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SELECTED MATHEMATICAL REVIEWS

related to the papers in the Nash Issue

MR0043432 (13,261g) 90.0X

Nash, John

Non-cooperative games.

Annals of Mathematics. Second Series **54** (1951), 286–295.

The general finite n-person game has previously been analyzed by considering the related two-person games obtained by partitioning the set of players into two disjoint subsets or coalitions [von Neumann and Morgenstern, Theory of Games and Economic Behavior, Princeton University Press, 1944; MR0011937]. This paper gives an entirely new treatment of such games in which cooperation (coalitions) are excluded. The main result is a generalization of the fundamental theorem of two-person zero-sum games (existence of a value) to general n-person games, and may be described as follows: Let $s_1, s_2, \cdots, s_n$ denote mixed strategies for players $1, 2, \cdots, n$, and let $p_i(s_1, s_2, \cdots, s_n)$ be the pay-off function to the ith player, where, as usual, each p_i is an n-linear function of the s_i's. An n-tuple of strategies $(s_1, s_2, \cdots, s_n)$ is called an "equilibrium point" if, for all $i, p_i(s_1, s_2, \cdots, s_n) \geq p_i(s_1, s_2, \cdots, s_i{}', \cdots, s_n)$, where $s_i{}'$ is any strategy of the ith player. Theorem: Every finite n-person game has an equilibrium point. Making use of this concept the author defines the notion of solution of a game (which, however, need not exist). Further sections are devoted to analyzing the structure of solutions and giving examples.

D. Gale

From MathSciNet, December 2016

MR0050928 (14,403b) 14.0X

Nash, John

Real algebraic manifolds.

Annals of Mathematics. Second Series **56** (1952), 405–421.

An algebraic variety defined over the real field in Euclidean n-space E_n may consist of several sheets. Any one of these which has no singular points is called an algebraic sheet. If M_d is a real closed differentiable manifold, it is known that it can be embedded differentiably in E_n, for a suitable value of n, and the principal theorem proved in this paper shows that this embedding can be approximated by an algebraic sheet $M_d{}'$ belonging to an algebraic variety V_d. If the other sheets of

V_d do not meet this sheet, M_d' is said to be a proper representation of M_d, and it is shown that a proper representation of M_d exists in E_n if $n \geq 2d+1$.

Let M_d be any real closed analytic manifold and suppose that on it there exists a ring R of analytic functions such that (a) in R there exists a set of functions which define an algebraic representation of M_d, (b) if $f_1, \cdots, f_{d+1} \in R$, there exists a real polynomial $\phi(X_1, \cdots, X_{d+1})$ such that $\phi(f_1, \cdots, f_{d+1}) = 0$ on M_d, (c) R is not contained in any larger ring having these properties. The pair (M_d, R) is called a real algebraic manifold, and two manifolds (M_d, R), (M_d', R) are said to be equivalent if there exists an isomorphism between R and R'. The second part of the paper develops a few properties of real algebraic manifolds, most of which correspond to well-known properties of algebraic geometry. The most important result obtained is that two real algebraic manifolds are equivalent if and only if they are analytically homeomorphic.

W. V. D. Hodge

From MathSciNet, December 2016

MR0065993 (16,515e) 53.0X

Nash, John

C^1 isometric imbeddings.

Annals of Mathematics. Second Series **60** (1954), 383–396.

This paper contains some surprising results on the C^1-isometric imbedding into an Euclidean space of a Riemannian manifold with a positive definite C^0-metric. The theorems are: 1) Any closed Riemannian n-manifold has a C^1-isometric imbedding in E^{2n} (the Euclidean space of dimension $2n$). 2) Any Riemannian n-manifold has a C^1-isometric immersion in E^{2n} and an isometric imbedding in E^{2n+1}. 3) If a closed Riemannian n-manifold has C^1-immersion or imbedding in E^k with $k \geq n+2$, it also has respectively an isometric immersion or imbedding in E^k. The basic idea is a perturbation process defined in a neighborhood and relative to two normal vector fields. The imbedded or immersed manifold is of course generally quite pathological.

S. Chern

From MathSciNet, December 2016

MR0075639 (17,782b) 53.1X

Nash, John

The imbedding problem for Riemannian manifolds.

Annals of Mathematics. Second Series **63** (1956), 20–63.

