted

$$\beta_i^R(N) = \text{rank } F_i.$$

They can be expressed as

$$\beta_i^R(N) = \dim_k \text{Tor}_i^R(N, k) = \dim_k \text{Ext}_R^i(N, k),$$

since the differentials in the complexes  $\mathbf{F} \otimes_R k$  and  $\text{Hom}_R(\mathbf{F}, k)$  are zero.

Set  $\text{Syz}_0^R(N) = N$ . For  $i \geq 1$ , the submodule

$$\text{Im}(\partial_i) = \text{Ker}(\partial_{i-1}) \cong \text{Coker}(\partial_{i+1})$$

of  $F_{i-1}$  is called the  *$i$ th syzygy module* of  $N$  and is denoted  $\text{Syz}_i^R(N)$ . Its elements are called  *$i$ th syzygies*. If  $g_1, \dots, g_p$  is a basis of  $F_i$ , then  $\partial_i(g_1), \dots, \partial_i(g_p)$  is a minimal system of generators of  $\text{Syz}_i^R(N)$ .

The general direction of research in this area is:

**Problem 1.12.** *How do the properties of the minimal free resolution of a finitely generated  $R$ -module  $N$  (in particular, the Betti sequence  $\{\beta_i^R(N)\}$ ) relate to the structure of  $N$  and the structure of  $R$ ?*

## 2. SYZYGIES OVER COMPLETE INTERSECTIONS

In this section, we assume that  $S$  is a regular local ring (for example,  $S = \mathbb{C}[[x_1, \dots, x_n]]$ ) with residue field  $k$ , and  $R$  is a quotient of  $S$ .

Hilbert's main result on syzygies is:

**Hilbert's Syzygy Theorem 2.1** (see [Pe, Theorem 15.2]). *Every finitely generated module over  $S$  has a finite minimal free resolution.*

In fact, we have a bound on the length of such resolutions, and in the case of  $S = \mathbb{C}[[x_1, \dots, x_n]]$  the length is at most  $n$ .

The situation changes dramatically for modules over non-regular rings, where the Auslander-Buchsbaum-Serre regularity criterion applies. In his textbook *Commutative ring theory*, Matsumura says that the Auslander-Buchsbaum-Serre regularity criterion is one of the top three results in commutative algebra. It is a homological criterion for a ring to be regular:

**Auslander-Buchsbaum-Serre Regularity Criterion 2.2** (see [Ei2, Theorem 19.12]). *The following are equivalent:*

- (1) *Every finitely generated  $R$ -module has a finite minimal free resolution.*
- (2) *The  $R$ -module  $k$  has a finite minimal free resolution.*
- (3)  *$R$  is regular.*

We would like to understand the structure of minimal free resolutions over  $R$ . The natural starting point is to find out what happens over a hypersurface ring  $R = S/(f)$ . In this case, one can use matrix factorizations:

**Definition 2.3** ([Ei1]). A *matrix factorization* of a non-zero element  $f \in S$  is a pair of square matrices  $(d, h)$  with entries in  $S$ , such that

$$dh = hd = f\text{Id}.$$

The notation is motivated by the fact that  $d$  is a differential and  $h$  is a homotopy on a certain resolution; see Theorem 2.4(3). The associated *matrix factorization module* (or *MF-module*) is

$$M := \text{Coker}(R \otimes d).$$

For example,

$$d = \begin{pmatrix} -x^3 & x^2y^2z \\ z & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 0 & x^2y^2z \\ z & x^3 \end{pmatrix}$$

is a matrix factorization for  $f = x^2y^2z^2 \in \mathbb{C}[[x, y, z]]$ .

Eisenbud introduced the concept of matrix factorization in 1980 and proved the following result:

**Theorem 2.4** ([Ei1]). *Let  $R = S/(f)$  be a hypersurface ring (where  $0 \neq f \in S$ ).*

- (1) *If  $(d, h)$  is a minimal matrix factorization of  $f$ , then*

$$\cdots \xrightarrow{d} R^b \xrightarrow{h} R^b \xrightarrow{d} R^b \xrightarrow{h} R^b \xrightarrow{d} R^b$$

*is the minimal free resolution of  $M = \text{Coker}(R \otimes d)$  over  $R$ , where  $b$  is the size of the matrices  $d$  and  $h$ .*

- (2) *Asymptotically, every minimal free resolution over  $R$  is of type (1): if  $\mathbf{F}$  is a minimal free resolution over  $R$ , then for  $s \gg 0$  the truncation*

$$\mathbf{F}_{\geq s} : \cdots \longrightarrow F_{s+2} \xrightarrow{\partial_{s+2}} F_{s+1} \xrightarrow{\partial_{s+1}} F_s$$

*is described by a matrix factorization.*

- (3) *Under the assumptions in (1), the minimal free resolution of  $M$  over  $S$  is*

$$0 \rightarrow S^b \xrightarrow{d} S^b.$$

The asymptotic view in (2) is necessary, since Eisenbud [Ei1] provided examples of minimal free resolutions with quite intricate behavior at the beginning.

Matrix factorizations have many other applications, for example in cluster tilting, CM modules, Hodge theory, knot theory, moduli of curves, quiver and group representations, singularity categories, and singularity theory; physicists discovered amazing connections with string theory—see [As] for a survey.

There are several open problems involving matrix factorizations, starting with the basic question, *What determines the minimal size of the matrices in the matrix factorizations of a given power series  $f$ ?*

Given the elegant and simple structure of minimal free resolutions over a hypersurface, one might hope that the structure of an infinite minimal free resolution is encoded in finite data. The Serre-Kaplansky problem embodies this hope for the generating function of the Betti numbers of the simplest possible module, namely the residue field  $k$ :

**Serre-Kaplansky Problem 2.5.** *Is the Poincaré series*

$$P_k^R(t) := \sum_{i \geq 0} \beta_i^R(k)t^i$$

*of the residue field  $k$  over  $R$  rational?*

This was one of the central questions in commutative algebra for many years. Work on the problem was strongly motivated by the expectation that the answer would be positive. Moreover, a result of Gulliksen showed that a positive answer for all such rings implies the rationality of the Poincaré series of any finitely generated module. However, in 1980 Anick [An1, An2] found a counterexample:

$$R = \mathbb{C}[x_1, \dots, x_5] / \left( (x_1, \dots, x_5)^3, x_1^2, x_2^2, x_4^2, x_5^2, \right. \\ \left. x_1x_2, x_4x_5, x_1x_3 + x_3x_4 + x_2x_5 \right).$$

Since then, many other examples have been found but the quest for rings with rational Poincaré series continues. Recently, Herzog and Huneke proved in [HH] the rationality of the Poincaré series when the ring has the form  $R = \mathbb{C}[x_1, \dots, x_n]/J^m$ , for any  $m \geq 2$  and any homogeneous ideal  $J$ ; the local case is still open.

