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Asymptotic geometric analysis, Part I, by Shiri Artstein-Avidan, Apostolos Giannopoulos, and Vitali D. Milman, Mathematical Surveys and Monographs, Vol. 202, American Mathematical Society, Providence, RI, 2015, xx+451 pp., ISBN 978-1-4704-2193-9

For a sequence $\{x_n\}_{n=1}^\infty$ of real numbers, the series $\sum_{n=1}^\infty x_n$ converges absolutely, i.e., $\sum_{n=1}^\infty |x_n| < \infty$, if and only if it converges unconditionally, i.e., $\sum_{n=1}^\infty \pm x_n$ converges for all choices of signs. This is true also if $\{x_n\}$ is a sequence in a finite-dimensional normed space. Answering a question of Banach [1], Dvoretzky and Rogers [4] showed that no infinite-dimensional Banach space continues to satisfy this property: in any such space there is an unconditionally convergent series $\sum_{n=1}^\infty x_n$ such that $\sum_{n=1}^\infty \|x_n\| = \infty$. That a Hilbert space contains such a sequence is easy—take any orthogonal sequence $\{x_n\}_{n=1}^\infty$ with $\sum_{n=1}^\infty \|x_n\|^2 < \infty$ and $\sum_{n=1}^\infty \|x_n\| = \infty$. The main result in [4] is that a somewhat similar phenomenon occurs in any *finite-dimensional* normed space: a consequence of what is now called the Dvoretzky–Rogers Lemma says that any such n-dimensional space $(X, \|\cdot\|)$ admits a Euclidean norm $|\cdot|$ which dominates $\|\cdot\|$ and such that there exists an orthonormal basis $\{x_i\}_{i=1}^n$ with $\sum_{i=1}^n \|x_i\| \geq \frac{n}{10}$ (the actual result is stronger). From this it is easy to deduce the existence of the required sequence in any infinite-dimensional Banach space.

Grothendieck got interested in the result of Dvoretzky and Rogers and gave in [6] a different proof for the existence, in any infinite-dimensional Banach space, of an unconditionally convergent series that is not absolutely convergent. In [7] he returned to this topic and discussed also the finite-dimensional result mentioned above. At the end of the paper Grothendieck suggested that the lattice of finite-dimensional subspaces of a Banach space should play a significant role in the study of the geometry of the Banach space. Among other things, he suggested introducing partial order on the set of Banach spaces by saying that X is of metric type at most that of Y if for every constant $\varepsilon > 0$, each finite-dimensional subspace of X is $1+\varepsilon$ isomorphic to a subspace of Y. This is in contrast to a similar notion of Banach (linear dimension) which is purely infinite-dimensional. In particular he raises the question of whether the Dvoretzky–Rogers Lemma can be strengthened so as to show that for every $n \in \mathbb{N}$ and every $\varepsilon > 0$ any normed space of sufficiently large dimension contains an n-dimensional subspace which is $1+\varepsilon$ isomorphic to an n-dimensional Euclidean space. This would imply that Hilbert spaces are of the smallest metric type. Why Grothendieck even suspected that this may hold is beyond the comprehension of this reviewer!

Several years later in a *tour de force* paper, Dvoretzky [3] solved Grothendieck's question in the affirmative: for each $n \in \mathbb{N}$ and $\varepsilon > 0$, there is an $N = N(n, \varepsilon) \in \mathbb{N}$ such that if $(X, \|\cdot\|)$ in an N-dimensional normed space, then there is a linear invertible map $T : \mathbb{R}^n \to X$ with

$$\|x\|_2 \leq \|Tx\| \leq (1+\varepsilon)\|x\|_2$$

for all $x \in \mathbb{R}^n$. Here $\|\cdot\|_2$ denotes the Euclidean norm in $\mathbb{R}^n$. (When this happens, we shall say that the space $T\mathbb{R}^n$ is *ε-Euclidean.*) One may find the equivalent

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geometrical statement more appealing: for each $n \in \mathbb{N}$ and $\varepsilon > 0$, there is an $N = N(n, \varepsilon) \in \mathbb{N}$ such that any symmetric about zero convex body K (i.e., open and bounded convex set such that $K = -K$) in $\mathbb{R}^N$ admits an n-dimensional "ε-spherical section"; i.e., admits a subspace L of dimension n and a radius r such that

$$B_r \subseteq K \cap L \subseteq (1+\varepsilon)B_r.$$

Here B_r is the Euclidean ball centered at zero and of radius r.