Continuing his work on the imbedding problem, begun in his former paper on C'-embeddings [Ann. of Math. (2) **60** (1954), 383–396; MR0065993], the author turns to the study of imbeddings of Riemannian manifolds of class C^3 or higher, and establishes that every such manifold can be embedded in a C^3-isometric way in a Euclidean space, whose dimension can be estimated. The main part of the proof rests on a perturbation theorem of the "open-mapping" variety. This may most easily be explained as follows: Let ϕ be a non-linear map of a Banach space X of functions or tensors into another such space Y. Let $\dot{y} = A(x, \dot{x})$ be the Fréchet derivative of ϕ at the point x, and suppose that the equation $\dot{y} = A(x, \dot{x})$ can be solved for $\dot{x}$ in terms of $\dot{y}$ in the form $\dot{x} = F(x, \dot{y})$, so that $\dot{y} = A(x, F(x, \dot{y}))$. In this case, ϕ may be regarded as having a "non-vanishing Jacobian" in a suitable abstract sense. Nash then proves that with suitable assumptions on the analytic nature of ϕ, A, F, much weaker however than continuity of F in x, the range of ϕ covers an open set. If F were known to be continuous in $\dot{y}$, this could be done as usual by solving the equation $x_t = F(x, \Delta y)$ for any desired variation Δy of y. In the present case, in which F acts on x as a partial differential operator of second order, Nash modifies the equation $x_t = F(x, \Delta y)$ to a system of integral equations as follows:

$$z(\theta) = z_0 + \int_{\theta_0}^{\theta} F(S_\theta z, M)\, d\theta,\ L(\theta) = \int_{\theta_0}^{\theta} u(\theta - \theta')\{A(z, F(S_\theta z, M) - M)\}\, d\theta.$$

Here u is a C^∞ function vanishing for $\theta < 0$ and identically one for $\theta > 0$; S_θ is a suitably chosen "smoothing operator" or "mollifier" such that $S_\theta z \to z$ as $z \to \infty$, and

$$M = M(L) = \frac{\partial}{\partial\theta}\{u(\theta - \theta_0)G + L\}.$$

If θ_0 is sufficiently large, these integral equations may be solved, and $\phi(z(\infty)) = \phi(z_0) + G$. Using the main perturbation theorem reported above, together with a number of devices taken from "C^1-isometric embeddings", the author is then able to establish Theorem 2: A compact Riemannian n-manifold with a C^k positive metric has a C^k isometric imbedding in any small volume of Euclidean $\frac{1}{2}n(3n+11)$-space, provided $3 \le k \le \infty$.

An especially interesting feature of the proof is that the bound on the number of dimensions required for the imbedding is obtained by the application of "algebraic-geometry" dimensionality arguments, whose use is justified by appeal to an earlier result of the author giving an algebraic imbedding for general differential manifolds [ibid. **56** (1952), 405–421; MR0050928].

The paper ends with the extension of Theorem 2 to noncompact manifolds, the result being as follows: Theorem 3: Any Riemannian n-manifold with a C^k positive metric, where $3 \le k \le \infty$, has a C^k isometric imbedding in $\frac{1}{2}n(n+1)(3n+11)$-dimensional euclidean space, in fact, in any small portion of this space.

The proof uses Theorem 2 and a special device for localizing the embedding problem on a non-compact manifold, using a special covering of the manifold by neighborhoods suitably defined in terms of a triangulation.

J. Schwartz

From MathSciNet, December 2016

MR0100158 (20 #6592) 35.00

Nash, J.

Continuity of solutions of parabolic and elliptic equations.

American Journal of Mathematics **80** (1958), 931–954.

In this paper, the writer considers bounded solutions $T(x,t)$ $(x = x^1, \cdots, x^n)$ of parabolic equations of the form $T_t = \nabla \cdot a \cdot \nabla T$, in which the eigenvalues of the matrix $a = \|a_{ij}(x,t)\|$ are always between two numbers c_1 and c_2, with $0 < c_1 < c_2$, and ∇ denotes the x-gradient. He first proves that any solution $T(x,t)$ such that

$|T(x,t)| \leq B$ for $t > t_0$ satisfies a Hölder condition of the form

$$|T(x_1,t) - T(x_2,t)| \leq AB(|x_1 - x_2|/(t-t_0)^{\frac{1}{2}})^{\alpha}, 0 < \alpha < 1,$$

where A and α depend only on n, c_1, c_2; he also shows that any such T satisfies a corresponding Hölder condition in the time variable. The writer bases his proof on a succession of inequalities for fundamental solutions $S(x,t;\overline{x},\overline{t})$ which are solutions for $t > \overline{t}$ and have a unit source at $(\overline{x},t)$. Using these results he shows that bounded solutions of the elliptic equations $\nabla \cdot a \cdot \nabla T = 0$ satisfy certain Hölder conditions on interior domains which depend only on $\sup |T|$, n, c_1, c_2, and the distance of the interior domain from the boundary of the domain of definition. This result generalizes to the case $n > 2$ an old result of the reviewer [Trans. Amer. Math. Soc. **43** (1938), 126–166], which he used to prove the differentiability of the solutions of certain variational problems and which Nirenberg [Comm. Pure Appl. Math. **6** (1953), 103–156, 395; MR0064986] used to establish the existence of the solutions of certain quasi-linear elliptic equations. Recently, E. DeGiorgi has proved (by completely different methods) similar results in the elliptic case only [Mem. Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. (3) **3** (1957), 25–43; MR0093649], but for solutions which may only be in L_2. The present writer also announces results concerning continuity on the boundary in the elliptic case.

