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SELECTED MATHEMATICAL REVIEWS

related to the paper in the previous section by
PETER KUCHMENT

MR0164098 (29 #1397) 34.51; 01.60

Ljapunov, A. M.

Исследоование одного из особенных случаев об устоирев чивости двитения. (Russian)

Izdat. Leningrad. Univ., *Leningrad*, 1963, 116 pp.

In this review $[u, v, \cdots]_k$ stands for a power series in $u, v, \cdots$, holomorphic at the origin and with no terms of degree $< k$. It may also denote (clear from the context) a vector whose components behave as stated relative to the components of $u, v, \cdots$. The letters z, P represent an n-vector and an $n \times n$ constant stable matrix (all characteristic roots have negative real parts). x, y denote scalars. In his opus magnum [Thesis, Kharkov, 1892; French transl., *Problème général de la stabilité du mouvement*, Ann. Fac. Sci. Univ. Toulouse (2) **9** (1907), 203–474; photolithoprinted, Princeton Univ. Press, Princeton, N.J., 1947; MR0021186] Liapunov investigated in full the stability problem of the system (1) $\dot{y} = Y(y, z)$, $\dot{z} = Pz + Z(y, z)$, where Y and Z are $[y, z]_2$. There is a unique solution $u(y)$ of $Pu + Z(y, u) = 0$ and the substitution $z \to z + u$ reduces (1) to $\dot{y} = gy^m + \cdots$, $m > 1$, $\dot{z}$ as before. The motion is asymptotically stable if m is odd and $g < 0$, and unstable otherwise.

Now in 1893 Liapunov published a large paper entitled "Investigation of one of the special cases of the stability of motion" [Mat. Sb. **17** (1893), 253–333; also his *Collected works* (Russian), Vol. II, Izdat. Akad. Nauk SSSR, Moscow, 1956; MR0156047]. In this paper he stated that he proposed to discuss the stability of a system (2) $\dot{x} = y + X(x, y, z)$, $\dot{y} = Y(x, y, z)$, $\dot{z} = Pz + Z(x, y, z)$, where X, Y, Z are $[x, y, z]_2$, but actually the paper only dealt with the preliminary problem of (3) $\dot{x} = y + X(x, y)$, $\dot{y} = Y(x, y)$, where X, Y are $[x, y]_2$. In fact, by a series of quite difficult transformations Liapunov succeeded in treating this case completely.

At the time of the recent publication of the *Complete works* there was discovered an extensive manuscript of Liapunov under the same title as the large paper in which he dealt with the full problem of (2) but was unable to terminate the research. Presumably this was the reason why the manuscript was withheld from publication. The present monograph presents this manuscript preceded by an excellent summary by V. P. Bassov.

Let $Y(x, y, 0) = Y_0(x) + Y_1(x)y + W(x, y)y^2$, $Z(x, y, 0) = Z_0(x) + Z_1(x)y + U(x, y)y^2$, where the coefficients are holomorphic in x or x, y at the origin. If the functions other than U, W are not identically 0, let their expansions be $Y_0(x) = gx^m + g_1x^{m+1} + \cdots$ $(m \geq 2)$; $Y_1(x) = ax^\alpha + a_1x^{\alpha+1} + \cdots$ $(\alpha > 1)$; $Z_0(x) = hx^p + \cdots$, $Z_1(x) = kx^\beta + \cdots$. Here g, g_1, a, a_1 are scalar constants and h, k are constant n-vectors. By transformations not affecting the form of the system one may obtain this situation: (a) If $Y_0(x) = Y_1(x) \equiv 0$, then also $Z_0(x) = Z_1(x) \equiv 0$; (b) if $Y_0(x) \equiv 0$, $Y_1(x) \not\equiv 0$, then $\beta > \alpha$; (c) if $Y_0(x) \not\equiv 0$, then $p > m$ and also $\beta > \alpha$ if $\alpha < m$ and $\beta \geq m$ if $\alpha \geq m$. In addition, one may always assume that $X(x, 0, 0) \equiv 0$. Liapunov distinguishes these four cases: (I) $Y_0(x) \equiv 0$; (II) m even; (III) m odd,

$g > 0$; (IV) m odd, $g < 0$. For the cases (I), (II), (III), and (IV) with $\alpha < m$ and α even, Liapunov was able to conclude the treatment along the same lines as his 1893 paper, always using his stability theorems, except that when it came to instability his own theorem on instability had to be replaced by what amounted to using the Četaev generalization. It is actually the full treatment of (IV) that Liapunov was unable to carry out to completion. It still remains as an open problem. [Additional reference: Četaev, *Stability of motion* (Russian), 2nd ed., GITTL, Moscow, 1955; MR0077746].

{For additional information pertaining to this item see [L. Kurakin, Vladikavkaz. Mat. Zh. **11** (2009), no. 3, front matter, 28–37; MR2559078].}

REVISED (May, 2010)

S. Lefschetz

From MathSciNet, April 2016

MR1154209 (93e:01035) 01A75; 34D05, 34D20, 70K15, 70K20, 93D05

Lyapunov, A. M.

The general problem of the stability of motion. (English)

International Journal of Control **55** (1992), *no.* 3, 521–790.

