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Foundation of free noncommutative function theory, by Dmitry S. Kaliuzhnyi-Verbovetskyi and Victor Vinnikov, Mathematical Surveys and Monographs, Vol. 199, American Mathematical Society, Providence, Rhode Island, 2014, vi+183 pp., ISBN 978-1-4704-1697-3, US $77.00

Noncommutative structures in mathematics (groups, rings, matrices, quaternions, ...) have been studied for centuries, but the last few decades have seen an explosion of new ideas and methods. The explosion was driven partly by the needs of various applications, such as automata theory and theoretical physics, and by a natural curiosity for a broader picture.

Most of these new ideas are based on different adjustments and generalizations, or *noncommutative deformations* of what is already known in the commutative mathematical universe, including the idea of the deformation quantization influenced by physicists (see, for example, [12]).

A competing approach is aimed at constructing the building of noncommutative mathematics "from scratch" by considering classical (rather elementary) problems in a purely noncommutative setting.

Among noncommutative deformations, one could list the studies of quantum groups and rings of differential operators or, more generally, noncommutative algebras of polynomial growth (see, for example, [2]). More subtle examples include Kontsevich's approach to noncommutative symplectic geometry [13] and Kapranov's definition of noncommutative Grassmannians based on replacing the transition maps by noncommutative formal series [11]. Other adjustments include the well-known and widely used Dieudonné determinant, which does not go that far from its commutative counterpart [1].

In the commutative world, determinants display two important features: they were used for Cramer's rules to solve systems of linear equations (long before matrices were introduced) and they possess the multiplicative property defining a one-dimensional representation of the group of invertible matrices over a commutative ring. One cannot keep these two features together in the noncommutative case (see [3]). To save the multiplicativity property, as Dieudonné did, the determinants must take values in the quotients of the given rings modulo commutators which makes Cramer's rules useless. Otherwise, one has to abandon the multiplicativity to follow a down-to-earth approach for treating systems of linear equations.

In comparison to noncommutative deformations, a more revolutionary approach to noncommutative geometry is based on various extensions of the commutative duality between spaces and algebras of functions on these spaces, a duality first discovered by I. Gelfand for topological spaces and then introduced by A. Grothendieck into commutative algebraic geometry. Here a *space*, or rather its algebra of functions, is replaced by an algebra of Noetherian type or by an appropriate category of sheaves, or sheaf-like algebraic or operator-algebraic structures (see, for example, [5, 7].) In other words, we have to expand to the noncommutative world the powerful methods that worked so effectively in the commutative environment.

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An alternative idea, advocated by Gelfand inter alia, is not to rely on the machinery of commutative methods, but to treat the noncommutative case in a completely new way. Gelfand suggested returning to initial sources, to try to solve a selection of classical problems in this completely noncommutative world and then proceed with new theories. In other words, the Gelfand recipe was to start with noncommutative high-school or freshmen-level mathematics with an expectation that the new theories will be different and probably even simpler than their commutative counterparts.

The first attempt at such an approach appears in Wedderburn's study of the elementary property of noncommutative continuous fractions [19]. Wedderburn writes that the noncommutative quantities used in his considerations may be matrices or differential operators but, in fact, he is working with free variables. It is also impossible not to mention the *Schur complement* introduced in 1917 for inverting square block-matrices (in an implicit form, the Schur complement was used by Laplace and Sylvester.)

In 1927 Heyting [10], trying to understand projective geometry over noncommutative skew-fields, suggested a noncommutative replacement of the determinant, which he called *designant*. This notion was forgotten for more than 60 years and was reborn again as a special case of quasi-determinants introduced by Gelfand and Retakh in [9]. The idea goes back to the Schur complement. It is a rather popular belief that nothing deep can come out of working with free variables (one of the arguments is that the corresponding algebras are of exponential and not polynomial growth); however, there exists a number of counterexamples contradicting this theory.

The down-to-earth approach is especially important when working within classical areas, like function theory and linear algebra. In a pure algebraic setting, this was promoted in the work by P. M. Cohn on the ring of noncommutative polynomials and the skew-field of noncommutative rational functions [6] and was studied by experts in polynomial and rational identities [14] and automata theory and formal languages [4, 15]. Various properties of tensor algebra and their free noncommuting variables could be found in any advanced course on abstract algebra. Analytic theories were, in particular, developed for applications in free probability [18] and control theory [8].

The book under review is the first thorough description of a theory of functions depending on free noncommuting variables. Matrix-valued and operator-valued functions were widely used in functional analysis, and an interest in analytic quaternionic functions was revived by A. Sudbery in 1979 [16]. But the study of analytic functions of several "pure" noncommuting variables began in the pioneering work of J. L. Taylor on noncommutative spectral theory [17]. He showed that such functions admit a good difference-differential calculus leading to a noncommutative version of the classical Taylor formula (the similarity of names is just a coincidence). The Taylor–Taylor formula (isn't it a nice terminology?) and the Taylor–Taylor series constitute the backbone of the book.

After introducing of the main subject, the authors describe left and right difference-differential operators of the first and the higher orders and the corresponding calculus, including higher-order directional difference-differential operators and noncommutative integrability. They discuss noncommutative analyticity, uniformly open topology over an operator space and uniformly analytic noncommutative functions. As in traditional textbooks in analysis, the authors pay special attention to the continuation and convergence of noncommutative power series in various topologies (finitely open, norm, and uniformly open). A separate chapter is devoted to noncommutative polynomials.

Overall, the book is the first and an excellent attempt to present a systematic and carefully written introduction to a theory of functions of free noncommuting variables. It offers a unified approach to a variety of free noncommutative quantities in different areas of mathematics, including noncommutative analysis, and it is highly recommended to experts. It would have been nice, however, to include in the book more applications of the theory, such as free probability, control theory, various interpolation and approximation theorems, the corona theorem, etc. (they are just listed in the introduction). Then the volume of the book could be, probably, twice as much as the current one, but it would enormously increase the number of potential readers, including graduate students.

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