Our goal is to study the structure of minimal free resolutions over a quotient ring  $R = S/(f_1, \dots, f_c)$ . Anick's example shows that we should impose some conditions on  $f_1, \dots, f_c$ . A main class of interest in algebra are the complete intersection rings, whose name comes from their role in algebraic geometry. A sequence  $f_1, \dots, f_c \in S$  is a *regular sequence* if each  $f_i$  is a non-zero divisor modulo the ideal  $(f_1, \dots, f_{i-1})$  generated by the preceding elements and  $(f_1, \dots, f_c) \neq S$ ; in this case the quotient  $S/(f_1, \dots, f_c)$  is called a *complete intersection*. We will focus on the following question:

**Question 2.6.** *What is the structure of minimal free resolutions over a complete intersection  $R = S/(f_1, \dots, f_c)$ ?*

The first work in this direction was Tate's elegant construction of the minimal free resolution of  $k$  in 1957 [Ta]. The next impressive result was Gulliksen's proof [Gu] in 1974 that the Poincaré series  $\sum_i \beta_i^R(N)t^i$  is rational for every finitely generated  $R$ -module  $N$ , and that the denominator divides  $(1-t^2)^c$ . For this purpose, he showed that  $\text{Ext}_R(N, k)$  can be regarded as a finitely generated graded module over a polynomial ring  $k[\chi_1, \dots, \chi_c]$ . This also implies that the even Betti numbers  $\beta_{2i}^R(N)$  are eventually given by a polynomial in  $i$ , and the odd Betti numbers are given by another polynomial. In 1989 Avramov [Av] proved that the two polynomials have the same leading coefficient and the same degree. He also identified the dimension of  $\text{Ext}_R(N, k)$  with a correction term in a natural generalization of the Auslander-Buchsbaum formula. In 1997 Avramov, Gasharov, and Peeva [AGP] showed that the truncated Betti sequence  $\{\beta_i^R(N)\}_{i \geq q}$  is either strictly increasing or constant for  $q \gg 0$  and proved further properties of the Betti numbers. Avramov and Buchweitz [AB] in 2000 described the minimal free resolutions of high syzygies over a codimension two complete intersection (the case when  $c = 2$ ). All results indicated that minimal free resolutions over a complete intersection are highly structured. Nevertheless, their structure remained mysterious for many years and no conjectures were guessed. In sharp contrast, non-minimal free resolutions were known from the work of Shamash [Sh] since 1969.

In 2015 Eisenbud and Peeva introduced the concept of *higher matrix factorization*  $(d, h)$  for a regular sequence  $f_1, \dots, f_c \in S$  in [EP]. The associated module  $M := \text{Coker}(R \otimes d)$  is called an *HMF-module* (that is, a higher matrix factorization

module). Furthermore:

- (1) They constructed the minimal free resolution of  $M := \text{Coker}(R \otimes d)$  over the complete intersection  $R$ .
- (2) They proved that asymptotically, every minimal free resolution over  $R$  is of type (1): if  $\mathbf{F}$  is a minimal free resolution over  $R$ , then for  $s \gg 0$  the truncation

$$\mathbf{F}_{\geq s} : \cdots \longrightarrow F_{s+2} \xrightarrow{\partial_{s+2}} F_{s+1} \xrightarrow{\partial_{s+1}} F_s$$

is described by a higher matrix factorization.

- (3) They constructed the finite minimal free resolution of  $M := \text{Coker}(R \otimes d)$  over the regular local ring  $S$ .

Thus, their results describe the minimal free resolutions of  $N$  as an  $S$ -module and as an  $R$ -module, when  $N$  is a high syzygy (of a given module) over the complete intersection  $R$ .

For more details about syzygies over complete intersections, we refer the reader to [EP].

### 3. INTRODUCTION TO INVARIANTS OF FINITE GRADED FREE RESOLUTIONS

In this section we work over a polynomial ring  $S = k[x_1, \dots, x_n]$  over a field  $k$  (e.g.,  $\mathbb{C}$ ), and set  $\mathbf{m} = (x_1, \dots, x_n)$ . The ring is standard graded with  $\deg(x_i) = 1$  for all  $i$ . Furthermore,  $M$  stands for a finitely generated graded  $S$ -module.

**Definition 3.1.** Let  $\mathbf{F}$  be the minimal graded free resolution of  $M$ . It is finite by Hilbert’s Syzygy Theorem 2.1. We may write

$$F_i = \bigoplus_{p \in \mathbb{Z}} S(-p)^{\beta_{i,p}}$$

for each  $i$ , where  $S(-p)$  is the rank one graded free module generated in degree  $p$ . The numbers  $\beta_{i,p}$  are called the *graded Betti numbers* of  $M$  and are denoted  $\beta_{i,p}^S(M)$ . These numbers form the *Betti table*  $\beta(M)$  in the following way: The columns are indexed from left to right by homological degree starting with homological degree zero. The rows are indexed increasingly from top to bottom starting with the minimal degree of an element in a minimal system of homogeneous generators of  $M$ . The entry in position  $i, j$  is  $\beta_{i,i+j} = \beta_{i,i+j}^S(M)$ .

A module  $M$  generated in degrees  $\geq 0$  has Betti table of the form

$$(3.2) \quad (\beta_{i,i+j}) = \begin{array}{c|cccc} & 0 & 1 & 2 & \dots \\ \hline 0: & \beta_{0,0} & \beta_{1,1} & \beta_{2,2} & \dots \\ 1: & \beta_{0,1} & \beta_{1,2} & \beta_{2,3} & \dots \\ 2: & \beta_{0,2} & \beta_{1,3} & \beta_{2,4} & \dots \\ 3: & \beta_{0,3} & \beta_{1,4} & \beta_{2,5} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{array}$$

In Example 1.6, the Betti table is

$$\begin{array}{c|cc} & 0 & 1 \\ \hline 0: & - & - \\ 1: & - & - \\ 2: & 2 & 1 \end{array}$$

where - stands for zero.