It is almost criminal that Lyapunov's fundamental monograph on stability had to wait 100 years for an English translation. It was published in 1892 [Kharkov. Mat. Obshch, Kharkov, 1892], at the same time as Poincare's *Les methodes nouvelles de la mecanique celeste*, Vol. I [Gauthier-Villars, Paris, 1892; Jbuch 24, 1130]. Lyapunov refers to the earlier work of Torricelli, Liouville, Tait, and to Poincare's works ("... in a large part of my researches, I was guided by the ideas developed in the [Poincare's] above-mentioned memoir").

A French translation by E. Davaux (reviewed and corrected by Lyapunov) appeared in 1907 [Ann. Fac. Sci. Toulouse (2) 9 (1907), 203–474], and was reprinted in 1949 [Ann. of Math. Stud., 17, Princeton Univ. Press, Princeton, NJ, 1947; MR0021186]. The present English translation by Fuller of Cambridge University is based on the French translation. It is amply annotated and even has an index of key concepts. It is preceded by a translator's introduction which contains a full discussion of the historical origins of this work and an analysis of Lyapunov's results. It also contains a useful short reference list of historical and expository articles.

The translation of Lyapunov's monograph is complemented by a short (10 pages) but very informative biography. (For example, I didn't know that Steklov was a student of Lyapunov's.) This is a translation (with very useful additional notes), by Barrett of Southampton, of the 1948 biography by Smirnov which was included in Lyapunov's collected works. In addition, Barrett provides a bibliography of Lyapunov's publications, including not just his scientific works but also his lecture notes from courses at Kharkov and his "Reviews, translations, obituaries, etc."

Readers should be aware (and Fuller points out) that the translation Stability of motion, by F. Abramovici and M. Shimshoni [Academic Press, New York, 1966; MR0208093], is not the same as the monograph under review—it is essentially the remaining case not discussed by Lyapunov in his 1892 monograph (the linear approximation has two zero eigenvalues), and was published posthumously.

The scientific and engineering community owes Fuller, Barrett, and the International Journal of Control a large debt of graditude for providing us with an English version of this seminal (and readable) work.

J. W. Macki
From MathSciNet, April 2016

MR0178246 (31 #2504) 35.45; 35.80
Agmon, Shmuel
Lectures on elliptic boundary value problems. (English)
Prepared for publication by B. Frank Jones, Jr. with the assistance of George W. Batten, Jr.
Van Nostrand Mathematical Studies, No. 2.
D. Van Nostrand Co., Inc., *Princeton, N.J.–Toronto–London*, 1965, v+291 pp.

Based on notes from a summer course in 1963, this book gives a very clear and thorough introduction to the theory. Many results of fundamental importance for the subject are here presented in a readily accessible form, often for the first time with detailed proofs and under minimal assumptions.

In the first part of the book, the author considers elliptic operators $A(x,D) = \sum c_\alpha(x)D^\alpha$ or bilinear forms $B[\varphi,\psi] = \sum \int_\Omega c_{\alpha\beta}(x)D^\alpha\varphi\overline{D^\beta\psi}\,dx$ in a domain $\Omega \subset R^n$. It is always assumed that A and B have at least bounded, measurable lower-order coefficients and continuous leading coefficients. The domain Ω possesses the restricted (i.e., local) cone property or, in some cases, the segment property. The definitions of the cone and segment properties are given in the introductory chapters, together with a discussion of the relations between the Sobolev spaces $H_m(\Omega)$ and $W_m(\Omega)$ of distributions with strong or weak derivatives of order $\leq m$ in Ω. Careful proofs are given of global forms of Rellich's lemma and the Sobolev inequalities. Then follow local existence and regularity theorems for elliptic equations (or overdetermined systems) with s-smooth coefficients, $s \geq 1$, the case $s = 1$ also covering Lipschitz-continuous coefficients. A general form of Poincaré's inequality is derived, as well as necessary and/or sufficient conditions for Gårding's inequality. The Lax-Milgram theorem for positive, continuous bilinear forms on a Hilbert space is stated and used to prove a global existence theorem for the generalized Dirichlet problem. Finally, a "fundamental regularity lemma" leads to a priori estimates and to results about the global regularity of the solutions of the Dirichlet problem in domains Ω of class C^{2m+k}, $k \geq 0$, $2m =$ order of A.

The next topic is the coerciveness of a bilinear form with respect to a linear space V with $\mathring{\mathrm{H}}_m(\Omega) \subset V \subset H_m(\Omega)$, $\mathring{\mathrm{H}}_m(\Omega)$ being the closure of $C_0{}^\infty(\Omega)$ in $H_m(\Omega)$. In some examples with $V = H_m(\Omega)$, the corresponding "natural boundary conditions" are derived from a Green's formula. Next follow some theorems from the Calderon-Zygmund theory of singular integral operators, and a proof of "Sobolev's representation formula". These tools are then used, together with Hilbert's Nullstellensatz, in the proof of the Aronszajn-Smith conditions for coercivity over $V = H_m(\Omega)$, when Ω is a bounded domain with the restricted cone property. (A useful corollary is that $(*)$ if u and its weak derivatives $D_j{}^k u$, $1 \leq j \leq m$, belong to $L^2(\Omega)$, then $u \in H_m(\Omega)$, and $\|u\|_{m,\Omega} \leq C\{\sum \|D_j{}^m u\|_{0,\Omega} + \|u\|_{0,\Omega}\}$.) The same methods are also used to prove the existence of a bounded linear extension of $H_m(\Omega)$ into $H_m(R^n)$—the Calderon extension theorem.

