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NASH'S WORK IN ALGEBRAIC GEOMETRY

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ABSTRACT. This article is a survey of Nash's contributions to algebraic geometry, focusing on the topology of real algebraic sets and on arc spaces of singularities.

Nash wrote two papers in algebraic geometry, one at the beginning of his career [Nash52] and one in 1968, after the onset of his long illness; the latter was published only much later [Nash95]. With these papers Nash was ahead of the times; it took at least 20 years before their importance was recognized. By now both are seen as starting points of significant directions within algebraic geometry. I had the privilege to work in these areas and discuss the problems with Nash. It was impressive that, even after 50 years, these questions were still fresh in his mind and he still had deep insights to share.

Section 1 is devoted to the proof of the main theorem of [Nash52], and Section 3 is a short introduction to the study of arc spaces pioneered by [Nash95]. Both of these are quite elementary. Section 2 discusses subsequent work on a conjecture of [Nash52]; some familiarity with algebraic geometry is helpful in reading it.

1. The topology of real algebraic sets

One of the main questions occupying mathematicians around 1950 was to understand the relationships among various notions of manifolds.

The *Hauptvermutung* (=Main Conjecture) formulated by Steinitz and Tietze in 1908 asserted that every topological manifold should be triangulable. The relationship between triangulations and differentiable structures was not yet clear. Whitney proved that every compact C^1-manifold admits a differentiable—even a real analytic—structure [Whi36], but the groundbreaking examples of Milnor [Mil56, Mil61] were still in the future, and so were the embeddability and uniqueness of real analytic structures [Mor58, Gra58].

Nash set out to investigate whether one can find even stronger structures on manifolds. It is quite likely that polynomials form the smallest class of functions that could conceivably be large enough to describe all manifolds.

Definition 1. A *real algebraic set* is the common zero set of a collection of polynomials

$$X := \{\mathbf{x} : f_1(\mathbf{x}) = \cdots = f_r(\mathbf{x}) = 0\} \subset \mathbb{R}^N.$$

Comments. For the purposes of this article, one can just think of these as subsets of $\mathbb{R}^N$, though theoretically it is almost always better to view $X = X(\mathbb{R})$ as the real points of the complex algebraic set $X(\mathbb{C})$, which consist of all complex solutions of

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the equations. Algebraic geometers would also specify the sheaf of regular functions on $X(\mathbb{C})$. To emphasize the elementary aspects, I will write about "real algebraic sets" instead of "real algebraic varieties".

One should properly define notions like smoothness and dimension in algebraic geometry, but here we can take shortcuts. Thus we say that X is *smooth* iff it is a smooth submanifold of $\mathbb{R}^N$ and we use the topological dimension of X. (If X is smooth in the algebraic geometry sense (cf. [Sha74, Sec.II.1]), then these give the right notions, but not always. For example, algebraic geometers refer to $(x^2 + y^2 = 0)$ as a plane curve with a singular point at the origin, but the above definitions do not distinguish it from the 0-dimensional set $(x = y = 0)$.)

For complex algebraic geometers it may seem bizarre to work only with real algebraic sets that are sitting in affine spaces; Example 4 (4.2) serves as an explanation.

The main theorem of [Nash52] is the following.

Theorem 2. *Every smooth, compact manifold M is diffeomorphic to a smooth, real algebraic set X.*

We refer to any such X as an *algebraic model* of M. To be precise, Nash proved only that M is diffeomorphic to a connected component of a smooth, real algebraic set X; the above stronger form was obtained by Tognoli [Tog73], building on [Wal57]. We will outline the proof of the following variant which says that embedded manifolds can be approximated by smooth real algebraic sets.

Theorem 3. *Let $M \subset \mathbb{R}^N$ be a smooth, compact submanifold. Assume that $N \geq 2\dim M + 1$. Then there are diffeomorphisms ϕ of $\mathbb{R}^N$, arbitrarily close to the identity, such that $\phi(M) \subset \mathbb{R}^N$ is a real algebraic set.*

Later results of [AK92a], building on [Tog88], show that, even if $N < 2\dim M+1$, the manifold M can be approximated by smooth real algebraic sets in $\mathbb{R}^{N+1}$.

Example 4. Let us start with a few examples, nonexamples and constructions of smooth, real algebraic sets.

(4.1) The standard sphere is $\mathbb{S}^n = (x_1^2 + \cdots + x_{n+1}^2 = 1) \subset \mathbb{R}^{n+1}$.

(4.2) By any general definition $\mathbb{RP}^n$ is a real algebraic set, and one can realize it as a subset of $\mathbb{R}^N$ using the embedding

$$\mathbb{RP}^n \hookrightarrow \mathbb{R}^{n(n+1)/2} : (y_0{:}\cdots{:}y_n) \mapsto \Big(\frac{y_i y_j}{\sum_k y_k^2} : 0 \leq i, j, \leq n\Big).$$

(This explains why Definition 1 works: any projective real algebraic set is also a real algebraic subset of some $\mathbb{R}^N$.)

(4.3) A connected component of a real algebraic set is usually not real algebraic. For example, the curve $C := \big(y^2 + x(x-1)(x-2)(x-3) = 0\big)$ consists of two ovals, and any polynomial $f(x, y)$ that vanishes on one of the ovals also vanishes on the other one. Indeed, we can write

$$f(x, y) = h(x, y)\big(y^2 + x(x-1)(x-2)(x-3)\big) + \big(ay + g(x)\big).$$

If $f(x, y)$ vanishes on one of the ovals, then the same holds for $ay + g(x)$. If (x_0, y_0) is on the oval, then so is $(x_0, -y_0)$, and if $ay + g(x)$ vanishes on both of them, then $a = 0$. Since $g(x)$ can vanish at finitely many x-values only, we conclude that $ay + g(x) \equiv 0$, hence f vanishes on both ovals.

(4.4) Any complex algebraic set can be viewed as real algebraic by replacing the complex coordinates z_j with their real and imaginary parts x_j, y_j. For instance, a complex plane curve given by one equation $f(z_1, z_2) = 0$ becomes a real surface in $\mathbb{R}^4$ given by two real equations,

$$\Re\big(f(x_1 + iy_1, x_2 + iy_2)\big) = 0 \quad \text{and} \quad \Im\big(f(x_1 + iy_1, x_2 + iy_2)\big) = 0.$$

(4.5) Blowing-up is an operation that replaces a point $x \in X$ with all the tangent directions at x; see [Sha74, Sec.II.4]. Thus if X is a real algebraic set of dimension n and x is a smooth point, then $B_xX \overset{\text{diff}}{\sim} X\#\mathbb{RP}^n$, the connected sum of X with $\mathbb{RP}^n$. In particular, we can get all nonorientable surfaces from the sphere $\mathbb{S}^2$ by blowing up points.