The size of a Betti table is given by the projective dimension and the regularity: The *projective dimension*

$$\mathrm{pd}_S(M) = \sup \left\{ i \mid \beta_i^S(M) \neq 0 \right\}$$

is the index of the last non-zero column of the Betti table  $\beta(M)$ , and thus it measures its length. The height of the table is measured by the index of the last non-zero row, and is called the (*Castelnuovo-Mumford*) *regularity* of  $M$ , which is

$$\mathrm{reg}_S(M) = \sup \left\{ j \mid \beta_{i, i+j}^S(M) \neq 0 \right\}.$$

Both projective dimension and regularity are very important and well-studied invariants. Hilbert's Syzygy Theorem 2.1 (see [Pe, Theorem 15.2]) provides a reasonable upper bound on projective dimension, namely

$$\mathrm{pd}_S(M) \leq n.$$

On the other hand, regularity can be quite large: In [MM] Mayr and Meyer construct an ideal generated by  $10r - 6$  forms of degree  $\leq 4$  in  $10r + 1$  variables and with regularity  $\geq 2^{2^r} + 1$ . One of the most interesting, challenging, and important conjectures on syzygies comes from algebraic geometry and provides a sharp upper bound on regularity of prime ideals:

**The Regularity Conjecture 3.3** (Eisenbud and Goto, [EG]). *If  $P \subset (x_1, \dots, x_n)^2$  is a prime homogeneous ideal in  $S$ , then*

$$\mathrm{reg}_S(P) \leq \deg(S/P) - \mathrm{codim}(S/P) + 1,$$

(where  $\deg(S/P)$  is the degree (also called multiplicity) of  $S/P$  and  $\mathrm{codim}(S/P)$  is the codimension).

The conjecture is proved for curves by Gruson, Lazarsfeld, and Peskine [GLP] and for smooth surfaces by Lazarsfeld and Pinkham [La, Pi]. Kwak [Kw] gave bounds for regularity in dimensions 3 and 4 that are only slightly worse than the optimal ones in the conjecture; his method yields new bounds up to dimension 14, but they get progressively worse as the dimension goes up.

The following two weaker forms of the above conjecture are open as well:

**Conjecture 3.4.** *If  $P \subset (x_1, \dots, x_n)^2$  is a prime homogeneous ideal in  $S$ , then  $\mathrm{reg}_S(P) \leq \deg(S/P)$ .*

**Conjecture 3.5.** *If  $P \subset (x_1, \dots, x_n)^2$  is a prime homogeneous ideal in  $S$ , then the maximal degree of a polynomial in a minimal homogeneous system of generators of  $P$  is  $\leq \deg(S/P)$ .*

For more details about regularity, we refer the reader to the expository paper [BM].

#### 4. STILLMAN'S QUESTION

In this section we focus on the regularity and projective dimension of a homogeneous ideal  $I$  in a standard graded polynomial ring  $S$  over a field  $k$ . Since  $\mathrm{pd}_S(S/I) = \mathrm{pd}_S(I) + 1$ , we may equivalently consider cyclic modules  $S/I$ .

A classical construction of Burch [Bu] and Kohn [Ko] shows that there exist three-generated ideals in polynomial rings whose projective dimension is arbitrarily



large. In particular, this means it is not possible to bound the projective dimension of an ideal purely in terms of the number of generators. Later Bruns showed in a very precise sense that all the pathology of minimal free resolutions of modules is exhibited by the resolutions of ideals with three generators [Br]. Yet, when applying Bruns' argument to create three-generated ideals with arbitrarily large projective dimension, the degrees of the generators are forced to grow linearly with the length of the resolution.

Motivated by computational complexity issues, Stillman posed the following question:

**Stillman's Question 4.1** ([PS, Problem 3.14]). *Fix a sequence of natural numbers  $d_1, \dots, d_g$ . Does there exist a number  $p$  such that  $\text{pd}_S(S/I) \leq p$ , where  $I$  is a homogeneous ideal in a polynomial ring  $S$  with a minimal system of generators of degrees  $d_1, \dots, d_g$ ? Note that the number of variables in  $S$  is not fixed.*

For example, the case  $g = 3$  and  $d_1 = d_2 = d_3 = 3$  of Question 4.1 asks for a bound on the projective dimension of homogeneous ideals generated minimally by three cubic forms in an unknown number of variables.

It is worth noting that we are assuming nothing about the ideal  $I$  other than the number and the degrees of the generators. One expects better results in the geometric case, as in Conjecture 3.3.

The study of Stillman's question is further motivated by the following parallel question:

**Question 4.2** ([PS, Problem 3.15]). *Fix a sequence of natural numbers  $d_1, \dots, d_g$ . Does there exist a number  $q$ , such that  $\text{reg}_S(S/I) \leq q$ , where  $I$  is a homogeneous ideal in a polynomial ring  $S$  with a minimal system of generators of degrees  $d_1, \dots, d_g$ ? The number of variables in  $S$  is not fixed.*

Caviglia proved that an affirmative answer to Stillman's Question 4.1 implies an affirmative answer to Question 4.2 and vice versa; see [MS, Theorem 2.4] and [Pe, Theorem 29.5]. We focus our attention on Question 4.1 for the rest of this section as all non-trivial known positive answers to Question 4.2 follow from Caviglia's result.