C. B. Morrey Jr.

From MathSciNet, December 2016

MR0656198 (83j:58014) 58C15; 46G05, 58D15

Hamilton, Richard S.

The inverse function theorem of Nash and Moser.

American Mathematical Society. Bulletin. New Series **7** (1982), *no.* 1, 65–222.

This paper is a careful exposition of the Nash-Moser inverse function theorem in a setting that allows one to apply the theorem without the need to prove a new version of it for each new kind of application. Along with this comes much useful extra equipment (a priori estimates) fitting perfectly into the setting. Convincing examples show the theorem with its different "outfits" at work.

The first part of the paper deals with calculus in Fréchet spaces. A mapping is smooth if all directional derivatives exist and are jointly continuous in all the variables involved. This is the weakest notion of differentiability utilizing continuity and admitting a chain rule. A very elegant proof of the so-called omega lemma is given. Then Fréchet manifolds are investigated, spaces of mappings and diffeomorphisms are equipped with Fréchet manifold structures, composition and inversion of mappings are shown to be smooth (some gaps here—but there are lots of correct proofs in the literature now). Spaces of compact submanifolds of manifolds and of regions with smooth boundary are shown to be Fréchet manifolds. We point out that the space of smooth mappings from X to Y is a Fréchet manifold if and only if X is compact—the same restriction is necessary for all the examples. Finally the stage is set for inverse function theorems. The inverse function theorem on Banach spaces is proved in such a way that all partial results that hold on Fréchet spaces are obtained there. Numerous counterexamples are given which show that the Nash-Moser theorem is in some sense the best possible; e.g., the exponential mapping of the Fréchet-Lie groups of diffeomorphisms is not surjective on any neighbourhood of the identity (following Omori). Nice examples are given to illustrate the use of the inverse function theorem on Banach spaces: finding geodesics with given endpoints; solving the classical Plateau problem; studying oscillatory motions in a smooth convex trough; a smooth vector field on a Fréchet space that factors over a Banach space admits a local flow (this is essential later in the proof of the Nash-Moser theorem).

The second part introduces the category of tame Fréchet spaces and tame smooth mappings, the category in which the Nash-Moser theorem inverts mappings. First of all one considers graded Fréchet spaces, i.e., spaces with distinguished increasing families of seminorms generating the topology (written $\| \; \|_n$, $n \in \mathbf{N}$). A linear mapping between graded Fréchet spaces is tame of degree r and base b if $\|Lx\|_n \leq C\|x\|_{n+r}$ for all $n \geq b$ with a constant that may depend on n. A graded Fréchet space is said to be tame if it is a tame direct summand in a space $\Sigma(B)$ of exponentially decreasing sequences (x_k) in some Banach space B, with grading $\|(x_k)\|_n = \sup_k e^{nk}\|x_n\|_B$, e.g. The point is that tame Fréchet spaces admit smoothing operators. Originally the smoothing operators of John Nash were convolutions with carefully chosen functions, on spaces of C^∞-functions on the torus. Via Fourier development one notices that cutting the Fourier series at a certain order does the same thing, and on the space $\Sigma(B)$ of exponentially decreasing sequences cutting the sequence at a certain order (depending smoothly on the order) is a very convenient family of smoothing operators. The following are tame Fréchet spaces: $C^\infty(X)$ for a compact manifold X, with or without boundary, spaces of smooth sections of vector bundles with compact base, Banach spaces, some spaces of holomorphic functions. Most of those are seen to be tame using the Fourier transform.