(4.6) Let $X(\mathbb{C})$ be a complex algebraic set, and let $x \in X(\mathbb{C})$ be an isolated singular point. Intersecting $X(\mathbb{C})$ with a small sphere around x results in a smooth real algebraic set called a *link* of $x \in X(\mathbb{C})$. Many interesting manifolds can be obtained this way. For instance,

$$\big(z_1^2 + z_2^3 + z_3^5 = 0\big) \cap \big(\textstyle\sum_i |z_i|^2 = 1\big)$$

is the Poincaré homology sphere, while

$$\big(z_1^2 + z_3^2 + z_3^2 + z_4^3 + z_5^{6r-1} = 0\big) \cap \big(\textstyle\sum_i |z_i|^2 = 1\big)$$

give all 28 differentiable structures on $\mathbb{S}^7$ for $r = 1, \dots, 28$. See [Mil68] for an introduction to links.

We start the proof of Theorem 2 with a result of Seifert [Sei36].

Special Case 5 (Hypersurfaces)**.** Assume that $M \subset \mathbb{R}^{n+1}$ is a smooth hypersurface. Alexander duality implies that M is 2-sided; that is, a suitable open neighborhood $U_M \supset M$ is diffeomorphic to $M \times (-1, 1)$. Projection to the second factor gives a proper, C^∞-submersion $F : U_M \to (-1, 1)$ whose zero set is M. By Weierstrass's theorem, we can approximate F by a sequence of polynomials $P_m(\mathbf{x})$. We hope that the real algebraic hypersurfaces $X_m := \big(P_m(\mathbf{x}) = 0\big)$ approximate M.

There are two points to clarify. First, as we will see in Discussion 14, the X_m approximate M if P_m converges to F in the C^1-norm. (That is, $P_m \to F$ and $\partial P_m/\partial x_i \to \partial F/\partial x_i$ for every i, uniformly on compact subsets of U_M.) This version of the Weierstrass theorem is not hard to establish; see [Whi34] or [dlVP08].

Second, the polynomials $P_m(\mathbf{x})$ could have unexpected zeros outside U_M. Thus we only obtain that *one* of the connected components of the hypersurfaces X_m approximate M.

We can avoid the extra components as follows. Instead of using U_M, we need to extend F to a large ball B of radius R containing it and do the approximation on B. Then we change the approximating polynomials P_m to

$$P_{m,s} := P_m + \big(\tfrac{1}{R^2}\textstyle\sum x_i^2\big)^s.$$

Note that, for $s \gg 1$, the function $P_{m,s}$ is very close to P_m inside B and is strictly positive outside B. Thus $X_{m,s} := \big(P_{m,s}(\mathbf{x}) = 0\big)$ is contained in B for $s \gg m$ and they give the required approximation of M.

More generally we obtain the following. (The above arguments correspond to the case $Y = \mathbb{RP}^{n+1}$ and $F = \emptyset$.)

Claim 5.1. Let Y be a compact, smooth, real algebraic set, let $F \subset Y$ be an algebraic hypersurface, and let $M \subset Y$ be a differentiable hypersurface. Assume that M is homologous to F, with $\mathbb{Z}_2$-coefficients. Then M can be approximated by algebraic hypersurfaces in Y.

Outline of proof. $F \subset Y$ defines an algebraic line bundle L on Y. The assumption implies that L has a smooth section σ whose zero set is M. We approximate σ by algebraic sections s_i in the C^1-norm; then the algebraic hypersurfaces $(s_i = 0) \subset Y$ approximate M. □

We can summarize the above approach in three steps.

First, we found a differentiable map $F : \mathbb{R}^N \to \mathbb{R}$ such that $M = F^{-1}(0)$. The situation is very special, but the key turns out to be that $\mathbb{R}$ is a smooth real algebraic set and $\{0\} \subset \mathbb{R}$ is a smooth real algebraic subset.

Then we approximated $F : \mathbb{R}^N \to \mathbb{R}$ by algebraic maps $P_{m,s} : \mathbb{R}^N \to \mathbb{R}$, and finally we showed that the real algebraic subsets $P_{m,s}^{-1}(0)$ approximate M.

This suggests the following approach to prove Theorems 2 and 3.

Plan of Proof 6. Given a compact manifold $M \subset \mathbb{R}^N$, we aim to find an algebraic approximation in three steps.

Step 1. Find a smooth, real algebraic set U, a smooth, real algebraic subset $Z \subset U$, and a differentiable map $g : \mathbb{R}^N \to U$ such that $M = g^{-1}(Z)$. (U will be some "universal" space, not much related to M.)

Step 2. Approximate $g : \mathbb{R}^N \to U$ by "algebraic" maps $h_i : \mathbb{R}^N \to U$.

Step 3. Show that the real algebraic subsets $h_i^{-1}(Z)$ approximate M.

We start with the second step, which is the most interesting.

Discussion 7 (Retraction of neighborhoods)**.** Let $M \subset \mathbb{R}^N$ be a compact manifold of dimension n. As we see below, if $p \in \mathbb{R}^N$ is close enough to M, then there is a unique point $\pi(p) \in M$ that is closest to p and we get a retraction $\pi : U_M \to M$ of some open neighborhood $M \subset U_M \subset \mathbb{R}^N$.

We will need to understand the regularity properties of π. The first version is classical.

Claim 7.1. If M is real analytic, then π is also real analytic.