Hilbert's Syzygy Theorem 2.1 shows that the projective dimension is bounded by the number of variables, but that does not shed light on Stillman's question since the number of variables is not fixed. While there are some cases in which Question 4.1 has an easy positive answer, the problem remains wide open. It is easy to see that  $\text{pd}(S/I) = 1$  when  $I$  is principal, and  $\text{pd}_S(S/I) = 2$  when  $I$  is minimally generated by two elements. If the number of terms appearing in the minimal generators of  $I$  is bounded (for example, ideals generated by monomials or toric ideals), then Question 4.1 has a positive answer because in this situation there is an immediate bound on the number of variables appearing in the minimal generators of  $I$ ; thus the minimal resolution of  $I$  may be computed over a smaller polynomial ring with a bounded number of variables, and so Hilbert's Syzygy Theorem 2.1 yields a bound. On the other hand, even in the case  $g = 3$  and  $d_1 = d_2 = d_3 = 2$  in which we consider the projective dimension of ideals generated by three quadric forms, it requires a non-trivial argument to show  $\text{pd}_S(S/I) \leq 4$ ; see [MS, Theorem 3.1]. One of the first positive results was proved by Engheta [En1, En2], who showed that the projective dimension of an ideal generated by three cubics is at most 36, while the expected upper bound is 5.

Currently, there are two tracks of research regarding Question 4.1 dealing with the following two problems:

- (1) *Establish bounds for large classes of ideals.*
- (2) *Find tight upper bounds for more specific cases.*

In the former direction, Ananyan and Hochster proved the following:

**Theorem 4.3** ([AH, Main Theorem 2.2]). *Let  $f_1, \dots, f_g \in S$  be polynomials of degree at most two, and let  $I = (f_1, \dots, f_g)$ . If  $k$  is infinite, then  $f_1, \dots, f_g$  are contained in a  $k$ -subalgebra of  $S$  generated by a regular sequence of at most  $C(g)$  forms, where  $C(g)$  is an integer depending on  $g$  that is asymptotic to  $2g^{2g}$ . Consequently,  $\text{pd}_S(S/I) \leq C(g)$ .*

Note that the projective dimension does not change when we enlarge the base field, hence the above projective dimension bound holds in general. More recently, Ananyan and Hochster have announced (personal communication) a positive answer to Question 4.1 in the case of ideals with generators of degree at most 4 and  $\text{char}(k) \neq 2, 3$ .

Contrasting with Theorem 4.3, Huneke, Mantero, McCullough, and Seceleanu proved in [HMMS1] that ideals generated by  $g$  quadrics with codimension two have projective dimension at most  $2g - 2$ ; moreover, known examples show this bound to be optimal. In [HMMS2], they also establish an optimal upper bound of 6 for ideals generated by four quadrics. These results are achieved by a mix of techniques including structure theorems for prime and primary ideals in polynomial rings over an algebraically closed field, residual intersection theory, liaison theory, results on 1-generic matrices, and homological techniques. They indicate that the upper bounds of Ananyan and Hochster are far from optimal. At this point, the growth rate of the optimal answer to Question 4.1 in the case of ideals generated by  $g$  quadrics is not known.

While these upper bounds seem large, families of ideals with large projective dimension constructed by McCullough in [Mc] and by Beder, McCullough, Núñez-Betancourt, Seceleanu, Snapp, and Stone in [BMNSSS] show that any positive answer to Question 4.1 must be large. McCullough constructed a family of ideals  $Q_g$  generated by  $2g$  quadrics with

$$\text{pd}_S(S/Q_g) = g^2 + g.$$

Beder et al. constructed a family of ideals  $T_g$  generated by three homogeneous elements of degree  $g^2$  with  $\text{pd}_S(S/T_g) = g^{g-1}$ . Both constructions use the fact that if  $S/I$  has a component primary to the maximal ideal (i.e.,  $S/I$  has depth 0), then  $\text{pd}_S(S/I) = \dim(S)$ .

The following example gives the Betti table of one of these ideals:

**Example 4.4.** Let  $S = k[x, y, a, b, c, d]$  and  $I = (x^2, y^2, ax + by, cx + dy)$ . Then the Betti table for  $S/I$  is

	0	1	2	3	4	5	6
0:	1	-	-	-	-	-	-
1:	-	4	-	-	-	-	-
2:	-	-	13	20	15	6	1

In particular,  $\text{pd}_S(S/I) = 6$ , showing that the bounds in [HMMS1] and [HMMS2] are optimal.

In some special cases, the projective dimension of  $S/I$  can be bounded by the number of elements in a minimal system of generators of  $I$  (which we may call the number of generators). The simplest such case is when  $I$  is generated by monomials and one can apply Taylor's (usually non-minimal) free resolution; see [Pe, Section 26]. Very recently, De Stefani and Núñez-Betancourt [DN, Theorem 7.3] proved that this bound holds if  $\text{char}(k) = p > 0$  and  $S/I$  is  $F$ -pure; a ring  $T$  of positive characteristic is called  $F$ -pure if the Frobenius morphism  $F : T \rightarrow T$  is pure, meaning  $F \otimes_T \text{id}_M : T \otimes_T M \rightarrow T \otimes_T M$  is injective for all  $T$ -modules  $M$ . The authors ask in [DN, Question 7.5] whether the same bound holds when  $S/I$  satisfies the characteristic-0 analog of  $F$ -purity.

For more details about Stillman's question, we refer the reader to the expository paper [MS].

## 5. BOIJ-SÖDERBERG THEORY

The previous section concerned properties of the Betti table of a homogeneous ideal  $I$  in a standard graded polynomial ring  $S = k[x_1, \dots, x_n]$  over a field  $k$ , notably its regularity and projective dimension.

This section is concerned with *modules*; more precisely, throughout we are concerned with *finitely generated graded modules* over the polynomial ring  $S$ , and their Betti tables as in (3.2). To classify Betti tables of modules is a task which seems beyond reach, but in 2006 two Swedish mathematicians, Boij and Söderberg [BS1], came up with a brilliant insight which has enabled us to get a fairly sharp view of the range of Betti tables that occurs. What they realized and formulated as two conjectures was the following: Given a table of natural numbers  $(\beta_{i,i+j})$  of the form (3.2), it may not be easy to determine if there exists a module  $M$  with this Betti table, i.e.,  $\beta_{i,i+j} = \beta_{i,i+j}^S(M)$ , but we are able to determine if there exists a large  $q \gg 0$  such that  $q \cdot (\beta_{i,i+j})$  (this means multiplying each entry in the table with  $q$ ) is the Betti table of a module. In other words, we know how to precisely determine which positive rays  $q \cdot (\beta_{i,i+j})$ , where  $q$  is a positive rational number, contain the Betti table of a module.