In the last part of the book the author studies spectral properties of certain abstract operators on a complex Hilbert space, with applications to elliptic operators of the type discussed before, not necessarily self-adjoint. A first chapter is concerned with linear operators A possessing a "direction of minimal growth", i.e., for which there are real numbers θ and s_0 such that the modified resolvent $A_\lambda = A(1-\lambda A)^{-1}$ exists and $\|A_\lambda\| = O(s^{-1})$ as $\lambda = se^{i\theta}$, $s_0 \leq s \to \infty$. (It is proved that $\|A_\lambda\|$ cannot decay faster than $O(|\lambda|^{-1})$ along any ray.) Also discussed are (compact) operators T with finite "double-norm" $|||T||| = \sum \|T\varphi_i\|^2$ (where $\{\varphi_i\}_1{}^\infty$ is any orthonormal basis for the Hilbert space X), which reduce to integral operators with Hilbert-Schmidt kernel when $X = L^2(\Omega)$. Let then T be a bounded linear operator on $L^2(\Omega)$ with range $R(T) \subset H_m(\Omega)$, for instance, the operator from $L^2(\Omega)$ into V defined by the Lax-Milgram theorem, for which $B[v, Tf] = (v, f)_{0,\Omega}$, $v \in V$, $f \in L^2(\Omega)$, when B is strongly coercive over V, $V = \overline{V} \subset H_m(\Omega)$. It is proved that T is bounded from $L^2(\Omega)$ to $H_m(\Omega)$, with norm $\|T\|_m < \infty$, and Sobolev's inequality is used to derive the estimate $|||T||| \leq C|\Omega|^{1/2}\|T\|_m{}^{n/2m}\|T\|^{1-n/2m}$, when $m > n/2$. Applying this inequality to T_λ, $\lambda = te^{i\theta}$, when θ is a direction of minimal growth, the author proves that the number of eigenvalues with modulus $\leq t$ is given by $N(t) = \sum_{|\lambda_j| \leq t} 1 \leq 4t^2|||T_\lambda|||^2 \leq c_1|\Omega|\|T\|_m{}^{n/m}t^{n/m}$. Next, it is proved that if $R(T)$, $R(T^*) \subset H_m(\Omega)$, $m > 2[n/2] + 1$, then T has a Hilbert-Schmidt kernel $K(x, y) \in H_k(\Omega \times \Omega)$, $k = k(m) > [n/2]$. (The proof uses $(*)$ and the observation that $\Omega \times \Omega$ has the restricted cone property if Ω has it.) It follows that if there is a direction θ of minimal growth for T_λ, then $\{\int_\Omega |K_\lambda(x, x)|^2\, dx\}^{1/2} = O(t^{-1+n/m})$, and $\int_\Omega K_\lambda(x, x)\, dx = \sum(\lambda_j - \lambda)^{-1}$ for $\lambda = te^{i\theta}$, $t > t_0$.

Now consider a self-adjoint extension $\mathcal{A}$ of an elliptic operator $A(x, D)$ of order m', with $C_0{}^\infty(\Omega) \subset D(\mathcal{A}^k) \subset H_{km'}(\Omega)$, $k = [n/m'] + 1$. Then the author puts $T = (\mathcal{A}^k)^{-1}$ (supposing that O is in the resolvent set of $\mathcal{A}$), compares K_λ locally with certain fundamental solutions, and concludes that $\int_\Omega K_\lambda(x, x)\, dx = ct^{n/m'-1}(1 + o(1))$, where c is well determined, when $\lambda = te^{i\theta}$, $t \to \infty$, $\theta \neq k\pi$. Then, a Tauberian theorem is used to derive asymptotic estimates for the largest of $N_+(t)$ and $N_-(t)$ (or both, if $n = 1$, m' odd), where $N_\pm(t) = \sum_j 1$, $0 < \pm\lambda_j \leq t$. A similar result is obtained when $A(x, D)$ is elliptic of order $m' = 2m$ with real leading coefficients. In particular, let $\mathcal{A} = \mathcal{A}_0 + \mathcal{B}$, where $\mathcal{A}_0$ is self-adjoint in $L^2(\Omega)$, $D(\mathcal{A}_0) \subset H_{m'}(\Omega)$, and where $D(\mathcal{A}_0) \subset D(\mathcal{B}) \subset L^2(\Omega)$, $\|\mathcal{B}u\|_{0,\Omega} \leq C\|u\|_{m'-1,\Omega}$. Then it is proved that the eigenvalues λ of $\mathcal{A}$ cluster in a "parabolic" region, $|\operatorname{Im}\lambda| \leq c(1 + |\operatorname{Re}\lambda|^{1-1/m'})$, around the real axis, an extension of a theorem by Carleman. For general elliptic operators A, finally, the idea used is to regard $A - \lambda$ as a reduced form of an elliptic operator in $n + 1$ dimensions.

The last chapter contains the following result, based on abstract theorems by Carleman and Dunford-Schwartz. Let $\mathcal{A}$ be closed and densely defined in $L^2(\Omega)$, $D(\mathcal{A}^k) \subset H_{km'}(\Omega)$, $k = [n/2m'] + 1$, with a sufficiently dense set of directions of minimal growth for the resolvent. Then the eigenfunctions are complete in $L^2(\Omega)$.

The book ends with an adequate index, but unfortunately it contains no bibliography and very few references to the literature.

J. Friberg

From MathSciNet, April 2016

MR0832922 (87k:35184) 35P15; 35J10, 47F05, 81C10

Deift, Percy A.; Hempel, Rainer

On the existence of eigenvalues of the Schrodinger operator $H - \lambda W$ in a gap of $\sigma(H)$.