Proof. This is a local question, so choose orthonormal coordinates such that $\pi(p)$ is the origin and $x_1, \dots, x_n$ are coordinates on the tangent space $T_{\pi(p)}M$. Thus we can write M as a graph

$$M = \big(x_1, \dots, x_n, \phi_{n+1}(x_1, \dots, x_n), \dots, \phi_N(x_1, \dots, x_n)\big), \tag{7.2}$$

where the ϕ_j vanish to order 2 at the origin. Then $\pi(p)$ is a critical point of the function

$$(x_1, \dots, x_n) \mapsto \textstyle\sum_{i=1}^n (x_i - p_i)^2 + \sum_{j=n+1}^N \big(\phi_j(\mathbf{x}) - p_j\big)^2, \tag{7.3}$$

and hence a solution of the system of equations

$$(x_i - p_i) + \textstyle\sum_{j=n+1}^N (\phi_j - p_j)\frac{\partial \phi_j}{\partial x_i} = 0 \quad \text{for} \quad i = 1, \dots, n. \tag{7.4}$$

By the implicit function theorem, the solution depends real-analytically on p, provided the Jacobian matrix of system (7.4) is invertible. We compute that the Jacobian is

$$\mathbf{1}_n + \textstyle\sum_{j=n+1}^N (\phi_j - p_j) \cdot \operatorname{Hessian}(\phi_j) + \sum_{j=n+1}^N \Big(\frac{\partial \phi_j}{\partial x_i} \cdot \frac{\partial \phi_j}{\partial x_k}\Big). \tag{7.5}$$

Since the ϕ_j vanish to order 2 at the origin, the last term is small, and so is $\phi_j - p_j$ if the p_j are small. □

Nash made two crucial observations: one can go from real analytic to real algebraic functions and this leads to an interesting and useful class of functions.

Claim 7.6. If M is real algebraic, then π is also real "algebraic". (We will make this claim precise in Definition 9.)

Proof. Claim 7.6 is again a local question, so we may assume that M is the common zero set of $r = N - n$ real polynomials: $M = (f_1 = \cdots = f_r = 0)$ with linearly independent gradients at $\pi(p)$.[1] Then $\pi(p)$ is a solution of the Lagrange multiplier equations,

$$\begin{aligned} \nabla\bigl(\textstyle\sum_{i=1}^{N}(x_i - p_i)^2\bigr) &= \textstyle\sum_{j=1}^{r}\lambda_j \nabla f_j, \quad \text{and} \\ f_j(x_1, \ldots, x_N) &= 0 \quad \text{for} \quad j = 1, \ldots, r. \end{aligned} \tag{7.7}$$

Next we discuss what can one conclude about the coordinate functions of π, using that (7.7) is a system of *polynomial* equations.

Definition 8 (Algebraic functions)**.** Going back at least to Euler, a 1-variable function $f(x)$ is called *algebraic* if it satisfies an equation

$$g_n(x)f^n + g_{n-1}(x)f^{n-1} + \cdots + g_0(x) \equiv 0, \tag{8.1}$$

where the $g_i(x)$ are polynomials (and g_n is not identically 0). For example $f := \sqrt[n]{g(x)}$ is algebraic since it satisfies the equation $f^n - g(x) \equiv 0$. Complex analysts always view $f(x)$ as a multivalued function, but for real variables it can happen that a sensible single-valued choice is possible. (For example, we all use $\sqrt[n]{x}$ as a single-valued real function for n odd.)

Similarly, an m-variable (possibly multivalued) function $f(x_1, \ldots, x_m)$ is *algebraic* if it satisfies an equation

$$g_n f^n + g_{n-1} f^{n-1} + \cdots + g_0 \equiv 0, \tag{8.2}$$

where the $g_i \in \mathbb{R}[x_1, \ldots, x_m]$ are polynomials (and g_n is not identically 0). We can reinterpret an algebraic function by introducing its "algebraic graph",

$$\Gamma_f := \bigl(g_n x_{m+1}^n + g_{n-1}x_{m+1}^{n-1} + \cdots + g_0 = 0\bigr) \subset \mathbb{R}^{m+1}.$$

Thus Γ_f is a real algebraic set, and we can view $f : \mathbb{R}^m \to \mathbb{R}$ as the composite

$$f : \mathbb{R}^m \xrightarrow{\sigma} \Gamma_f \xrightarrow{\pi} \mathbb{R}. \tag{8.3}$$

Here σ is a (usually nonalgebraic) section that is continuous if f is continuous and π is the last coordinate projection (hence algebraic).

Any polynomial $F(x_1, \ldots, x_{m+1})$ defines x_{m+1} as a (multivalued) algebraic function of $x_1, \ldots, x_m$, provided x_{m+1} actually appears in F. It is a less trivial statement that something similar holds for systems of polynomial equations.

Claim 8.4. Let $F_j(x_1, \ldots, x_n, y_1, \ldots, y_m) = 0$ for $j = 1, \ldots, r$ be a system of polynomial equations. Assume that there is some open subset $B \subset \mathbb{R}^n$ such that for every $(b_1, \ldots, b_n) \in B$ the system

$$F_j(b_1, \ldots, b_n, y_1, \ldots, y_m) = 0 \quad \text{for} \quad j = 1, \ldots, r$$

[1] This is the algebraic definition of smoothness, so we did not gain much by avoiding it earlier.

has finitely many (but at least one) complex solutions. Then the system defines $y_1, \dots, y_m$ as algebraic functions of $x_1, \dots, x_n$.

Comments. Traditionally Claim 8.4 was proved using resultants as part of elimination theory; see [Kro1881]. In modern algebraic geometry books it is viewed as a special case of Chevalley's theorem—that the image of an algebraic set is constructible—applied to the coordinate projections. See [CLO92, Chap.III] for a detailed introduction.

Applying Claim 8.4 to the system (7.7), we see that the coordinates of $\pi(p)$ are algebraic functions of p. We saw that $\pi(p)$ is also a real analytic function of p, suggesting the following definition.

Definition 9 (Nash functions)**.** Let X be a smooth real algebraic set. A function $\phi : X \to \mathbb{R}$ is a *Nash function* if it is both algebraic and real analytic.

For example, $\sqrt[n]{1 + x^2 + y^2}$ is a Nash function on $\mathbb{R}^2$.

It is not hard to see that if ϕ is algebraic and C^∞, then it is a Nash function, but lower order differentiability is not enough. (For example, $x \mapsto \sqrt[3]{x^{10}}$ is algebraic and C^3, but not Nash.)

A map $\phi : X \to Y$ between real algebraic sets is *Nash* if the coordinate functions of the composite $X \to Y \hookrightarrow \mathbb{R}^N$ are Nash for one (equivalently, every) embedding $Y \hookrightarrow \mathbb{R}^N$.

Following (8.3) we can thus realize a Nash map $\phi : X \to Y$ by a diagram

$$X \underset{\text{finite fibers}}{\overset{\text{real analytic}}{\rightleftarrows}} X' \xrightarrow{\text{algebraic}} Y \tag{9.1}$$

where $\pi : X' \to X$ is a surjection of real algebraic sets with finite fibers, $\sigma : X \to X'$ is a real analytic section, $h : X' \to Y$ is a real algebraic map and $\phi = h \circ \sigma$.

We can now make Claim 7.6 precise: if M is real algebraic, then the orthogonal projection $\pi : U_M \to M$ is Nash.