The *local theory of Banach spaces* deals with two subjects: the "isomorphic" study of finite-dimensional normed spaces and the study of properties of (infinite-dimensional) Banach spaces through their lattice of finite-dimensional subspaces. Since all normed spaces of the same dimension are isomorphic, the study of finite-dimensional spaces is only significant if we introduce a parameter K and identify two spaces X and Y when they are K isomorphic, i.e., when there is an invertible linear operator T from X onto Y with $\|T\|\|T^{-1}\| \leq K$. The result of Dvoretzky and Rogers described in the first paragraph of this review is probably the first significant example of a result concerning infinite-dimensional Banach spaces whose proof is local as it uses properties of finite-dimensional subspaces in a significant way. The book of Artstein-Avidan, Giannopoulos, and Milman is devoted to the first subject, studying the isomorphic theory of finite-dimensional spaces *per se* and does not deal with the relation of this topic to the study of infinite-dimensional Banach spaces. Before we confine ourselves to this topic, let us illustrate by one more example (which may not be the most important one, but is close to the heart of this reviewer) what we mean by studying infinite-dimensional spaces through their lattice of finite-dimensional subspaces. A major problem in the theory of Banach spaces is to determine the isomorphic classes of complemented subspaces of the "classical" spaces. Here a subspace $Y \subseteq X$ is said to be complemented in X if it is the range of a bounded linear projection, i.e., $Y = PX$ where the linear operator $P : X \to X$ satisfies $P = P^2$ and $\|P\| < \infty$. Then X is isomorphic to the direct sum of Y and $(I - P)X$. The characterization of the complemented subspaces of a given space is known only for a few simple spaces. In particular, it is not known for the $L_p(0,1)$ spaces (unless $p = 2$). Here $L_p(0,1)$ is the space of real measurable functions on $(0,1)$ for which $\|f\|_p = (\int_0^1 |f(t)|^p)^{1/p} < \infty$. A theorem of Lindenstrauss, Pełczynski, and Rosenthal [9,10] says that for $1 < p \neq 2 < \infty$ Y is isomorphic to an infinite-dimensional complemented subspace of $L_p(0,1)$ if and only if it is a Hilbert space or it is paved by ℓ_p^n spaces; i.e., for some $K < \infty$ each finite-dimensional subspace E of Y is contained in a subspace F of Y which is K isomorphic to ℓ_p^n for some finite n. (ℓ_p^n is $\mathbb{R}^n$ with the norm $\|(x_1, \ldots, x_n)\|_p = (\sum_{i=1}^n |x_i|^p)^{1/p}$.)

As I have already said, the book and the rest of this article deal with the other aspect of the local theory of Banach spaces, investigating properties of finite-dimensional normed spaces of isomorphic character. This is now commonly referred to as asymptotic geometrical analysis, hence the title of the book. A common feature in this theory is that the results usually start being true or significant only for spaces of sufficiently large dimension. Another feature is that the results have geometrical flavor, stemming from the fact that finite-dimensional normed spaces stand in simple one-to-one correspondence with symmetric (about the origin) convex bodies in Euclidean spaces. Dvoretzky's theorem stated above is a good example of such a property, and to a large extent it marks the start of this line of investigation.

The original proof of Dvoretzky's theorem is very involved. Several simplified proofs were given in the beginning of the 1970s. The first and most influential one is by Vitali Milman [11] and I shall discuss it in some detail below. The others were by Figiel [5], which is probably the shortest proof but gives poor estimates for the function $N(n, \varepsilon)$, and by Szankowski [14], which actually gives the right dependence of $N(n, \varepsilon)$ on n (obtained earlier by Milman). There is also a paper by Tzafriri [15] which uses much "softer" methods and proves the theorem but only for some fixed ε and with no estimate on N.

The proofs of Dvorezky, Milman, Figiel, and Szankowski are all probabilistic and show that for an appropriate natural probability measure, with high probability, an n-dimensional subspace of a given $N(n, \varepsilon)$-dimensional normed space is ε-Euclidean. Milman's proof has three advantages over the other ones: 1. It gives the right estimate of $N(n, \varepsilon)$ as a function of n. (The estimate of $N(n, \varepsilon)$ as a function of ε is still far from being resolved.) 2. Given an N-dimensional normed space X, the proof gives an estimate of the dimension of its largest ε-Euclidean subspace, based on a certain parameter of the space. In many interesting situations this estimate is much better than in the worst-case scenario. 3. The proof revolves about the notion of *concentration of measure* to which I shall return shortly. This is by far the most important feature of Milman's proof: this notion and its use turned out to be very productive in similar and not-so-similar situations.

I shall not review the content of the book in detail. The Preface, which can be freely accessed online, contains a chapter-by-chapter detailed description of its content. Instead, I shall concentrate on a few parts of the text.

Milman's proof of Dvoretzky's theorem is exposed in detail in Chapter 5 of the book, which starts on page 161. If one is interested only in this proof, there are of course much shorter expositions in the literature. The purpose here is almost the opposite: the book expands very much on each tool used on the way. Given a norm $\|\cdot\|$ on $\mathbb{R}^N$, one can define M to be the average of this norm on the Euclidean sphere S^{N-1} and b to be the maximal value of $\|\cdot\|$ on S^{N-1}; equvivalently, the Lipschitz norm of the function $\|\cdot\| : S^{N-1} \to \mathbb{R}$. The critical parameter of the norm $\|\cdot\|$ is NM^2/b^2 and the critical parameter of an N-dimensional normed space X is the maximum of NM^2/b^2 over all norms $\|\cdot\|$ on $\mathbb{R}^N$ such that X is isometric to $(\mathbb{R}^N, \|\cdot\|)$. (This definition is for the purposes of this review only, and it deviates a bit from that of the book.) Milman's theorem, from which Dvoretzky's theorem follows, says that X contains an ε-Euclidean subspace of dimension $c(\varepsilon)$ times the critical parameter of X.