Communications in Mathematical Physics **103** (1986), *no.* 3, 461–490.

Consider the Schrödinger operator $H_0 = -\Delta + V$ in $L^2(\mathbf{R}^n)$, $n \geq 1$, and define $H_\lambda = H_0 - \lambda W$ for some short-range interaction W. The basic question addressed in this paper is: Given $E \in \mathbf{R} \setminus \sigma(H_0)$, $\sigma(H_0)$ denoting the spectrum of H_0, does there exist $\lambda \in \mathbf{R}$ such that $E \in \sigma(H_\lambda)$? If W decays and W does not change sign or λ is below $\sigma(H_0)$ the question can be answered by standard Birman-Schwinger techniques.

Let $S \subset \mathbf{R}$. The authors state that (H_0, W, S) is complete if for each $E \in \mathbf{R} \setminus \sigma(H_0)$ there exists $\lambda = \lambda(E) \in S$ such that $E \in \sigma(H_\lambda)$. Let $\bigcup_{k=1}^N O_k = \mathbf{R} \setminus \sigma(H_0)$, $1 \leq N \leq \infty$, O_k open, disjoint. Then (H_0, W, S) is said to be essentially complete, provided that for each k there exists at most one energy $E_k \in O_k$ such that $E_k \notin \bigcup_{\lambda \in S} \sigma(H_\lambda)$. The E_k are called exceptional levels. It is easily seen that if $\bigcup_{\lambda > 0} \sigma(H_\lambda)$ is dense in $\mathbf{R} \setminus \sigma(H_0)$, then there is at most one exceptional value in each gap. The main general result is that if $V, W\colon \mathbf{R}^n \to \mathbf{R}$ are bounded, $W(x) \geq h > 0$ for x in some ball B and $|x|^2 W(x) \to \infty$ as $|x| \to \infty$, then $(H_0, W, [0, \infty))$ is essentially complete. Furthermore, it is proved that if W has compact support and E is an exceptional level, then E is an eigenvalue of the operator H_0 in $L^2(\mathbf{R}^n \setminus \operatorname{supp} W)$ with Dirichlet boundary conditions.

If W is assumed to satisfy $(\inf_{A_{n,i}} W)^{-1} \operatorname{osc}_{A_{n,i}} W \to_{i \to \infty} 0$ for any $n \in \mathbf{N}$, where $\operatorname{osc}_A W = \sup_A W - \inf_A W$ and $A_{n,i} = \{x\colon (i-1)n \leq |x| \leq (i+2)n\}$, then $(H_0, W, \mathbf{R})$ is also essentially complete.

Finally it is proved under various assumptions in one dimension ($n = 1$) that exceptional levels do not occur. However it is also shown by an explicit example that exceptional levels may exist.

Helge Holden

From MathSciNet, April 2016

MR1232660 (94h:35002) 35-02; 35C15, 35P10, 47N20

Kuchment, Peter

Floquet theory for partial differential equations. (English)

Operator Theory: Advances and Applications, 60.

Birkhauser Verlag, *Basel*, 1993, xiv+350 pp.

This is a very substantial monograph on the theory of linear partial differential equations with periodic coefficients. There is a significant distinction between the case of ordinary and partial differential periodic equations. The main tool of the theory of periodic ordinary differential equations is the classical Floquet theory. There are two standard ways of proving the Floquet representation theorem: the first, by converting the periodic system into a system with constant coefficients; the second, by using the monodromy operator. These methods do not apply to periodic partial differential equations.

On the other hand, there is an obvious similarity between constant coefficient and periodic partial differential equations: in both cases the operators are equivariant under natural abelian group actions. In the first case we are dealing with the Lie group $\mathbf{R}^n$ acting on itself. After applying the Fourier transform in an appropriate space of test functions, we obtain an operator of multiplication by a polynomial in a space of analytic functions. Now it is possible to apply methods of complex analysis. In the periodic case the operator is equivariant with respect to the action of $\mathbf{Z}^n$ on $\mathbf{R}^n$. The approach of the author is to develop an appropriate Fourier transform, to obtain an operator of multiplication by an operator-valued function and then to apply the analytic Fredholm theory.

Holomorphic Fredholm operator function theory is introduced in Chapter 1. Chapter 2 provides the definition and properties of the transform that plays the role of the classical Fourier transform. The Floquet theory for hypoelliptic equations is presented in Chapter 3. Chapter 4 is devoted to properties of solutions of periodic equations. Chapter 5 treats evolution equations and Chapter 6 briefly discusses some related problems.

In particular, results are given about completeness of the set of Floquet solutions, the Floquet expansion and the structure of the Floquet variety. Among other topics, the solvability of the nonhomogeneous equation, the existence of bounded and decreasing solutions, and the structure of the set of positive solutions and the spectrum are also discussed.

The present book is based on research papers of the author and his co-authors who have worked on the theory over a period of time. The material is presented at an advanced graduate level. A few proofs are only sketched and technical points are sometimes skipped. The book includes several applications with frequent comments and an extensive list of references to the recent literature.

Yehuda Pinchover

From MathSciNet, April 2016

MR1472485 (2000i:35002) 35-02; 35J10, 35P05, 47A55, 47N50, 81Q10, 81Q15

Karpeshina, Yulia E.

Perturbation theory for the Schrödinger operator with a periodic potential. (English)

Lecture Notes in Mathematics, 1663.