Nash maps have very good approximation properties.

Corollary 10. *Let X, Y be smooth, real algebraic sets, and let $X^0 \subset X$ be an open subset. Let $g : X^0 \to Y$ be a differentiable map. Then g can be approximated by Nash maps on compact subsets of X^0.*

Proof. Choose an embedding $Y \hookrightarrow \mathbb{R}^N$ and use the Weierstrass theorem to approximate the composite map $X^0 \to Y \hookrightarrow \mathbb{R}^N$ by polynomial maps. The image of the approximation still lies in a small neighborhood $U_Y \supset Y$. We can thus compose the approximation with the orthogonal projection $U_Y \to Y$ to get the required approximation of g. Note that the orthogonal projection $U_Y \to Y$ is a Nash map, so the approximation of g is by Nash maps. □

Corollary 11. *Any diffeomorphism between smooth, compact real algebraic sets $X \overset{\text{diff}}{\sim} Y$ can be approximated by Nash diffeomorphisms.*

Before we turn to Step 1 of Plan of Proof 6, we need to review Grassmannians.

Definition 12 (Real Grassmannians)**.** Let W be a real vector space of dimension N. Given $0 \le r \le N$, let $\operatorname{Grass}(r, W)$ denote the set of r-dimensional linear subspaces of W. There are many ways to see that $\mathrm{Grass}(r, W)$ has a natural structure as a compact manifold, even a smooth real algebraic set.

After fixing an inner product on W, we can identify a subspace $V^r \subset W$ with the orthogonal projection matrix $P \in \mathrm{End}(W)$ whose kernel is V^r. Thus $\mathrm{Grass}(r, W)$ is a subset of $\mathrm{End}(W) \cong \mathbb{R}^{N^2}$, and its equations are given by $P^2 = P, P = P^t, \mathrm{tr}\, P = N - r$. We also have the universal family of subspaces

$$\mathrm{Univ}(r, W) \subset \mathrm{Grass}(r, W) \times W$$

parametrizing pairs (P, w) such that $Pw = 0$. The first coordinate projection $\mathrm{Univ}(r, W) \to \mathrm{Grass}(r, W)$ is a vector bundle of rank r and $P \mapsto (P, \mathbf{0})$ is the zero section.

Proof 13 (We complete Step 1 of Plan of Proof 6)**.** We have $M^n \subset \mathbb{R}^N$ and its neighborhood $U_M \supset M$ with orthogonal projection $\pi : U_M \to M$ as in Discussion 7. For a point $q \in M$, we decompose $T_q\mathbb{R}^N = T_qM + N_qM$ as the orthogonal sum of the tangent space and the normal space of M. Set $Y := \mathrm{Univ}(r, \mathbb{R}^N)$, let $Z \subset Y$ denote the zero section, and define $g : U_M \to Y$ by the formula

$$g(p) := \big(N_{\pi(p)}M - \pi(p), p - \pi(p)\big) \in \mathrm{Univ}(r, \mathbb{R}^N).$$

In other words, we take the normal space $N_{\pi(p)}M$ at $\pi(p)$ and translate it by $-\pi(p)$ to get a linear subspace of $\mathbb{R}^N$ of dimension r, and hence a point in $\mathrm{Grass}(r, \mathbb{R}^N)$. It is clear that $g(p) \in Z$ iff $p - \pi(p) = 0$; that is, iff $p \in M$. Thus $g^{-1}(Z) = M$. □

The completion of Step 3 of Plan of Proof 6 uses only basic differential topology.

Discussion 14 (Isotopies between fibers)**.** Let $g : X \to Y$ be a proper C^∞-submersion between smooth, connected manifolds. By Ehresmann's fibration theorem the fibers $g^{-1}(y)$ are all diffeomorphic. Moreover, if $y_1, y_2 \in Y$ are close enough, then there is a diffeomorphism of the pairs $\big(g^{-1}(y_1) \subset X\big) \overset{\text{diff}}{\sim} \big(g^{-1}(y_2) \subset X\big)$ that is close to the identity.

Assume next that we have another $h : X \to Y$ that is close enough to g in the C^1-topology. We see next that there is a family of maps $g_t : X \to Y$ for $t \in (-\epsilon, 1+\epsilon)$ such that $g_0 = g,\ g_1 = h$, and $G(x, t) := \big(g_t(x), t\big)$ defines a proper, C^∞-submersion

$$G : X \times (-\epsilon, 1+\epsilon) \to Y \times (-\epsilon, 1+\epsilon). \tag{14.1}$$

Thus we conclude that $g^{-1}(y) \overset{\text{diff}}{\sim} h^{-1}(y)$.

In order to construct G, assume that $Y \subset \mathbb{R}^N$. First, take the family of linear combinations $tg + (1-t)h : X \to \mathbb{R}^N$; the image of $X \times (-\epsilon, 1+\epsilon)$ lies in a small neighborhood $U_Y \supset Y$. Then we can use orthogonal projection $U_Y \to Y$ (as in Discussion 7) to get a family of maps $g_t : X \to Y$.

We also need the following more general version of the above.

Let $g : X \to Y$ be a proper, differentiable map between differentiable manifolds, and let $Z \subset Y$ be a submanifold. We say that g is *transversal* to Z if for every $z \in Z$ and $x \in g^{-1}(z)$ the composite $T_xX \xrightarrow{g_*} T_zY \to N_zZ$ is surjective, where N_zZ is the normal space of Z in Y at z. If g is transversal to Z, then $g^{-1}(Z) \subset X$ is a submanifold.

Assume next that we have $G(x, t)$ as in (14.1) such that $G(\ , t) : X \to Y$ is transversal to Z for every t. Then G restricts to a submersion

$$G^{-1}\big(Z \times (-\epsilon, 1+\epsilon)\big) \to (-\epsilon, 1+\epsilon),$$

and we obtain the following.

Claim 14.2. Let X, Y be differentiable manifolds, and let $Z \subset Y$ be a submanifold. Let $g : X \to Y$ be a proper, differentiable map that is transversal to Z.

