BOOK REVIEWS

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Function theory on symplectic manifolds, by Leonid Polterovich and Daniel Rosen, CRM Monograph series, vol. 34, American Mathematical Society, Providence, Rhode Island, 2014, xii+203 pp., ISBN 978-1-4704-1693-5

As the authors state in the preface, "the book is a fusion of a research monograph on function theory on symplectic manifolds and an introductory survey on symplectic topology". This is exactly correct. It is possible to read this book without any prior symplectic geometric background, because the book skillfully presents all the necessary background material and ideas, and can serve as an introductory text. In the main part of the book some necessary background symplectic topological notions are introduced and their key properties are postulated. The authors then return to building more rigorous symplectic topological foundations in the last three chapters of the book. But the main part of this book introduces the reader to a new very interesting area of symplectic topology largely discovered by Leonid Polterovich with coauthors, especially with Michael Entov. This new area has fascinating connections with quantum mechanics, which are also discussed in the book.

Symplectic structure on a manifold M is a closed non-degenerate differential 2-form ω. According to a theorem of Darboux, it is locally equivalent to the form $\omega_0 = \sum_1^n dx_i \wedge dy_i$ on $\mathbb{R}^{2n}$. Thus the symplectic form is a skew-symmetric cousin of the Euclidean metric. But while the group of isometries of the Euclidean metric is finite dimensional, the symplectic isometries, called symplectomorphisms, form a huge infinite-dimensional group $\mathrm{Diff}(M,\omega)$. So, symplectic geometry has a more topological, rather than geometric, flavor. Hence, the term *symplectic topology.* Symplectic structure naturally appears on the phase space of a classical mechanical system, which makes symplectic geometry the natural language for Hamiltonian mechanics, and symplectic topology is specifically designed to address its qualitative problems.

With any smooth function $H : M \to \mathbb{R}$, one can associate a *Hamiltonian vector field* X_H that is ω-dual to the differential dH of the function H, i.e., $\omega(\cdot, X_H) = dH(\cdot)$ and which generates (if it is integrable) a 1-parametric subgroup of $\mathrm{Diff}(M,\omega)$. In turn, these subgroups generate a subgroup $\mathrm{Ham}(M) \subset \mathrm{Diff}(M,\omega)$, called the group of *Hamiltonian diffeomorphisms.* The Lie algebra of $\mathrm{Ham}(M)$ is the functional space $C^\infty(M)/\mathbb{R}$ of smooth functions defined up to an additive constant.

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According to a theorem of A. Banyaga [3], the group $\mathrm{Ham}(M)$ for a closed M is *simple* and coincides with the commutator subgroup of the identity component of $\mathrm{Diff}(M,\omega)$. If M is non-compact, then according to another result of Banyaga, the subgroup $\mathrm{Ham}_0(M)$ of compactly supported symplectomorphisms and its universal cover $\widetilde{\mathrm{Ham}_0(M)}$ are *perfect* groups. There exists a unique non-trivial homomorphism $\mathrm{Cal} : \widetilde{\mathrm{Ham}_0(M)} \to \mathbb{R}$, called *Calabi homomorphism* [7]; its kernel coincides with the commutator subgroup $[\mathrm{Ham}_0(M), \mathrm{Ham}_0(M)]$. The Calabi homomorphism can be defined by the formula $\mathrm{Cal}(f) = \int_0^1 \int_M H_t \omega^n$, where H_t is compactly supported Hamiltonian generating f. It turns out that the definition does not depend on the choice of the generating Hamiltonian H_t.