Springer-Verlag, *Berlin*, 1997, viii+352 pp.

The book under review gives a detailed account of a method, based on perturbation theory, to prove the so-called Bethe-Sommerfeld conjecture: Let $n \geq 2$. Then any (semi-bounded) periodic Schrödinger operator $H_V = -\Delta + V(x)$ acting in $L_2(\mathbf{R}^n)$ has at most a finite number of spectral gaps, or, put differently, there exists a real number E_0 (which depends on the periodic potential V) such that the interval $[E_0, \infty)$ is contained in the spectrum of H_V. Of course, this requires appropriate regularity assumptions on the periodic potential V.

The first proofs for some important cases have been obtained previously by B. E. J. Dahlberg and E. Trubowitz [Comment. Math. Helv. **57** (1982), no. 1, 130–134; MR0672849] for $n = 2$ and M. M. Skriganov [Invent. Math. **80** (1985), no. 1, 107–121; MR0784531] for $n = 2, 3$. The present book follows a different approach which is based on analytic perturbation theory applied to the so-called "Bloch manifold" or "Bloch variety" of the unperturbed operator $H_0 = -\Delta$. The Bloch variety of a periodic Schrödinger operator H_V consists of all eigenvalues of the differential operators $H_V(\theta) = -\Delta + V(x)$, acting in a fundamental period cell, with θ-periodic boundary conditions, for θ ranging over a cell of the dual lattice [cf., e.g., M. Reed and B. Simon, *Methods of modern mathematical physics. IV. Analysis of operators*, Academic Press, New York, 1978; MR0493421 (Section XIII.16)]. The Bloch manifold will have self-intersections which are related to von Laue diffraction conditions in physics. By the decomposition theorem of Floquet-Gel′fand, the spectrum of H_V in $\mathbf{R}^n$ is obtained by projecting the Bloch variety onto the energy axis. This projection leads to plenty of overlap in the case of H_0 at high energy, and one may thus hope that the projection of the Bloch manifold for H_V will still cover some interval $[E_0, \infty)$.

Therefore, one has to control the movement of eigenvalues for $H_{\alpha V}(\theta)$ as α increases from 0 to 1. On the technical level, one would like to show that there is a power series for the eigenvalues of $H_{\alpha V}(\theta)$ which converges for $|\alpha| \leq 1$. This is not possible in general because isolated, simple eigenvalues of $H_{\alpha V}(\theta)$ may collide with nearby eigenvalues, as the parameter α increases from 0 to 1, and nasty problems with small denominators will arise. Hence the method is easier to implement if the eigenvalues are spread away from one another at high energy. In view of Weyl's law on the asymptotic distribution of eigenvalues it is then not too surprising that the easiest cases will be for operators of the form $(-\Delta)^l + \alpha V,\ |\alpha| \leq 1$, with $2l > n$. Here the author in fact shows that the power series for the perturbed eigenvalues from the Bloch manifold will converge for $|\alpha| \leq 1$, for a sufficiently large subset of the Bloch manifold chosen away from the self-intersections (Section 2). However, the condition $2l > n$ excludes Schrödinger operators ($l = 1$, $n = 3$ or $n = 2$).

In Section 3 of the book, the case $4l > n + 1$ is discussed, where von Laue diffraction conditions come into play and the interaction of a pair of eigenvalues is built into the expansion.

The most ambitious case considered here is for the Schrödinger operator in $\mathbf{R}^3$ (i.e., $l = 1$ and $n = 3$). Here a rather sophisticated analysis of generalized von Laue diffraction conditions is necessary to obtain the Bethe-Sommerfeld conjecture (Section 4). On the other hand, it is now possible to treat some singular potentials V (e.g., Coulomb potentials), as long as they are given in the form of trigonometric polynomials.

The last part (Section 5) of the book deals with the interaction of a free (plane) wave with a semi-crystal, i.e., with a periodic structure filling a half-space. Here a new type of an inverse problem is discussed (how to find the periodic potential from the high-energy reflection coefficients, knowing that the potential is a trigonometric polynomial).

The book starts from the Bloch manifold for the free Laplacian and develops a natural perturbation theory for this object, taking into account the important physical difference between the refractive and the nonrefractive regime. The introduction is rather long, but readable and gives a complete picture of the ideas and the content (similarly, the introductions to the individual sections give a flavor of what is going to happen). The actual hard core of the book is very detailed and presents virtually all relevant estimates and technicalities.

This book is a research monograph presenting the reader with a detailed and systematic exposition of the content of a large number of research papers devoted to a proof of the Bethe-Sommerfeld conjecture and the perturbation theory of Bloch manifolds.

Rainer Hempel

From MathSciNet, April 2016

MR1903839 (2003f:82043) 82B44; 34L40, 34M60, 47B80, 81Q05, 81Q20

Fedotov, Alexander; Klopp, Frédéric

Anderson transitions for a family of almost periodic Schrödinger equations in the adiabatic case.

Communications in Mathematical Physics **227** (2002), *no.* 1, 1–92.