Let $h : X \to Y$ be another differentiable map that is close enough to g in the C^1-topology. Then $g^{-1}(Z) \stackrel{\text{diff}}{\sim} h^{-1}(Z)$. Moreover, there is a diffeomorphism of the pairs $\big(g^{-1}(Z) \subset X\big) \stackrel{\text{diff}}{\sim} \big(h^{-1}(Z) \subset X\big)$ that is close to the identity. □

Proof 15 (Conclusion of the proofs of Theorems 2 and 3)**.** We start with $M^n \subset \mathbb{R}^N$ and set $r = N - n$. Let $U_M \supset M$ be a small open neighborhood of M. In Proof 13 we wrote down a differentiable map $g : U_M \to \mathrm{Univ}(r, \mathbb{R}^N)$ such that g is transversal to the zero section $Z = \mathrm{Grass}(r, \mathbb{R}^N) \subset \mathrm{Univ}(r, \mathbb{R}^N)$ and $g^{-1}(Z) = M$.

Since $\mathrm{Univ}(r, \mathbb{R}^N)$ is real algebraic, Corollary 10 shows that g has a Nash approximation $\phi : U_M \to \mathrm{Univ}(r, \mathbb{R}^N)$. By Claim 14.2 we obtain that $\phi^{-1}(Z) \subset U_M \subset \mathbb{R}^N$ is an approximation of M.

The proof is complete if $\phi^{-1}(Z)$ is a real algebraic set, but for the moment we only know that it is given as the common zero set of Nash functions. In the current situation diagram (9.1) becomes

$$\mathbb{R}^N = X \underset{\pi}{\overset{\sigma}{\rightleftarrows}} X' \xrightarrow{\quad h \quad} \mathrm{Univ}(r, \mathbb{R}^N),$$

where σ is real analytic and h is algebraic. Our arguments show that $\phi^{-1}(Z) \subset X'$ is a real algebraic set and it has a connected component V_1' such that $\pi(V_1')$ is an approximation of M. However, $\phi^{-1}(Z)$ usually has other components V_i', and the $\pi(V_i')$ may well intersect $\pi(V_1')$. Now we use the assumption $N \geq 2n + 1$: after a suitable perturbation π' of π, the images $\pi'(V_i')$ become disjoint. (Actually, as we see in Example 18, $\pi\big(\phi^{-1}(Z)\big) \subset X$ may have real points that are images of imaginary points of $\phi^{-1}(Z)$; these are not much harder to deal with.)

We have completed the proof of Nash's original form of the theorem.

In order to go further, [Wal57, Tog73] use a result of Thom on unoriented cobordisms: every compact manifold M^n is cobordant to a product of real projective spaces [Tho54]. In particular, every compact manifold M^n is cobordant to a smooth, compact, real algebraic set F^n. We can glue together two copies of this cobordism to get a compact manifold N^{n+1} such that $M^n \subset N^{n+1}$ and $F^n \subset N^{n+1}$ are homologous.

Next, we apply the Nash-version of Theorem 2 to N^{n+1}. We thus get a smooth, compact real algebraic set Y^{n+1} plus a diffeomorphism $\tau : N^{n+1} \hookrightarrow Y^{n+1}$ onto a connected component of Y^{n+1}. Note that $\tau(M^n)$ and $\tau(F^n)$ are still homologous; the other connected components of Y^{n+1} play no role. We would like to apply Claim 5.1 to obtain an algebraic approximation of $\tau(M^n)$, but this would require that $\tau(F^n)$ be an algebraic subset of Y^{n+1}, and this is usually not the case. However, a small modification of the above proofs gives that if a smooth subset $F \subset N^{n+1}$ is algebraic, then we can choose Y^{n+1} and $\tau : N^{n+1} \hookrightarrow Y^{n+1}$ such that $\tau(F) \subset Y^{n+1}$ is also algebraic. Then Claim 5.1 applies and we get Theorem 2. □

Further work. Having proved that all compact manifolds can be described by polynomial equations, it is natural to ask two further questions.

Question 16. Which topological structures on a manifold have algebraic realizations?

For example, one can look for algebraic realizations of a manifold M for which all (co)homology or all vector bundles have algebraic representatives. The first obstructions were found by [BD84]. It turned out to be especially interesting to understand when all homotopy classes of maps between various manifolds have algebraic representatives.

Question 17. Which subsets $P \subset \mathbb{R}^N$ can be approximated by real algebraic subsets?

Algebraic sets can be triangulated [vdW30], and so it is natural to work with simplicial complexes. There are some obstructions, for example, the Euler characteristic of links is even for every real algebraic set [Sul71]; see Example 18. There do not seem to be plausible complete answers or conjectures.

Akbulut, Benedetti, Bochnak, King, Kucharz, Shiota, Tognoli, and many others contributed to these questions; see the monographs [Shi87, AK92b, BCR98] for detailed treatments and references. A related question of [Nash52] is answered in [Kur88].

Example 18. Start with $\mathbb{R}^3$. It is a real algebraic set where the link of every point is $\mathbb{S}^2$ and $\chi(\mathbb{S}^2) = 2$?

Next take the involution $\tau : \mathbf{x} \mapsto -\mathbf{x}$ and consider the quotient $\mathbb{R}^3/(\tau)$. It would seem that the link of the origin is $\mathbb{S}^2/(\tau) \sim \mathbb{RP}^2$ and $\chi(\mathbb{RP}^2) = 1$, in violation of the above mentioned parity claim of [Sul71].

Let us see more carefully what happens. Choose coordinates x_1, x_2, x_3 on $\mathbb{R}^3$. The ring of τ-invariant polynomials on $\mathbb{R}^3$ is generated by the $x_i x_j$ for $1 \le i \le j \le 3$. Fix $\mathbb{R}^6$ with coordinates y_{ij} for $1 \le i \le j \le 3$. We can thus realize the quotient $\mathbb{R}^3/(\tau)$ as the image of $\mathbb{R}^3$ under the map

$$\mathbb{R}^3 \to \mathbb{R}^6 : y_{ij} = x_i x_j \quad \text{for} \quad 1 \le i \le j \le 3.$$

Let $W \subset \mathbb{R}^6$ denote the image as a real algebraic set. What are the real points of W? We obviously have $\mathbb{R}^3/(\tau)$, but we soon notice that if the x_i are all purely imaginary, then the products $x_i x_j$ are all real. Thus in fact $W(\mathbb{R})$ is the union of two pieces, $\mathbb{R}^3/(\tau)$ and $\sqrt{-1}\mathbb{R}^3/(\tau) \sim \mathbb{R}^3/(\tau)$. Hence the link of the origin is not $\mathbb{RP}^2$ but two copies of $\mathbb{RP}^2$, and it has Euler characteristic 2.

The equations defining W can be given in a very transparent form as

$$\operatorname{rank} \begin{pmatrix} y_{00} & y_{01} & y_{02} \\ y_{01} & y_{11} & y_{12} \\ y_{02} & y_{12} & y_{22} \end{pmatrix} \le 1,$$

which we can unwind to get six quadratic equations.