A symplectomorphism $f : (M,\omega) \to (M,\omega)$ preserves the volume form ω^n, and hence the group of symplectomorphisms is a subgroup of the group $\mathrm{Diff}(M,\omega^n)$ of volume preserving diffeomorphisms of M. In the beginning of the 1970s Mikhail Gromov proved an alternative: $\mathrm{Diff}(M,\omega)$ is either C^0-closed (*rigidity*) or C^0-dense in $\mathrm{Diff}(M,\omega^n)$ (*flexibility*). Moreover, the alternative must have the same resolution for all symplectic manifolds. The flexible resolution of the alternative would imply that compared to volume preserving maps, symplectic maps have no special qualitative properties, such as additional fixed points or constraints on symplectic embeddings of one domain to another. In the 1980s Gromov's alternative was resolved in favor of rigidity by Gromov himself [16], as well as by the author of this review [10, 11]. The proof of Arnold's conjecture for the $2n$-torus in the work of C. Conley and E. Zehnder in [9] also implied the flexible resolution of the alternative. The method of J-holomorphic curves invented by Gromov to establish symplectic rigidity was very remarkable in its own right and yielded a large number of other fundamental results. Many of them were discussed in the original Gromov work. For instance, Gromov defined the first specifically symplectic invariant, now called *Gromov width*, which allowed him to prove his famous symplectic non-squeezing theorem: a ball of radius 1 cannot be symplectically embedded to a polydisc whose smallest radius is < 1, regardless of its volume. Remarkable development of the subject quickly followed Gromov's work: Floer homology theory, Hofer geometry, and many other great results which continue to be discovered today.

The introduction in 1990 by Helmut Hofer [17] of a bi-invariant metric on the group of Hamiltonian diffeomorphisms was a seminal step in the development of symplectic topology and led to the foundation of a new area, now called *Hofer geometry*. Existence of a bi-invariant metric on a huge non-compact infinite-dimensional group is a remarkable phenomenon manifesting symplectic rigidity. It is also quite remarkable that such a bi-invariant metric is essentially unique. In 2010 L. Buhovsky and Y. Ostrover proved the following result (see [5]): *any bi-invariant non-degenerate Finsler type distance metric on* $\mathrm{Ham}(M)$ *generated by a norm that is continuous in the* C^∞*-topology is equivalent to the Hofer metric.* The Hofer metric, according to the authors of the book under review, is one of "three wonders" of symplectic topology (the other two are C^0-closedness of the group $\mathrm{Diff}(M,\omega)$ and additional fixed point phenomenon for symplectomorphisms, the subject of V. I. Arnold's conjectures [2]), which are discussed in the first chapter. And it is the the starting point of the functional development which is the main subject of the book.

A symplectic form ω on a manifold M yields a *Poisson bracket* $\{f,g\}$ for functions on M, given by the formula $\{f,g\} = \omega(X_f, X_g) = dg(X_f) = -df(X_g)$. In the general vein of symplectic C^0-rigidity, Entov and Polterovich proved [13], extending earlier results of Cardin and Viterbo [8] and Zapolsky [20], that the Poisson bracket is lower semi-continuous in the C^0-topology, i.e.,

$$\liminf_{(\overline{F},\overline{G}) \overset{C^0}{\to} (F,G)} \{\overline{F},\overline{G}\} = \{F,G\}.$$

The second chapter of the book discusses various proofs of this result and its refinements, and in particular, the following Buhovsky's estimate [4] of the precise rate of this convergence. Denote

$$||\{F,G\}||_\delta = \inf_{||F-\overline{F}||,||G-\overline{G}||\leq\delta} ||\{\overline{F},\overline{G}\}||,$$

where $||\cdot||$ is the C^0-norm of a function on a compact manifold M. Then *there exists* $\delta_0(F,G) > 0$ *such that for all* $\delta \in (0, \delta_0(F,G))$ *one has*

$$||\{F,G\}|| - ||\{F,G\}||_\delta \leq C\Psi(F,G)^{1/3}\delta^{2/3},$$

where $C \leq 100$ *is a constant independent of* F *and* G*, and*

$$\Psi(F,G) := \|\{\{\{F,G\},F\},F\} + \{\{\{F,G\},G\},G\}\|.$$

Moreover, for generic F,G *the power* $\delta^{2/3}$ *is sharp.*

Exploration of Poisson bracket rigidity is one of the main themes throughout the book. For instance, the authors discuss an interesting invariant $\mathrm{pb}_4(X_0,X_1,Y_0,Y_1)$ of quadruples of compact sets X_0,X_1,Y_0,Y_1 of a symplectic manifold M, based on properties of the Poisson bracket. The invariant was defined by Buhovsky, Entov, and Polterovich in [6]. We return to its definition later on in the review.