The explicit concentration of measure phenomenon used in the proof of Milman's version of Dvoretzky's theorem is a result of P. Levy [8] which says that a real function f of Lipschitz constant 1 on the sphere S^{N-1} is basically a constant: on a complement of a set of probability $e^{-N\varepsilon^2/100}$, the oscillation of f is at most ε. (This result is applied to the function $f(x) = \|x\|/b$, whose Lipschitz constant is one and so is well concentrated near its average, M/b.) Similar results, with possibly different estimates of the "degree of concentration" of a Lipshitz function, hold for many other natural metric probability spaces. Chapter 3 of the book discusses this phenomenon in detail.

The concentration of measure phenomenon in a given metric probability space is equivalent to an approximate isoperimetric inequality, with particular estimates, in the same space. By this we mean a good lower bound on the probability of the ε boundary, or equivalently the ε expansion, of a set A of a given probabilty. By the ε boundary (resp. the ε expansion) of A we mean the set of all points in the complement of A (resp. in the whole space) which are of distance at most ε from A. Clearly, if the Lipschitz 1 function defined by $f(x) =$ the distance of x from A is well concentrated, then we get a good lower bound on the size of the ε boundary of A. The converse also holds and is not hard to prove. This explains the equivalence of the two phenomena. I remark in passing that Dvoretzky's original proof also pertains to a lower bound on the probability of expansions of sets in certain metric probability spaces. Approximate isoperimetric inequalities are of course intimately connected with the classical isoperimetric inequalities. The classical Brunn–Minkowski inequality provides a way to prove the classical isoperimetric inequality, and many variants of it. It can also be used to prove an approximate isoperimetric inequality equivalent to Levy's concentration result. The first chapter of the book revolves mainly around the Brunn–Minkowski inequality and its variants. In the spirit of the book, several proofs of the inequality are given. The two appendices explore more of classical convexity and in particular contain also a proof of the Alexander–Fenchel inequality which arguably is the most profound theorem of classical convexity.

To get Dvoretzky's theorem from Milman's, one needs to find a good norm $\|\cdot\|$ on $\mathbb{R}^N$ such that X is isometric to $(\mathbb{R}^N, \|\cdot\|)$ and M/b is large. This amounts to finding a good Euclidean structure, equivalently ellipsoid, in X. The right one for this purpose is the "ellipsoid of maximal volume" (this ellipsoid plays a role already in Dvoretzky's proof and in the Dvoretzky–Rogers Lemma). For other purposes, other Euclidean structures are better. Chapter 2 of the book explores this topic, again from a much broader prospective. In Chapter 8, another Euclidean structure, more modern and due to Milman, is introduced and explored with unexpected corollaries.

Before touching upon one more subject the book deals with, let us say that one common feature of (most of) the results in the book is that Euclidean structure lies in the heart of them. Either the results themselves involve Euclidean structure, for example Dvoretzky-like theorems, or the proofs do. Chapter 8 contains two very good examples of the later: the Bourgain–Milman theorem estimating the product of the volumes of a symmetric convex body and its polar, and the so-called reversed Brunn–Minkowsky inequality. I will not enter into the discussion of this as well as of many other subjects the book deals with. I would like to remark that the theory also has nice directions, of flavor similar to the material in the book, which do not involve Euclidean structure at all. One example of such a subject is determining the high-dimensional subspaces of the classical finite-dimensional normed spaces, in particular of the ℓ_p^n spaces. This is not discussed in the current volume. I hope it will be in the next one.

One more subject I would like to touch upon is the one presented in Chapter 10. It revolves around the still open *slicing conjecture*: given a convex body in $\mathbb{R}^n$ of volume 1, is there a hyperplane section through its center of mass whose $(n-1)$-dimensional volume is bounded from below by a universal constant? This and related questions about the distribution of the Lebesgue measure on convex bodies is still a very active field of research. The book deals with some of the main relatively recent results here, and the authors promise to return to it in the next volume of the book. I hope it will be written soon. I mention in passing that there is another recent book on this subject [2].

As I have said above, there are other expository works covering portions of the material around Dvoretzky's theorem and other topics of the current book. The two book references, by now quite old, are [12] and [13]. In contrast to these older references, which are relatively brief, the current book is much more comprehensive and goes at a slower pace. It is a very blessed addition to the literature on this exciting topic and should be on the bookshelf of every person even remotely interested in the subject. I sincerely hope that its publication, and possibly courses taught using it, will bring new young blood into this wonderful subject, which is still sizzling with activity.

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Gideon Schechtman
Department of Mathematics
Weizmann Institute of Science
Rehovot, Israel
E-mail address: gideon@weizmann.ac.il