2. The Nash conjecture on rational parametrizations

Theorem 2 says that every compact manifold can be specified by a few polynomials. It is, however, usually difficult to recover a manifold from the equations. This is hard computationally (solving over-determined systems of polynomial equations is unstable) but also theoretically since it is not even easy to decide whether a given system of equations has any solutions.

It is much easier to work with algebraic sets that admit a parametrization by "simple" functions. There are several variants of this notion, the most frequently used one involves *rational* functions; that is, quotients of polynomials.

Theoretically, one should work with *rationally connected varieties* (see [Kol01b] for a general introduction), but the following definition is intuitively clearer.

Definition 19. A *rational parametrization* of a real algebraic set $X^n \subset \mathbb{R}^N$ is a rational map $\Phi : \mathbb{R}^n \dashrightarrow X^n$ with dense image, preferably in the Euclidean topology, though algebraic geometers would frequently use the Zariski topology. In many cases one can do even better and get Φ that is injective on a dense, open subset of $\mathbb{R}^n$; the latter is the version used by Nash.

Given a rational parametrization, we can easily write down points of X. Φ is given by its coordinate functions $\Phi_i = g_i/h_i$, where the g_i and h_i are polynomials. We can thus pick a random point $p \in \mathbb{R}^n$ and compute the values $g_i(p)/h_i(p)$.

At the end of his paper, Nash asks the following question. Nash himself was rather undecided about it—writing that "this would be a very powerful theorem if it could be proved"—but later authors were more definitive.

Conjecture 20 (Nash conjecture)**.** *Let M be a connected compact manifold. Then there is a smooth real algebraic set X such that $X(\mathbb{R}) \overset{\text{diff}}{\sim} M$ and X has a rational parametrization.*

This turned out to have been one the shortest-lived conjectures; it was disproved 38 years before its formulation by Comessatti.

Theorem 21 ([Com14])**.** *A compact, topological surface S has an algebraic model with a rational parametrization iff S is*

(1) *either a sphere, a torus,*
(2) *or it is nonorientable.*

Outline of proof. We assume some familiarity with algebraic surfaces; for example [Bea96, Chaps. I–V].

One of the basic results about complex algebraic surfaces says that a smooth, compact, complex algebraic surface has a rational parametrization iff it is obtained as follows.

- Start with the basic examples: $\mathbb{CP}^2$ and with $\mathbb{CP}^1$-bundles over $\mathbb{CP}^1$.
- Blow up points on the basic examples and repeat.

Comessatti developed a similar description of smooth, compact, real algebraic surfaces that admit a rational parametrization. Again we have the same two steps:

- Start with the basic examples.
- Blow up points on the basic examples and repeat.

By (4.5), blowing up a point is the same as taking a connected sum with $\mathbb{RP}^2$. If we blow up any point, the resulting surface is nonorientable and thus we have case (2) of Theorem 21. It remains to list all basic examples and show that they are all spheres and tori topologically.

It is not hard to prove that the basic examples are either Del Pezzo surfaces or minimal conic bundles which implies that the list is finite, up to homeomorphisms. Comessatti gives a quite intricate argument to arrive at a complete list; see [Kol01a, Sec. 5] for a modern treatment. A general introduction to real algebraic surfaces is in [Sil89]. □

Example 22. It is easy to parametrize the sphere. The inverse of the stereographic projection from the south pole of $(x^2 + y^2 + z^2 = 1)$ is given by

$$(x, y) \quad \mapsto \quad \left(\tfrac{2x}{1+x^2+y^2}, \tfrac{2y}{1+x^2+y^2}, \tfrac{1-x^2-y^2}{1+x^2+y^2}\right).$$

Next consider the torus obtained by rotating the circle $(x-b)^2+z^2=a^2$ in the (x,z)-plane around the z-axis. A parametrization is given by

$$(s^2+t^2=1)\times(u^2+v^2=1)\mapsto\big(u(as+b),v(as+b),at\big).$$

Combining it with the stereographic parametrizations of the circles gives

$$\begin{aligned}(s,u) &\mapsto \left(\tfrac{2s}{1+s^2},\tfrac{1-s^2}{1+s^2},\tfrac{2u}{1+u^2},\tfrac{1-u^2}{1+u^2},\right)\\ &\mapsto \left(\tfrac{2u(2as+b+bs^2)}{(1+u^2)(1+s^2)},\tfrac{(1-u^2)(2as+b+bs^2)}{(1+u^2)(1+s^2)},\tfrac{a-as^2}{1+s^2}\right).\end{aligned}$$

Exercise for algebraic geometers. Describe the singularities of the above torus and also its normalization.

It turned out to be quite subtle to generalize Comessatti's method to higher dimensions. Mori's program (see [Kol14] or [KM98]) says that the basic steps are quite similar to the surface case.

• There are basic examples. The main ones are Fano varieties, for these $-K_X$ is ample. In dimension 3 there are also families of rational curves over rational surfaces and families of Del Pezzo surfaces over $\mathbb{CP}^1$. (Unfortunately, in both cases some fibers can be quite singular.)

• Starting with any X that admits a rational parametrization (more generally, that it is rationally connected), there is a sequence of divisorial contractions and flips

$$X=X_0 \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_n = X^{\text{basic}}$$

until we get one of the basic examples.

There are two new, rather substantial, difficulties.

First, it seems rather hard—maybe even hopeless—to get a complete description of the basic examples. If X is a Fano variety, we have general finiteness statements, but, at the moment, we can say only that Fano 3-folds yield at most $10^{10^{500}}$ different topological types. (This is a ridiculous bound but the correct answer is likely to be in the millions; see the related long lists in [DIK00, Kas10].) Families of rational curves over rational surfaces and families of Del Pezzo surfaces over $\mathbb{CP}^1$ could give infinitely many exceptions. Finiteness results for these cases were first established in [Kol99b, Kol00] and extended by Mangolte in joint works with Catanese, Huisman and Welschinger; see [Man14] for a survey.

Second, the sequence of divisorial contractions and flips is not well understood in general. In fact, at first sight we should really worry about "flips". We usually think of flips as analogs of Dehn surgery on 3-manifolds: a flip of a real algebraic 3-fold removes a copy of $\mathbb{S}^1$ and puts it back differently.[2] It is known that Dehn surgeries can completely change a 3-manifold, so a sequence of flips could lead to complete loss of control over the topology of X. It is a very lucky happenstance that if we start with a real algebraic 3-fold X such that $X(\mathbb{R})$ is orientable, then the flips that appear do not affect the topology of the real part. The following result, proved in [Kol98, Kol99a], asserts that the real topology changes very mildly under the minimal model program.