Next, the book discusses the notion of a *quasi-morphism* and its symplectic ramifications. This notion belongs to geometric group theory. A quasi-morphism is a real valued function μ on a group G which is a homomorphism up to a bounded error, i.e., there exists a constant $C > 0$ such that

$$|\mu(gh) - \mu(g) - \mu(h)| \leq C \text{ for any } g,h \in G.$$

A prototypical example of a quasi-morphism is the *Poincaré rotation number*, which is a quasi-morphism on the universal cover $\widetilde{\mathrm{Diff}}(S^1)$ of the group of orientation preserving diffeomorphisms of the circle.

In the symplectic context, an important example of a quasi-morphism is the Maslov index $\mu : \widetilde{\mathrm{Sp}}(2n,\mathbb{R}) \to \mathbb{R}$, where we denote by $\widetilde{\mathrm{Sp}}(2n,\mathbb{R}) \to \mathbb{R}$ the universal cover of the group of linear symplectic transformations of $\mathbb{R}^{2n}$. Given a symplectic matrix S, we take its polar decomposition $S = PU$, where matrix P is positive definite self-adjoint and U is unitary, and then take the complex determinant $\det(U) \in S^1$. The constructed map $\mathrm{Sp}(2n,\mathbb{R}) \to S^1$ lifts to the required Maslov quasi-morphism $\mu : \widetilde{\mathrm{Sp}}(2n,\mathbb{R}) \to \mathbb{R}$.

Recall that a surjective Calabi homomorphism $\mathrm{Cal} : \widetilde{\mathrm{Ham}_0}(M) \to \mathbb{R}$ is defined on the group $\widetilde{\mathrm{Ham}_0}(M)$ for a non-compact M, while for a closed M there are no non-trivial homomorphisms $\mathrm{Ham}(M) \to \mathbb{R}$. However, it turned out that in many cases the group $\mathrm{Ham}(M)$ admits *Calabi quasi-morphisms*, which were discovered by Entov and Polterovich [12] and which satisfy the following *Calabi property*: for any f supported in a *displaceable* subset $U \subset M$, they coincide with $\mathrm{Cal}(f|_U)$, where displaceability means existence of a Hamiltonian diffeomorphism $h \in \mathrm{Ham}(M)$ such that $h(U) \cap U = \varnothing$. Existence of Calabi quasi-morphisms is another manifestation of symplectic rigidity. Similar to the Hofer metric they can be defined via the theory of spectral invariants, i.e., invariants defined in terms of critical values of the *action functional* $S_\phi(\widetilde{\gamma}) = \int_0^1 \Phi_t(\gamma(t))dt - \int_D \omega$. Here ϕ is a diffeomorphism from $\widetilde{\mathrm{Ham}}_0(M)$; Φ_t, $t \in [0,1]$, is a time dependent Hamiltonian generating ϕ and $\widetilde{\gamma}$ is a pair (γ, D), where $\gamma : [0,1] \to M$ is a loop and D is a homotopy class of 2-discs which span γ; this pair is an element of the universal cover $\widetilde{\Lambda}_0(M)$ of the space of contractible loops of M. The link between quasi-morphisms and spectral invariants is discussed in detail in the book.

The book then introduces the first connection of symplectic topology with quantum mechanics. In von Neumann formalism of quantum mechanics the *state* of a physical system is a functional ρ on the space $\mathcal{L}(H)$ of Hermitian operators (observables) of a Hilbert space H which satisfies three axioms: $\rho(\mathrm{Id}) = 1$ (*normalization*), $\rho(A) \geq 0$ if A is non-negative (*positivity*), and $\rho(\alpha A + \beta B) = \alpha\rho(A) + \beta\rho(B)$, $\alpha, \beta \in \mathbb{R}$, $A, B \in \mathcal{L}(H)$ (*linearity*). A. Gleason proved in [15] that when $\dim H \geq 3$ the relaxed definition of a *quasi-state*, when one postulates linearity *only for commuting pairs of operators* A, B, is equivalent to the original von Neumann definition. In classical mechanics a natural analog of quantum quasi-states are *symplectic quasi-states*. These are functionals $\zeta : C(M) \to \mathbb{R}$ on the space $C(M)$ of continuous functions on a symplectic manifold M which satisfies the analogous three conditions: $\zeta(1) = 1$ (*normalization*); $\zeta(F) \geq 0$ if $F \geq 0$ (*positivity*); and $\zeta(\alpha F + \beta G) = \alpha\zeta(F) + \beta\zeta(G)$ for Poisson commuting functions $F, G \in C(M)$ and real α, β (*quasi-linearity*). Here we call continuous functions Poisson commuting if they can be uniformly approximated by pairs of Poisson commuting functions. Entov and Polterovich discovered (see [14]) an *anti-Gleason phenomenon* in classical mechanics: there exist symplectic quasi-states which are not states, i.e., which are not necessarily linear on non-Poisson commuting pairs of functions. Again, the source of such examples is the theory of action spectral invariants. The theory of symplectic quasi-states is related to the theory of *topological quasi-states* introduced by A. Aarnes [1]. This link is also explored in the book. The book discusses several interesting applications of the theory of symplectic quasi-states to symplectic topology, and in particular to Hofer geometry and Lagrangian intersection theory.