In this paper, the authors carry out an elaborate study of the parameter-dependent quasiperiodic Schrödinger operator

$$H_{\phi,\epsilon}\psi(x) := -\frac{d^2}{dx^2}\psi(x) + \{V(x-\phi) + \alpha\cos(\epsilon x)\}\psi(x),$$

where $V \in L^2_{\rm loc}(\mathbf{R})$ is periodic with period 1, $\alpha > 0$, $\phi \in \mathbf{R}/\mathbf{Z}$ and $\epsilon > 0$ is such that $2\pi/\epsilon$ is irrational. The purpose of the study is to investigate the spectral properties of $H_{\phi,\epsilon}$ in the low-energy region when the parameter ϵ is small (adiabatic case). It is proved firstly that at low energies the spectrum of $H_{\phi,\epsilon}$ is confined in the union of intervals whose lengths are exponentially small in $1/\epsilon$ and which are separated by distances of the order of ϵ. Then the authors prove the coexistence of zones where the spectrum of $H_{\phi,\epsilon}$ is singular and other zones where most of the spectrum is absolutely continuous. The regions which are intermediate to these zones and in which the spectral property of $H_{\phi,\epsilon}$ is not determined become small as $\epsilon \to 0$, showing an asymptotically sharp transition from singular to absolutely continuous spectrum. Thus $H_{\phi,\epsilon}$ gives the first instance of a quasi-periodic Schrödinger operator which exhibits an Anderson transition.

The precise statement of the results of this paper is as follows. Let $H_0 := -d^2/dx^2 + V(x)$ be the auxiliary periodic Schrödinger operator. The spectrum $\sigma(H_0)$ of H_0 consists of a sequence of bands $[E_{2n-1}, E_{2n}]$, $n = 1, 2, \ldots$. It is assumed that the first gap (E_2, E_3) is open. By "low energy" is meant the vicinity of E_1, and the authors confine themselves to the energies satisfying $E_1 - \alpha < E < (E_1 + \alpha) \wedge (E_2 - \alpha)$. The study of spectral properties of $H_{\phi,\epsilon}$ is essentially reduced to the investigation of the behavior at infinity of the solution of the equation $H_{\phi,\epsilon}\psi = E\psi$, which in turn is reduced to the investigation of the behavior of the solution of the "monodromy equation" $F_{n+1} = M(E, \phi + nh)F_n$, $F_n \in \mathbf{C}^2$, $n \in \mathbf{Z}$, where $h = 2\pi/\epsilon \pmod 1$ and M is the "monodromy matrix" defined in the following way. The pair of solutions (ψ_1, ψ_2) of $H_{\phi,\epsilon}\psi = E\psi$ is called a consistent basis if their Wronskian is nonzero and does not depend on ϕ and if $\psi_j(x, \phi)$ is 1-periodic in ϕ. If (ψ_1, ψ_2) is a consistent basis, then $\psi_j(x + 2\pi/\epsilon, \phi + 2\pi/\epsilon)$, $j = 1, 2$, are again solutions of $H_{\phi,\epsilon}\psi = E\psi$, so that there is a 2×2 matrix $M(E, \phi)$ such that

$$\begin{pmatrix}\psi_1(x+2\pi/\epsilon, \phi+2\pi/\epsilon)\\ \psi_2(x+2\pi/\epsilon, \phi+2\pi/\epsilon)\end{pmatrix} = M(E,\phi)\begin{pmatrix}\psi_1(x,\phi)\\ \psi_2(x,\phi)\end{pmatrix}.$$

It is shown that there is a one-to-one correspondence between solutions f of $H_{\phi,\epsilon}\psi = E\psi$ and solutions F of the monodromy equation, which is given by

$$\begin{pmatrix} f(x+2\pi n/\epsilon, \phi)\\ f'(x+2\pi n/\epsilon, \phi)\end{pmatrix} = \Psi(x, \phi - nh)\sigma F_{-n},$$

where $\Psi = \begin{pmatrix}\psi_1 & \psi_2\\ \psi_1' & \psi_2'\end{pmatrix}$ and $\sigma = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$. In particular, the Lyapunov exponents $\Theta(E, \phi)$ and $\theta(E, \phi)$, corresponding respectively to the Schrödinger equation and the monodromy equation, are related by $\Theta = (\epsilon/2\pi)\theta$.