[2]Dehn surgeries are usually defined using tubular neighborhoods but the two versions are equivalent.

Theorem 23. *Let X be a smooth, compact, real algebraic 3-fold such that $X(\mathbb{R})$ is orientable. Let $X = X_0 \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_n$ be any run of the minimal model program. Then*

$$X(\mathbb{R}) \stackrel{\text{diff}}{\sim} \overline{X_n(\mathbb{R})} \# a\mathbb{RP}^3 \# b(\mathbb{S}^1 \times \mathbb{S}^2)$$

for some $a, b \geq 0$. A slight twist is that X_n may be singular. Thus $X_n(\mathbb{R})$ need not be a manifold, but in our case it is obtained from a manifold $\overline{X_n(\mathbb{R})}$ by identifying finitely many point pairs and contracting some embedded spheres to points.

These results imply that only finitely many orientable, hyperbolic 3-manifolds admit a real algebraic model with a rational parametrization.

Example 24. We saw in (4.5) that blowing up a smooth point corresponds to connected sum with $\mathbb{RP}^3$.

A more complicated 2-step blow-up is needed to realize $\#(\mathbb{S}^1 \times \mathbb{S}^2)$. To get a local model, we start with $\mathbb{R}^3$ and blow up the curve $C := (x_1 = x_2^2 + x_3^2 = 0)$. ($C$ has the origin as its only real point.) The resulting 3-fold has a unique singular point p. After blowing it up, we get a smooth algebraic set. It is not hard to check that topologically we obtain a connected sum with $\mathbb{S}^1 \times \mathbb{S}^2$. Following this local model, if X is any 3-dimensional smooth real algebraic set, then by two blow-ups we can get a new smooth real algebraic set X' such that $X' \stackrel{\text{diff}}{\sim} X\#(\mathbb{S}^1 \times \mathbb{S}^2)$.

A completely different method to approach the Nash conjecture was developed by Viterbo and Eliashberg. It rests on the observation that $X(\mathbb{R}) \subset X(\mathbb{C})$ is a Lagrangian submanifold for the natural symplectic structure. One should think of $X(\mathbb{C})$ as having positive Ricci curvature, suggesting that a Lagrangian submanifold should not have negative curvature. One of the consequences is the following unpublished result of Viterbo.

Theorem 25. *Let X be a smooth, compact, real algebraic set that admits a rational parametrization. Then $X(\mathbb{R})$ is not diffeomorphic to a hyperbolic 3-manifold.*

The technical details are quite subtle and long; see [Kha02] for a detailed discussion and references. [MW12] extended the method to 3-manifolds corresponding to some of the other geometries in the Thurston classification.

Remark 26 (Nonprojective Nash conjecture [Kol02])**.** The analogy between flips and Dehn surgeries can be used to prove that for every compact, connected 3-manifold M^3 there is a sequence of smooth, real-algebraic blow-ups and blow-downs

$$\mathbb{RP}^3 = X_0 \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_r$$

such that X_r is diffeomorphic to M^3. It has been known that, starting with dimension 3, sequences of smooth blow-ups and blow-downs can result in smooth, compact but nonprojective varieties, called Moishezon manifolds or algebraic spaces. This shows how crucial it is to use projectivity in Theorem 23 and the existence of a symplectic form in Theorem 25.

3. Arcs on algebraic sets

Nash's second paper in algebraic geometry, written in 1968 but published as [Nash95] is quite informal. Through a series of examples and comments it discusses "some interesting possible truths which were encountered." We give fewer details since there are several recent surveys about arc spaces. A very elementary treatment is [JK16], while more complete discussions are in [PS15, dF16].

We change notation: from now on X, etc., stands for *complex* algebraic (or analytic) sets.

Definition 27. Let $f_i(\mathbf{x})$ be polynomials (or analytic functions), and let

$$X := \big\{\mathbf{x} : f_1(\mathbf{x}) = \cdots = f_r(\mathbf{x}) = 0\big\} \subset \mathbb{C}^N \tag{27.1}$$

be the corresponding complex algebraic (or analytic) set. Let $p \in X$ be a point. For most of the theory the examples

$$X := \big(x^a + y^b + z^c = 0\big) \subset \mathbb{C}^3 \quad \text{and} \quad p := (0,0,0)$$

are quite representative. An (analytic) *arc* in X passing through p is a holomorphic map

$$\Phi : \mathbb{D} \to X \quad \text{such that} \quad \Phi(0) = p,$$

where $\mathbb{D} \subset \mathbb{C}$ is a small disc containing 0. Equivalently, Φ is given by N analytic functions $\phi_k(t)$ such that

$$f_i\big(\phi_1(t), \ldots, \phi_N(t)\big) \equiv 0 \quad \text{for} \quad i = 1, \ldots, r. \tag{27.2}$$

Technically, it is usually easier to allow the ϕ_k to be formal power series. Thus a *formal arc* on X is given by N formal power series $\phi_k(t)$ such that

$$f_i\big(\phi_1(t), \ldots, \phi_N(t)\big) \equiv 0 \quad \text{for} \quad i = 1, \ldots, r. \tag{27.3}$$

We can write $\phi_k = \sum_j a_{kj} t^j$ and expand (27.3) by powers of t:

$$f_i\big(\phi_1(t), \ldots, \phi_N(t)\big) = \textstyle\sum_\ell H_{i\ell}(a_{kj}) t^\ell, \tag{27.4}$$

where each $H_{i\ell}$ is a polynomial involving only finitely many of the a_{kj}. We can thus view the *arc space* of X as an infinite dimensional algebraic set

$$\operatorname{Arc}(X, p) := \big\{(a_{kj}) : H_{i\ell}(a_{kj}) = 0 : \forall i\ell\big\} \subset \mathbb{C}^\infty, \tag{27.5}$$

where the a_{kj} for $1 \le k \le r, 0 \le j < \infty$ are coordinates on $\mathbb{C}^\infty$.

There are many foundational issues in dealing with infinite dimensional algebraic sets, but these are surprisingly well behaved. Instead of definitions, let us see some examples.