A (nonlinear) mapping P between graded Fréchet spaces is tame (of base b and degree r) if $\|P(f)\|_n \leq C(1 + \|f\|_{n+r})$ for $n \geq b$, locally in f. P is called a smooth tame map if P is smooth and all its derivatives are tame. It turns out that all partial differential operators (nonlinear) on manifolds are smooth tame, so the chart changes of manifolds of mappings are smooth tame; thus all manifolds of mappings considered are tame manifolds. Composition and inversion of diffeomorphisms are smooth tame mappings. One of the main requirements for a smooth tame map P to be locally invertible by the Nash-Moser theorem is that $DP(x)$ be invertible in a whole neighbourhood and that the family of inverses $VP(x) = DP(x)^{-1}$ be smooth and tame. To check this is the hardest part in applying the theorem; it usually involves some a priori estimates for linear partial differential operators. The main body of the second part is devoted to checking exactly this for a number of important cases of tame families of linear mappings: certain ordinary linear differential equations; elliptic linear partial differential equations (using interpolation and Gårding's inequality), invertible, or having kernel and cokernel of finite dimension and admitting an elliptic invertible modification; elliptic boundary value problems (with linear coercive boundary conditions); linear differential operators of degree 1, mapping sections of a vector bundle to sections of the dual bundle, with symmetric symbol, nowhere characteristic at the boundary, admitting a "positive weight function" ("symmetric systems").

The third and last part of the paper is devoted to the Nash-Moser inverse function theorem proper, which is stated as follows: Let F and G be tame spaces and $P\colon U \subseteq F \to G$ a smooth tame map. Suppose that the equation for the derivative $DP(f)h = k$ has a unique solution $U = VP(f)k$ for all f in U and all k, and that the family of inverses $VP\colon U \times G \to F$ is a smooth tame map. Then P is locally invertible and each local inverse P^{-1} is a smooth tame map. There are partial results if DP has only a smooth tame family of left or right inverses. The proof is by the Newton method, written as an ordinary differential equation for the error term, modified by smoothing operators in such a way that the vector field of this ordinary differential equation factors over a Banach space and thus admits a local solution. By a priori estimates this solution is shown to exist for all times and converges exponentially to a solution of $P(f) = g$. That the inverse is smooth and tame is easy.

Then, instructive and convincing applications are given, using all the a priori estimates from the second part: embedding compact oriented surfaces with strictly positive curvature into $\mathbf{R}^3$ (elliptic equation with finite-dimensional null space); solving the shallow water equations on any compact Riemannian manifold (A) with arbitrary initial conditions for short time, and (B) for long time with small initial conditions (symmetric system); the space of compact submanifolds of total measure r of a finite-dimensional Riemannian manifold is a smooth tame submanifold of the Fréchet manifold of all compact submanifolds of X. If a tame Fréchet Lie group acts tamely on a tame connected Fréchet manifold, and the infinitesimal action is everywhere surjective with tame right inverse, then the group acts transitively. This is applied to show that neighbouring symplectic forms representing the same cohomology class of a compact manifold are conjugate by diffeomorphisms. The same holds for contact structures that are near enough.

If X is a compact smooth manifold, then the diffeomorphism group $\mathcal{D}(X)$ acts transitively on the space $\mathcal{M}(X)$ of smooth positive measures of total mass 1, and this makes $\mathcal{D}(X)$ into a principal fibre bundle over $\mathcal{M}(X)$ with fibre $\mathcal{D}_\mu(X)$ over $\mu \in \mathcal{M}(X)$, the group of μ-preserving diffeomorphisms.

Then the Nash-Moser theorem for exact sequences is presented (without proof) and applied to imbed annuli with strictly negative curvature in "proper position" into $\mathbf{R}^3$, by giving an embedding of one boundary component first (symmetric system).

Next the Nash-Moser theorem with quadratic error is presented (the proof depends on another paper of the author), and then applied to a (abstract) situation where the error comes from the choice of a smooth tame connection on a tame vector bundle in the plane which is in turn applied to a free boundary problem in the plane (there exists a unique perfect flow around any convex obstacle in the plane which is stagnant outside a compact set and has arbitrarily given outer velocity and circulation).

Peter Michor

From MathSciNet, December 2016

MR1381967 (98f:14011) 14E15

Nash, John F., Jr.

Arc structure of singularities.

A celebration of John F. Nash, Jr.

Duke Mathematical Journal **81** (1995), *no.* 1, 31–38.

From the text: "We prove the following results: Proposition 1. Corresponding to any algebraic subset W of a variety, there are a finite number of families of associated arcs $x(t)$ where $x(0)$ is on W.

"Proposition 2. Given a resolution of the variety, each arc family will correspond to a specific component of the image of W in the resolution.

"Corollary. There are essential components for W which must appear as components of the image of W in any resolution, equivalence of components being the birational correspondence of their monoidal transforms.

"Observation. If, in a resolution, a component of the image of the singular set of the singular variety is a hypersurface not birationally equivalent to a ruled variety, then it is an essential component and will appear in every resolution."

From MathSciNet, December 2016