The two conjectures, called the Boij-Söderberg conjectures, were proven shortly after in full generality by Eisenbud and Schreyer [ES1].

**Example 5.1.** It is not difficult to show that the table

$$(5.2) \quad \begin{array}{c|cccc} & 0 & 1 & & \\ \hline 0: & 1 & 2 & - & - \\ 1: & - & - & 2 & 1 \end{array}$$

is not the Betti table of any module. However if we multiply it by 2, the Betti table

$$\begin{array}{c|cccc} & 0 & 1 & & \\ \hline 0: & 2 & 4 & - & - \\ 1: & - & - & 4 & 2 \end{array}$$

is realized by the minimal free resolution

$$(5.3) \quad 0 \rightarrow S(-4)^2 \rightarrow S(-3)^4 \rightarrow S(-1)^4 \xrightarrow{\begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_1 & 2x_2 & 3x_3 & 4x_4 \end{pmatrix}} S^2,$$

where  $S = k[x_1, x_2, x_3, x_4]$ . It can be shown that if we multiply the table (5.2) with any natural number  $q \geq 2$ , we get the Betti table of a module.

The resolution (5.3) has the property of being a pure resolution:

**Definition 5.4.** A graded free resolution  $\mathbf{F}$  is *pure* if each  $F_i$  in the resolution is a free module  $F_i = \oplus S(-d_i)^{\beta_{i,d_i}}$  generated in a single degree  $d_i$ .

The Betti tables of pure resolutions, called pure Betti tables, turn out to be the basic building blocks for understanding Betti tables up to rational multiple.

If  $M$  is a module of codimension  $c$  over  $S$ , it follows by the Auslander-Buchsbaum theorem, mentioned just before Definition 1.11, that the length of its minimal free resolution is  $\geq c$ . We now consider the class of modules where the length of the minimal free resolution is as small as possible, equal to the codimension  $c$ . This is the class of so-called Cohen-Macaulay modules. If such a module  $M$  has pure resolution

$$(5.5) \quad S(-d_c)^{\beta_{c,d_c}} \rightarrow \dots \rightarrow S(-d_0)^{\beta_{0,d_0}},$$

it is not difficult to show that the tuple  $(\beta_{0,d_0}, \dots, \beta_{c,d_c})$  of Betti numbers is uniquely determined up to scalar multiple. The first part of the Boij-Söderberg conjectures states the following:

**Theorem 5.6.** *For every sequence of integers  $\mathbf{d} : d_0 < d_1 < \dots < d_c$ , called a degree sequence, there exists a Cohen-Macaulay module  $M$  with pure minimal free resolution (5.5).*

This was proven first when  $\text{char } k = 0$  by Eisenbud, Fløystad, and Weyman in [EFW], where two constructions of pure resolutions were given, and then in any characteristic in [ES1] by Eisenbud and Schreyer. Later, efficient machines for such constructions were given by Berkesch, Erman, Kummini, Sam in [BEKS] and by Fløystad in [F12, Section 3].

Let  $\pi(\mathbf{d})$  be the *Betti diagram* obtained from the Betti table of the pure resolutions (5.5), by adjusting up to scalar multiple, so that all the  $\beta_{i,d_i}$  are integers with no common factor, for instance in Example 5.1:

$$\pi(0, 1, 3, 4) = \begin{bmatrix} 1 & 2 & - & - \\ - & - & 2 & 1 \end{bmatrix}.$$

The set of positive rational rays of Betti tables of modules  $q \cdot (\beta_{i,i+j}^S(M))$ , where  $q > 0$ ,  $q \in \mathbb{Q}$ , actually forms a convex cone. This is simply because if  $c_1$  and  $c_2$  are natural numbers and  $M_1$  and  $M_2$  are modules, then

$$c_1 \left( \beta_{i,i+j}^S(M_1) \right) + c_2 \left( \beta_{i,i+j}^S(M_2) \right)$$

is the Betti table of the module  $M_1^{c_1} \oplus M_2^{c_2}$ . This raises the question as to what the extremal rays in this cone are. It is not hard to see that rays generated by pure Betti tables are extremal. It turns out that there are no others: The extremal rays are precisely those generated by pure Betti tables. Moreover, there is a very simple procedure to write a Betti table as a convex combination of the extremal Betti diagrams.

**Example 5.7.** Consider the resolution

$$0 \rightarrow S(-3) \oplus S(-4) \xrightarrow{\begin{pmatrix} 0 & x \\ x & -y^2 \\ -y & 0 \end{pmatrix}} S(-3) \oplus S(-2)^2 \xrightarrow{\begin{pmatrix} y^3 & xy & x^2 \end{pmatrix}} S.$$

Its Betti table is

$$\begin{bmatrix} 1 & - & - \\ - & 2 & 1 \\ - & 1 & 1 \end{bmatrix}.$$

This Betti table can be written as a positive linear combination of pure Betti diagrams. The only possible candidates for pure Betti diagrams in such a linear combination are

$$\pi(0, 2, 3) = \begin{bmatrix} 1 & - & - \\ - & 3 & 2 \\ - & - & - \end{bmatrix}, \quad \pi(0, 2, 4) = \begin{bmatrix} 1 & - & - \\ - & 2 & - \\ - & - & 1 \end{bmatrix}, \quad \pi(0, 3, 4) = \begin{bmatrix} 1 & - & - \\ - & - & - \\ - & 4 & 3 \end{bmatrix}.$$

In fact, one works out that

$$(5.8) \quad \begin{bmatrix} 1 & - & - \\ - & 2 & 1 \\ - & 1 & 1 \end{bmatrix} = \frac{1}{2}\pi(0, 2, 3) + \frac{1}{4}\pi(0, 2, 4) + \frac{1}{4}\pi(0, 3, 4).$$

Note that

$$(0, 2, 3) < (0, 2, 4) < (0, 3, 4)$$

in the natural partial order on degree sequences, where  $\mathbf{d} < \mathbf{d}'$  if each  $d_i \leq d'_i$ .

In the decomposition (5.8) above, it is not hard to argue by general principles that the Betti table on the left-hand side can be written as a linear combination of the Betti diagrams on the right-hand side. What is not at all obvious from the start is that the coefficients  $\frac{1}{2}, \frac{1}{4}, \frac{1}{4}$  will be non-negative. In fact, this is the second part of the Boij-Söderberg conjectures.