We mentioned above a Poisson bracket based invariant $\mathrm{pb}_4(X_0, X_1, Y_0, Y_1)$ of quadruples of compact subsets $X_0, X_1, Y_0, Y_1 \subset M$ of a symplectic manifold M. It is defined by the formula

$$\mathrm{pb}_4(X_0, X_1, Y_0, Y_1) = \inf\{F, G\},$$

where the infimum is taken over all compactly supported C^∞-functions $F, G : M \to \mathbb{R}$ which satisfy the following conditions:

$$F|_{X_0} \leq 0,\ F|_{X_1} \geq 1,\ G|_{Y_0} \leq 0,\ G|_{X_1} \geq 1.$$

The class of such pairs of functions is non-empty whenever $X_0 \cap X_1 = Y_0 \cap Y_1 = \varnothing$. Non-trivial bounds for this invariant come from the theory of quasi-states. In turn, this invariant allows us to prove interesting dynamical properties, such as existence of Hamiltonian trajectories between points of two subsets of the phase space of a Hamiltonian system. For instance, let V be compact domain in T^*T^n that contains the 0-section, and let D_1 and D_2 be intersections of two distinct fibers of T^*T^n with V. Then *for any Hamiltonian* $G : V \to \mathbb{R}$ *that vanishes on* ∂V *and is greater than or equal to* 1 *on the* 0*-section there is a Hamiltonian trajectory of* G *intersecting* D_0 *and* D_1.

Another interesting Poisson bracket based invariant can be associated to a finite open cover $\mathcal{U} = \{U_i\}_{i=1,\dots,N}$, $\bigcup_1^N U_i = M$ of a symplectic manifold M. Define

$$\mathrm{pb}(\mathcal{U}) = \inf_{\vec{F}} \max_{x,y \in Q^N} \left\| \left\{ \sum_1^N x_i F_i, \sum_1^N y_j F_j \right\} \right\|.$$

Here the maximum is taken over $x = (x_1, \dots, x_N)$, $y = (y_1, \dots, y_N)$ in the unit cube $Q^N = \{0 \le x_i \le 1,\ i = 1, \dots, N\}$, and the infimum is taken over all partition of unity $\vec{F} = \{F_1, \dots, F_N\}$ subordinated to the covering $\mathcal{U}$. It turns out that symplectic topology yields non-trivial lower bounds for the invariant $\mathrm{pb}(\mathcal{U})$. For instance, for displaceable sets the book defines a class of invariants, called *spectral widths*, which could be viewed as variations of Gromov's width invariant. If w is one of these invariants, define $w(\mathcal{U}) := \max_i w(U_i)$. It turns out that if the partition $\mathcal{U}$ consists of displaceable open sets, then $\mathrm{pb}(\mathcal{U}) \ge \frac{c(N)}{w(\mathcal{U})}$, where the constant $c(N)$ depends only on the number N of the elements of the partition.

Interestingly, the invariant $\mathrm{pb}(\mathcal{U})$ provides yet another link of symplectic topology with quantum mechanics, this time with the theory of quantum noise. This discussion, which follows Polterovich's papers [18, 19], culminates the main part of the book, before the authors return in the last three chapters to the building of the necessary additional symplectic topological background.

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