To obtain the spectral results in the adiabatic case, it is necessary to have good asymptotic estimates for the coefficients of M as $\epsilon \to 0$. For this purpose, suppose we take a consistent basis so that the monodromy matrix M takes the form $M = \left(\begin{smallmatrix} a & b \\ a^* & b^* \end{smallmatrix}\right)$, where $a = a(\phi, E)$, $b = b(\phi, E)$ and $a^*(\phi, E) = \overline{a(\overline{\phi}, \overline{E})}$, $b^*(\phi, E) = \overline{b(\overline{\phi}, \overline{E})}$. If we let a_0 and $\tilde{a}(\phi)$ [resp. b_0 and $\tilde{b}(\phi)$] be the zeroth and the first Fourier coefficients of the 1-periodic function $a(\phi)$ [resp. $b(\phi)$], then it is proved (Theorem 2.2 of the paper) that, by suitably choosing the consistent basis, $M(E, \phi)$ is analytic in E and ϕ, and that the Fourier coefficients introduced above have the asymptotic expressions $a_0 = (1/t_h)e^{i\Phi/\epsilon}(1 + o(1))$, $\tilde{a}(\phi) = -(t_v/t_h)e^{2i\pi(\phi-\phi_0)}(1 + o(1))$ and $b_0 = ia_0(1 + o(1))$, $\tilde{b}(\phi) = i\tilde{a}(\phi)(1 + o(1))$ as $\epsilon \to 0$. The proof of these asymptotic formulas is based on the previous work of the present authors on the complex WKB method [Asymptot. Anal. **27** (2001), no. 3-4, 219–264; MR1858917], and comprises the main body of the paper under review. Here the phase Φ and the tunneling coefficients t_h, t_v appearing in the above formulae are roughly described as follows: A solution ψ of the periodic Schrödinger equation $H_0\psi = E\psi$ is called a Bloch solution if it satisfies $\psi(\cdot + 1) = \lambda\psi(\cdot)$ for some constant λ; $\lambda = \lambda(E)$ is called the Floquet multiplier, and the Bloch quasi-momentum $k(E)$ is defined, modulo 2π, by $\lambda(E) = \exp(ik(E))$. Fixing an analytic branch of $k(E)$, the authors define the complex momentum $\kappa(\varphi) := k(E - \alpha\cos\varphi)$. For low energy E, $\kappa(\varphi)$ has a unique branch point φ_1 in $(0, \pi)$. Let φ_2 and φ_3 be the branch points, lying on the line $\pi + i\mathbf{R}_+$, closest to π and such that $\operatorname{Im}\varphi_2 < \operatorname{Im}\varphi_3$. Let κ_* be the branch of the complex momentum which takes values in $[0, \pi]$ on the set $[\varphi_1, 2\pi - \varphi_1] \cup [\overline{\varphi}_2, \varphi_2]$. The authors then define the phase $\Phi = (1/\epsilon)\int_{\varphi_1}^{2\pi-\varphi_1} k_* d\varphi$ and the actions $S_h = -i\int_{-\varphi_1}^{\varphi_1} k_* d\varphi$ and $S_v = i\int_{\overline{\varphi}_2}^{\varphi_2}(\kappa_* - \pi)d\varphi$. Finally the tunneling coefficients are defined by $t_h = \exp(-\frac{1}{\epsilon}S_h)$ and $t_v = \exp(-\frac{1}{\epsilon}S_v)$. It is shown that for low energy E such that

$$E \in J_\delta := \{E\colon E_1 - \alpha + \delta < E < (E_2 - \alpha - \delta) \wedge (E_1 + \alpha - \delta)\},$$

δ being a fixed positive number, $S_h(E)$, $S_v(E)$ and $\Phi(E)$ are positive and $\Phi'(E) > 0$.

It is now possible to describe the spectral nature of $H_{\phi,\epsilon}$ and how it is derived from the asymptotic formulae of Theorem 2.2. To begin with, let Σ_ϵ denote the spectrum of the operator $H_{\phi,\epsilon}$. Since $\{H_{\phi,\epsilon}\colon \phi \in \mathbf{R}/\mathbf{Z}\}$ is an ergodic family of self-adjoint operators, Σ_ϵ is independent of the parameter ϕ. By a perturbation argument, it is shown that Σ_ϵ is contained in the set $\Sigma := \sigma(H_0) + [-\alpha, \alpha]$. In fact, Σ_ϵ is "dense" in Σ for small ϵ in the following sense: for any compact subset K of Σ, there is a constant $C > 0$ such that for all $\epsilon > 0$ and $E \in K$, Σ_ϵ intersects the interval $(E - C\sqrt{\epsilon}, E + C\sqrt{\epsilon})$. After giving this preliminary description of Σ_ϵ, the authors turn to the investigation of its precise nature in the low-energy region J_δ. Namely let $E^{(l)}$ ($l \in \mathbf{N}$) be defined by the "quantization condition" $(1/\epsilon)\Phi(E^{(l)}) = \pi(l + 1/2)$. Then for small $\epsilon > 0$, one has $c_1\epsilon \le E^{(l)} - E^{(l-1)} \le c_2\epsilon$, for some constants c_1 and c_2, because of $\Phi'(E) > 0$, and the total number of l's such that $E^{(l)} \in J_\delta$ is of order $1/\epsilon$. It is then shown that to these l's there correspond intervals $I_l \subset J_\delta$ with the following properties: (1) $\Sigma_\epsilon \cap J_\delta$ is contained in $\bigcup_l I_l$; (2) each I_l lies in an $o(\epsilon)$ neighborhood of $E^{(l)}$; (3) the length of I_l is exponentially small as $\epsilon \to 0$, i.e. $|I_l| = 2\epsilon(t_v(E^{(l)}) + t_h(E^{(l)}))/\Phi'(E^{(l)})(1 + o(1))$. It is also shown that if $N(dE)$ is the measure corresponding to the integrated density of states of $H_{\phi,\epsilon}$, then (4) $\int_{I_l} N(dE) = \epsilon/2\pi$. In order to prove the above results on the location of Σ_ϵ, the authors note that if E is such that the equation $H_{\phi,\epsilon}\psi = E\psi$ has two independent solutions with exponential behavior at $\pm\infty$, then E belongs to the resolvent set of $H_{\phi,\epsilon}$. But the existence of such a pair of solutions is equivalent to the existence of solutions with similar behavior (monotonous Bloch solutions) of the monodromy equation. At this stage, the authors apply a general result due to V. S. Buslaev and Fedotov [Algebra i Analiz **7** (1995), no. 4, 74–122; MR1356532] which gives a sufficient condition for the existence of monotonic Bloch solutions. Using Theorem 2.2, it is shown that there are intervals $\{I_l\}$ that have the properties (1), (2), (3) and are such that for $E \in J_\delta \setminus \bigcup_l I_l$ the condition of this general theorem is satisfied. To obtain the result concerning the density of states, the authors note that there is a function $w(z)$ which is analytic in the upper half-plane and which relates the Lyapunov exponent and the integrated density of states by the formulae

$$\Theta(E) = -\operatorname{Re} w(E + i0);\ \text{and}\ N(E) = \operatorname{Im} w(E + i0).$$

The existence of such a function $w(z)$ is a general fact about an ergodic family of Schrödinger operators, and in the present case it is obtained by computing the Lyapunov exponents for the related monodromy equation. The desired result is then obtained by looking at the increment of $\operatorname{Im} w(z)$ along a contour which straddles I_l and which is contained in the upper half-plane.