**Theorem 5.9** ([ES1]). *Let  $M$  be a Cohen-Macaulay module of codimension  $c$  with Betti table  $\beta(M)$ . Then there is a unique chain of degree sequences*

$$\mathbf{d}^1 < \mathbf{d}^2 < \cdots < \mathbf{d}^p$$

such that the Betti table  $\beta(M)$  is uniquely a linear combination

$$c_1\pi(\mathbf{d}^1) + \cdots + c_p\pi(\mathbf{d}^p),$$

where the coefficients  $c_i$  are positive rational numbers.

In the above notation, we call

$$c_1\pi(\mathbf{d}^1) + \cdots + c_p\pi(\mathbf{d}^p)$$

the *BS-decomposition* (Boij-Söderberg decomposition).

Theorem 5.9 has been stated above for Cohen-Macaulay modules. However the theorem still holds if we consider the class of modules of codimension  $\geq c$ , as proved by Boij and Söderberg in [BS2]. The linear combination then involves degree sequences of length  $\geq c$ , with suitable extension of the partial order. In particular, for  $c = 0$  one has the class of all finitely generated graded modules.

A natural question is the following:

**Question 5.10.** *How does the BS-decomposition relate to the structure of the module?*

For instance, does the decomposition imply the existence of certain submodules? In general the BS-decomposition does not reflect structural properties of the module. The decomposition does indeed have a distinctive numerical flavor, the most obvious being that the coefficients in the decomposition are usually not integers. Consider for instance a general complete intersection of a form of degree 1 and a form of degree 2, and let  $I$  be the ideal in  $S$  they generate. The quotient ring  $S/I$  has a Betti table whose BS-decomposition is

$$\begin{bmatrix} 1 & 1 & - \\ - & 1 & 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 & 3 & - \\ - & - & 1 \end{bmatrix} + \frac{1}{3} \begin{bmatrix} 1 & - & - \\ - & 3 & 2 \end{bmatrix}.$$

The quotient ring corresponds to an irreducible codimension two variety in projective space, which is a degree 2 hypersurface in a hyperplane. The two Betti diagrams on the right do not correspond to any submodules of  $S/I$ , nor of  $(S/I)^3$ .

In order to be interesting, Question 5.10 has to be given a certain perspective. One line of inquiry is to consider combinatorially defined ideals and modules. Then the coefficients in the BS-decomposition can be interpreted combinatorially as counting something; see Nagel, Sturgeon [NS] and Engström, Stamps [ES].

Eisenbud, Erman, and Schreyer [EES] consider a module  $M$ , where the degree sequences  $\mathbf{d}^i$  in the decomposition are sufficiently “apart”, and then deduce the existence of a submodule  $M' \subseteq M$  corresponding to the first term in the BS-decomposition, or more generally a filtration  $M^0 \subseteq \dots \subseteq M^p = M$  such that  $M^i/M^{i-1}$  has Betti table  $c_i\pi(\mathbf{d}^i)$ . More generally, an interesting question is *which rays of Betti tables do not contain any module whose annihilator is a prime ideal*. In other words, the diagrams in the ray force the module to have some distinguished submodule(s).

Two other tracks of research deal with the following problems:

- (1) *Consider rings  $R$  other than polynomial rings. What are the extremal rays of the cone of Betti tables in this situation? How does the decomposition work?*
- (2) *Make a polynomial ring  $S$  more finely graded (multigraded), and consider the class of modules with this finer grading. What are the extremal rays of the cone of Betti tables in this situation? How does the decomposition work?*

The first question has been investigated by Berkesch, Burke, Erman, Gibbons in [BBEG] and by Kummini, Sam in [KS] for simple rings of low dimension or embedding dimension. In [EE, Section 9] Eisenbud and Erman consider finite extensions of polynomial rings.

A largely unexplored direction is to consider BS-decompositions for non-commutative rings. Here, there are many rings which, like the polynomial rings, have finite global dimension.

The second question has been given a start in the toric setting for Cox rings in [EE, Section 11], and for polynomial rings in two variables in [BF] by Boij, Fløystad, and in [BEK] by Berkesch, Erman, Kummini.

The survey article [Fl1] gives a more detailed introduction to Boij-Söderberg theory.

6. THE CONNECTION TO VECTOR BUNDLES ON PROJECTIVE SPACES

In this section we assume the reader is familiar with some basic notions in algebraic geometry: coherent sheaves on projective spaces and cohomology of sheaves.

In order to prove the second part of the BS-conjectures, Theorem 5.9, Eisenbud and Schreyer discovered a very interesting and surprising connection to algebraic vector bundles (locally free sheaves) on projective spaces, and their cohomology tables. To prove Theorem 5.9 they investigated the facets of what was the conjectured positive cone of Betti tables of Cohen-Macaulay modules, and the equations of the supporting hyperplanes of these facets. They found that the coefficients in these equations were cohomological dimensions of a certain natural class of vector bundles on projective spaces, a class they called vector bundles with supernatural cohomology.

The projective space  $\mathbb{P}^m$  has line bundles  $\mathcal{O}_{\mathbb{P}^m}(d)$  where  $d \in \mathbb{Z}$ . If  $\mathcal{F}$  is a coherent sheaf on  $\mathbb{P}^m$ , we get the twisted sheaves  $\mathcal{F}(d) = \mathcal{F} \otimes_{\mathcal{O}_{\mathbb{P}^m}} \mathcal{O}_{\mathbb{P}^m}(d)$ . There are sheaf cohomology groups  $H^i(\mathbb{P}^m, \mathcal{F}(d))$ , and we denote by  $\gamma_{i,d}$  the vector space dimensions of these groups over the field  $k$ . The cohomology table of  $\mathcal{F}$  is the indexed table  $(\gamma_{i,d})$  where  $i = 0, \dots, m$  and  $d \in \mathbb{Z}$ . In analogy with the display of the Betti table (3.2), we display a cohomology table by putting  $\gamma_{i,d}$  in row  $i$  and column  $i+d$ :

$$\begin{array}{cccccc}
 & \vdots & \vdots & \vdots & \vdots & \\
 \cdots & \gamma_{2,-3} & \gamma_{2,-2} & \gamma_{2,-1} & \gamma_{2,0} & \cdots \\
 \cdots & \gamma_{1,-2} & \gamma_{1,-1} & \gamma_{1,0} & \gamma_{1,1} & \cdots \\
 \cdots & \gamma_{0,-1} & \gamma_{0,0} & \gamma_{0,1} & \gamma_{0,2} & \cdots \\
 \hline
 & -1 & 0 & 1 & 2 & \cdots
 \end{array}$$