Example 28 (Arcs on $(xy = z^n)$)**.** We start with some obvious arc families. Fix $0 < m < n$ and write $x(t) = t^m(a_0 + a_1 t + \cdots)$ and $z(t) = t(c_0 + c_1 t + \cdots)$. If $a_0 \neq 0$, then these uniquely determine an arc

$$\begin{array}{rcl} x(t) & = & t^m(a_0 + a_1 t + \cdots), \\ y(t) & = & t^{n-m}(a_0 + a_1 t + \cdots)^{-1}(c_0 + c_1 t + \cdots)^n, \\ z(t) & = & t(c_0 + c_1 t + \cdots). \end{array}$$

Thus we get $n-1$ arc families which have the very pleasing property that they can be specified by free parameters a_i, c_i; no equations need to be solved.

It is not hard to see that the union of the above $n-1$ arc families is dense in $\operatorname{Arc}\big((xy = z^n), 0\big)$, for any sensible topology. Once the correct definitions about the infinite dimensional algebraic sets $\operatorname{Arc}(X, p)$ are established, we conclude that these $n-1$ families correspond to the $n-1$ *irreducible components* of $\operatorname{Arc}\big((xy = z^n), 0\big)$.

Example 29 (Arcs on $(x^2 + y^3 = z^3)$)**.** We again start by writing down some "free" families, but here they are less obvious.

Suppose we have an arc where $\operatorname{ord} z(t)$ (the order of vanishing of $z(t)$ at $t = 0$) is smallest. Then we can divide through by $z(t)^2$ to get

$$\left(\tfrac{x(t)}{z(t)}\right)^2 + \left(\tfrac{y(t)}{z(t)}\right)^3 z(t) = z(t), \tag{29.1}$$

and both fractions are power series. Thus $z = (x/z)^2(1 - (y/z)^3)^{-1}$ provided that $1 - (y/z)^3$ is invertible. Working backwards, $x/z := u(t)$ and $y/z := v(t)$ yield the arcs

$$x(t) = \tfrac{t^3u^3}{1-v^3}, \quad y(t) = \tfrac{t^2u^2v}{1-v^3}, \quad z(t) = \tfrac{t^2u^2}{1-v^3}, \tag{29.2}$$

where u, v are arbitrary power series but $v^3(0) \neq 1$.

We have to deal with the remaining case when the starting terms of $y(t), z(t)$ differ by a cube root of unity ϵ. We suspect that there are solutions where $\operatorname{ord} y(t) = \operatorname{ord} z(t) = 1$ and $\operatorname{ord} x(t) = 2$. We rewrite the defining equation as $x^2 = z^3 - y^3$ and factor the right-hand side:

$$x^2 = (z - \epsilon y)(z^2 + \epsilon zy + \epsilon^2 y^2). \tag{29.3}$$

Divide through by y^2 and write it as

$$\left(\tfrac{x}{y}\right)^2 = \left(\tfrac{z-\epsilon y}{y}\right)\left(y\tfrac{z^2+\epsilon zy+\epsilon^2y^2}{y^2}\right). \tag{29.4}$$

Note that if $\operatorname{ord} y(t) = \operatorname{ord} z(t) = 1$ and $\operatorname{ord} x(t) = 2$, then $\operatorname{ord}(x/y) = 1$ and the last term has also has order 1. Furthermore, by our choice of ϵ, $\operatorname{ord}(z - \epsilon y) \geq 2$ hence $\operatorname{ord}\big((z - \epsilon y)/y\big) \geq 1$. Thus (29.4) looks like the equation $z^2 = xy$ that we already dealt with in Example 28. It has arcs

$$\tfrac{x}{y} = tw(t), \quad \tfrac{z-\epsilon y}{y} = tu(t), \quad \text{and} \quad y\tfrac{z^2+\epsilon zy+\epsilon^2y^2}{y^2} = tu(t)^{-1}v(t)^2. \tag{29.5}$$

For each $\epsilon^3 = 1$, we get an arc family

$$\begin{array}{rcl} x(t) & = & tw(t)y(t), \\ y(t) & = & tw(t)^2\big(3\epsilon^2u(t) + 3\epsilon tu(t)^2 + t^2u(t)^3\big)^{-1}, \\ z(t) & = & \big(\epsilon + tu(t)\big)y(t), \end{array} \tag{29.6}$$

whenever $u(0) \neq 0$.

It is not at all obvious, but true, that the four arc families given by (29.2) and (29.6) are dense in $\operatorname{Arc}\big((x^2 + y^3 = z^3), 0\big)$ and also that none of them is contained in the closure of the others. Thus we conclude that $\operatorname{Arc}\big((x^2 + y^3 = z^3), 0\big)$ has four irreducible components.

After computing more examples, Nash observed that the process of finding "free" arc families very closely resembles resolution of singularities, and that any resolution of singularities of X can be used to write $\operatorname{Arc}(X, p)$ as a disjoint union of "free" arc families. (See [Kol07] for an introduction to resolutions.)

A puzzling aspect is that X has many different resolutions and usually one gets too many arc families. Many of these must be contained in the closure of the others. Nash identified the following as the central difficulty of the subject.

Problem 30. Describe the irreducible components of $\operatorname{Arc}(X, p)$ and the corresponding "free" arc families that cover an open, dense subset of $\operatorname{Arc}(X, p)$.

Nash then proposed that for surfaces these irreducible components correspond exactly to the exceptional curves that appear on the minimal resolution of singularities.

In higher dimensions there is no minimal resolution. After some quite insightful examples, Nash suggested that the irreducible components of $\mathrm{Arc}(X, p)$ correspond exactly to the exceptional divisors that appear on every possible resolution of the singularity $p \in X$. The latter formulation became known as the *Nash conjecture* on arc spaces.

Hironaka, Lejeune and Jalabert and their students popularized the question early on and made progress, mainly for surfaces. The most significant recent advance is the proof of Nash's conjecture for surfaces by Fernández de Bobadilla and Pe Pereira [FdBPP12]. Higher dimensional generalizations are proved by de Fernex and Docampo [dFD16].

The first counterexample—in dimension 4—was found by Ishii and Kollár [IK03], followed by simpler ones by de Fernex [dF13] and Johnson and Kollár [JK13]. Now we know that for the singularity

$$\big(x^2 + y^2 + z^2 + w^5 = 0\big) \subset \mathbb{C}^4,$$

the arc space has only one irreducible component yet there are two exceptional divisors that appear on every possible resolution. It is quite remarkable that Nash proposed this equation as the first possible counterexample to his question.

Interest in arc spaces further increased after Kontsevich based his theory of motivic integration on them (unpublished lecture in 1995, see [DL99, Bat99]) and Mustaţă used them to solve some problems in birational geometry [Mus02].

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