Now let $\lambda(E) = t_v(E)/t_h(E)$ and let $\Delta S(E) = -\epsilon \log \lambda(E) = S_v(E) - S_h(E)$. For each fixed $\delta > 0$, define the sets $J_\delta^\pm$ by $J_\delta^+ := \{E \in J_\delta | \Delta S(E) > \delta\}$ and $J_\delta^- := \{E \in J_\delta | \Delta S(E) < -\delta\}$. The authors then prove that if $\epsilon > 0$ is a small Diophantine number, each $I_l \subset J_\delta^+$ contains the absolutely continuous spectrum $\Sigma_{\rm ac}$ of $H_{\phi,\epsilon}$ in such a way that $|I_l \cap \Sigma_{\rm ac}| = |I_l|(1 + o(1))$ as $\epsilon \to 0$ (Theorem 2.4). On the other hand, for small $\epsilon > 0$ and for each $E \in J_\delta^-$, the Lyapunov exponent $\Theta(E)$ for the equation $H_{\phi,\epsilon}\psi = E\psi$ is strictly positive (Theorem 2.5), which implies the singularity of the spectrum of $H_{\phi,\epsilon}$ in J_δ^- by the Ishii-Pastur theorem. In these two theorems, it is possible to let δ depend on ϵ as $\delta = \epsilon^\beta$ with $0 < \beta < 1$, provided some modifications are made in the error terms appearing in the precise statements of the theorems. Thus the transition from the absolutely continuous to the singular spectrum is asymptotically sharp. In order to prove Theorem 2.4, the authors write the monodromy matrix M in the form $M = M_0 + M_1(\phi)$, where

$$M_0 = \begin{pmatrix} a_0 & b_0 \\ b_0^* & a_0^* \end{pmatrix},\ M_1(\phi) = \begin{pmatrix} \tilde{a}(\phi) & \tilde{b}(\phi) \\ \tilde{b}^*(\phi) & \tilde{a}^*(\phi) \end{pmatrix}.$$

Then, by Theorem 2.2, M_1 is of size of order λ, which is small when $E \in J_\delta^+$ and $\epsilon > 0$ is small. The authors then prove a KAM-theoretic theorem which guarantees that every solution of $H_{\phi,\epsilon}\psi = E\psi$ is bounded if the parameter ϵ satisfies a certain Diophantine condition. This means that the Lyapunov exponent $\Theta(E)$ vanishes for these values of ϵ and $E \in J_\delta^-$. Hence I_l contains the absolutely continuous spectrum by Kotani's theorem. On the other hand, when $E \in J_\delta^-$, Theorem 2.2 shows that one can write

$$M = \lambda e^{2\pi i(\phi-\phi_0)} \left[\begin{pmatrix} -1 & i \\ 0 & 0 \end{pmatrix} + o(1) \right],$$

where $\lambda = \lambda(E)$ is large when ϵ is small. In this case, the idea of Herman and Sorets-Spencer can be applied to prove the positivity of the Lyapunov exponent for the monodromy equation, whence follows that of the Schrödinger equation under consideration.

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Weinstein, Michael I.; Fefferman, Charles L.

Honeycomb lattice potentials and Dirac points.

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The paper deals with spectral features of Schrödinger operators which exhibit a honeycomb lattice symmetry. As exemplified by the 2010 Nobel Prize in Physics for research on graphene, such models are of relevance not only in mathematical physics, but in technological applications as well. The results of the paper concern the Fourier analysis, or, to be more precise, the Floquet-Bloch theory for the above-mentioned Schrödinger operator. The behaviour of the band dispersion function $k \mapsto \mu(k)$ (k is the quasi-momentum from the Brillouin zone) near the corners of the hexagonal Brillouin zone is studied. The aim is to establish that the function has conical singularities, i.e. $|\mu(k) - \mu(K_*)| \approx \pm c \cdot |k - K_*|$, $c \neq 0$, at any of the corner vertices K_*. Such vertex quasi-momenta are called Dirac points.

The authors consider the following model: Let $V\colon \mathbb{R}^d \to \mathbb{R}$ be a smooth potential, which is periodic with respect to the honeycomb lattice, reflection symmetric with respect to the origin, rotation symmetric with respect to the origin for the angle $2\pi/3$, and has non-vanishing 'first' Fourier coefficient $V_{(1,1)}$. Let ε be a real parameter, and $H^\varepsilon = -\Delta + \varepsilon V$ a family of Schrödinger operators. The authors show that H^ε exhibits conical singularities of the band dispersion function at each corner of the hexagonal Brillouin zone, with the possible exception of a set of ε which is countable and closed. For sufficiently small $\varepsilon_0 > 0$ the intersection of the exceptional set with $(-\varepsilon_0, \varepsilon_0)$ contains zero as the only point.

The authors identify for small ε the conical singularities as intersections of specific dispersion surfaces. They go on and study the stability of conical singularities of the band dispersion functions under small perturbations of the potential.

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