**Example 6.1.** Let  $P$  be a point in the projective space  $\mathbb{P}^2$ , and let  $\mathcal{I}_P \subseteq \mathcal{O}_{\mathbb{P}^2}$  be the ideal sheaf of this point. There is a short exact sequence

$$0 \rightarrow \mathcal{I}_P \rightarrow \mathcal{O}_{\mathbb{P}^2} \rightarrow \mathcal{O}_P \rightarrow 0,$$

where  $\mathcal{O}_P$  is the structure sheaf of the point  $P$ . From the associated long exact cohomology sequence, we compute the following cohomology table of the ideal sheaf  $\mathcal{I}_P$  (a single  $\cdot$  denotes a zero entry):

$$\begin{array}{cccccccc}
 \cdots & 6 & 3 & 1 & \cdot & \cdot & \cdot & \cdots \\
 \cdots & 1 & 1 & 1 & 1 & \cdot & \cdot & \cdots \\
 \cdots & \cdot & \cdot & \cdot & \cdot & 2 & 5 & 9 \cdots \\
 \hline
 & -3 & -2 & -1 & 0 & 1 & 2 & 3 \cdots
 \end{array}$$

**Example 6.2.** Let  $\mathcal{E}$  be the kernel sheaf of the map

$$\mathcal{O}_{\mathbb{P}^2}(-2) \oplus \mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-1) \xrightarrow{[x_0^2, x_1, x_2]} \mathcal{O}_{\mathbb{P}^2}.$$

Since the above map is surjective at any point of  $\mathbb{P}^2$ , the kernel  $\mathcal{E}$  is a locally free sheaf of rank 2. Again the long exact cohomology sequence enables the computation of the cohomology table of  $\mathcal{E}$ :

$$\begin{array}{cccccccc}
 \cdots & 19 & 11 & 5 & 1 & \cdot & \cdot & \cdots \\
 \cdots & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdots \\
 \cdots & \cdot & \cdot & \cdot & \cdot & 1 & 5 & 11 & 19 \cdots \\
 \hline
 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 \cdots
 \end{array}$$

In Example 6.2 the middle row contains a finite number of non-zero entries, while this is not the case in Example 6.1. For a cohomology table of a coherent sheaf on  $\mathbb{P}^n$ , the property that all rows between the (top)  $n$ th row and bottom 0th row contain a finite number of non-zero entries is equivalent to the sheaf being a locally free sheaf, also called an algebraic vector bundle, thus confirming that the sheaf  $\mathcal{E}$  is a vector bundle.

Among vector bundles there is a nice class constituting the analogs of Cohen-Macaulay modules with pure resolutions.

**Example 6.3.** The sheaf of differentials  $\Omega_{\mathbb{P}^2}$  is the kernel sheaf of the map

$$\mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-1) \xrightarrow{[x_0, x_1, x_2]} \mathcal{O}_{\mathbb{P}^2}.$$

It is a vector bundle with cohomology table:

$\dots$	15	8	3	.	.	.	.	.	$\dots$
$\dots$	.	.	.	1	.	.	.	.	$\dots$
$\dots$	.	.	.	.	3	8	15	24	$\dots$
	-2	-1	0	1	2	3	4	5	$\dots$

We see that in each column there is exactly one non-zero entry.

**Definition 6.4.** An algebraic vector bundle on a projective space is a vector bundle with *supernatural cohomology* if each column in its cohomology table has exactly one non-zero entry.

These vector bundles are the analogs on the vector bundle side of Cohen-Macaulay modules with pure resolutions. Here is a more detailed characterization of such vector bundles.

*Fact 6.5.* A vector bundle  $\mathcal{E}$  on  $\mathbb{P}^m$  has supernatural cohomology if and only if there is a sequence of integers  $r_m < r_{m-1} < \dots < r_1$ , called a *root sequence*, such that:

- The Hilbert polynomial

$$p(d) := \sum (-1)^i \dim_k H^i(\mathbb{P}^m, \mathcal{E}(d))$$

is the following, where  $\rho$  is the rank of  $\mathcal{E}$ :

$$\frac{\rho}{m!} \prod_{i=1}^m (d - r_i).$$

- For a given  $i$ , the dimension of the cohomology group  $H^i(\mathbb{P}^m, \mathcal{E}(d))$  is  $|p(d)|$  in the range  $r_{i+1} < d < r_i$  while it is zero otherwise (we let  $r_{m+1} = -\infty$  and  $r_0 = +\infty$ ).

There are several natural constructions of such vector bundles for any root sequence; see [Fl1, Sections 3.2 and 3.3].

**Example 6.6.** The sheaf of differentials  $\Omega_{\mathbb{P}^2}$  has root sequence  $(-1, 1)$ . The twist of the structure sheaf  $\mathcal{O}_{\mathbb{P}^2}(d)$  is a vector bundle with supernatural cohomology and root sequence  $(-d - 2, -d - 1)$ .



Given a free resolution  $\mathbf{F}$  of length  $\leq m + 1$  and a coherent sheaf  $\mathcal{F}$  on  $\mathbb{P}^m$ , Eisenbud and Schreyer [ES1] defined pairings  $\langle \mathbf{F}, \mathcal{F} \rangle_{e,t}$  where  $e \in \mathbb{Z}$  and  $t = 0, \dots, m$ , defined in terms of the Betti table of  $\mathbf{F}$  and the cohomology table of  $\mathcal{F}$ . To give the flavor, here is an example:

**Example 6.7.** Let  $m = 1, e = t = 1$ . Then the pairing  $\langle \mathbf{F}, \mathcal{F} \rangle_{1,1}$  is

$$\begin{aligned} & \sum_d \beta_{0,d} \gamma_{0,-d} - \sum_{d \leq 1} \beta_{1,d} (\gamma_{0,-d} - \gamma_{1,-d}) - \sum_{d > 1} \beta_{1,d} \gamma_{0,-d} \\ & + \sum_{d \leq 2} \beta_{2,d} (\gamma_{0,-d} - \gamma_{1,-d}) + \sum_{d > 2} \beta_{2,d} \gamma_{0,-d}. \end{aligned}$$

A conceptual understanding of these pairings is developed in [EE].

**Theorem 6.8** ([ES1]). *If  $\mathbf{F}$  is a minimal free resolution, then  $\langle \mathbf{F}, \mathcal{F} \rangle_{e,t}$  is non-negative.*

So for any coherent sheaf  $\mathcal{F}$ , the Betti table  $(\beta_{i,i+j}^S(M))$  of a module  $M$  is in the positive half-space of the hyperplane  $\langle -, \mathcal{F} \rangle_{e,t} = 0$ . It turns out that supporting hyperplanes of the cone of Betti tables of Cohen-Macaulay modules of codimension  $c$  are given by these pairings where  $\mathcal{F}$  are vector bundles with supernatural cohomology on the projective space  $\mathbb{P}^{c-1}$ .

The pairings  $\langle \mathbf{F}, \mathcal{F} \rangle_{e,t}$  between Betti tables and cohomology tables of vector bundles enables one to shift focus from Betti tables to the cohomology tables of vector bundles. The positive rays  $q \cdot (\gamma_{i,d}(\mathcal{E}))$ , with  $q > 0, q \in \mathbb{Q}$ , of vector bundles  $\mathcal{E}$  on the projective space  $\mathbb{P}^m$  form a cone, since for vector bundles  $\mathcal{E}_1$  and  $\mathcal{E}_2$  and natural numbers  $c_1$  and  $c_2$ , the linear combination  $c_1(\gamma_{i,d}(\mathcal{E}_1)) + c_2(\gamma_{i,d}(\mathcal{E}_2))$  is the cohomology table of the vector bundle  $\mathcal{E}_1^{c_1} \oplus \mathcal{E}_2^{c_2}$ . While the supporting hyperplanes for the cone of Betti tables of Cohen-Macaulay modules of codimension  $c$  were  $\langle -, \mathcal{E} \rangle_{e,t}$  where  $\mathcal{E}$  varies over vector bundles on  $\mathbb{P}^{c-1}$  with supernatural cohomology, the supporting hyperplanes of the facets of the cone of cohomology tables of vector bundles on  $\mathbb{P}^{c-1}$  are the hyperplanes  $\langle \mathbf{F}, - \rangle_{e,t}$  where  $\mathbf{F}$  varies over the *pure resolutions* of Cohen-Macaulay modules of codimension  $c$ . There arises the following analog of Theorem 5.9:

**Theorem 6.9.** *For any vector bundle  $\mathcal{E}$  on  $\mathbb{P}^{c-1}$  there is a unique chain of root sequences  $\mathbf{r}^1 < \mathbf{r}^2 < \dots < \mathbf{r}^p$  such that the cohomology table of  $\mathcal{E}$  is uniquely a positive rational linear combination*

$$c_1 \gamma(\mathbf{r}^1) + c_2 \gamma(\mathbf{r}^2) + \dots + c_p \gamma(\mathbf{r}^p)$$

*of cohomology tables of vector bundles with supernatural cohomology.*

**Example 6.10.** The sheaf of differentials  $\Omega_{\mathbb{P}^2}$  has supernatural cohomology with root sequence  $(-1, 1)$  and its twist  $\Omega_{\mathbb{P}^2}(-1)$  (whose cohomology table is that of  $\Omega_{\mathbb{P}^2}$  but shifted one step to the right) has root sequence  $(0, 2)$ . Denote their cohomology tables as  $\gamma(-1, 1)$  and  $\gamma(0, 2)$ .

The vector bundle  $\mathcal{E} \oplus \mathcal{O}_{\mathbb{P}^2}(-2)^2$  does not have supernatural cohomology as we see from Example 6.2. Its cohomology table is readily seen to be the positive linear combination

$$\gamma(-1, 1) + \gamma(0, 2).$$

For certain special classes of vector bundles, a categorification of the decomposition in Theorem 6.9 of the cohomology table of a vector bundle is developed in [ErSa].

**Question 6.11.** *What can one say about the cone of cohomology tables of coherent sheaves on a projective space in general?*

This is considerably more difficult, and our understanding of this needs to be developed. Eisenbud and Schreyer [ES3] generalize the theorem above to coherent sheaves, but have to allow infinite decompositions, and cannot even guarantee rational coefficients. Fløystad [Fl2] conjectures what the cohomology tables of coherent sheaves should be if regularity conditions on a coherent sheaf  $\mathcal{F}$  and its derived dual are specified.

One last thing should be mentioned: An *Ulrich sheaf* on a  $q$ -dimensional scheme  $X \subset \mathbf{P}^m$  may be defined as a coherent sheaf  $\mathcal{U}$  on  $X$  such that  $\pi_*\mathcal{U} \cong \mathcal{O}_{\mathbf{P}^q}^r$  for some  $r > 0$  and a general linear projection  $\pi : X \rightarrow \mathbf{P}^q$ . This is equivalent to the cohomology table of  $\mathcal{U}$  being a multiple of the cohomology table of the structure sheaf  $\mathcal{O}_{\mathbf{P}^q}$  on the projective  $q$ -space. As an example, consider a curve  $X$  of genus  $g$  embedded in  $\mathbf{P}^m$ . A line bundle  $\mathcal{U}$  on  $X$  is an Ulrich sheaf if  $\mathcal{U}(-1)$  has degree  $g-1$  and no global sections. In [ES4, Theorem 4.2] Eisenbud and Schreyer show that if a projective variety  $X$  has an Ulrich sheaf, then the positive cone of cohomology tables of coherent sheaves on  $X$  is the same as the positive cone for a projective space. A prominent conjecture in projective algebraic geometry states that *there exists an Ulrich sheaf on any projective variety* [ES2, p.543]. If this holds, it shows that the positive cone of cohomology tables does not depend on the geometry of the underlying variety, but rather represents some universal mathematical structure. For more on Ulrich sheaves, we refer the reader to the expository paper [ES5].

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