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SELECTED MATHEMATICAL REVIEWS

related to the work of JOHN TATE

MR0044509 (13,427c) 09.0X

Artin, Emil; Tate, John T.

A note on finite ring extensions.

J. Math. Soc. Japan **3** (1951), 74–77.

Let R and S be two commutative rings, $R \subseteq S$. Then S is called a module-finite extension of R, if it is an R-module with a finite set of generators, that is, if there exists a finite number of elements of S such that every element of S can be represented as a linear combination of them with coefficients in R. On the other hand, S is said to be a ring-finite extension of R, if it can be written in the form $S = R[\xi_1, \xi_2, \cdots, \xi_n]$. The following theorem is proved. Let R be a Noetherian ring with unit element, let S be a ring-finite extension, and let T be an intermediate ring, $R \subseteq T \subseteq S$, such that S is a module-finite extension of T. Then T is a ring-finite extension of R. As an application, the following theorem of Zariski [Bull. Amer. Math. Soc. **53**, 362–368 (1947); MR0020075] is obtained. If a ring-finite extension of a field is a field, then it is algebraic and hence module-finite. As shown by Zariski, the Nullstellensatz is an immediate consequence of this result. It is further shown that a Noetherian integral domain R with a unit element has ring-finite extensions which are fields if and only if the quotient field F of R is a ring-finite extension of R. The ring-finite extension fields of R are then exactly the module-finite extension fields of F. The condition that F is a ring-finite extension of R is equivalent to each of the following conditions. I. There exists an element $a \neq 0$ of R which is contained in all proper prime ideals of R. II. There exists only a finite number of minimal prime ideals of R. III. There exists only a finite number of prime ideals of R, and every one of them is maximal.

R. Brauer

From MathSciNet, June 2017

MR0049950 (14,252b) 10.0X

Tate, John

The higher dimensional cohomology groups of class field theory.

Ann. of Math. (2) **56** (1952), 294–297.

It is shown that the Galois cohomology group $H^r(G, A)$ in the idèle-class group A of an algebraic number field (or the multiplicative group of a p-adic number field) is canonically isomorphic to $H^{r-2}(G, Z)$ where Z is the additive group of rational integers ($r > 2$). Let first G be a finite group and A any abelian G-group. Let α be a 2-cohomology class of G in A, and $\overline{A}$ be the Artin splitting group for α. It is proved that the following two axioms are equivalent: (1) $H^1(U, A) = 0$, and $H^2(U, A)$ is cyclic of the same order as U, generated by the restriction of α to U, for all subgroups U of G; (2) $H^1(U, \overline{A}) = H^2(U, \overline{A}) = 0$ for all subgroups U of G. The proof depends on two exact sequences $\overline{A}/A \simeq I$, $R/I \simeq Z$, where R is the group ring of G over Z, considered as a G-module, and I is its ideal generated by the elements $\sigma - 1$ ($\sigma \in G$). These exact sequences entail exact sequences for 0-, 1-, 2-cohomology groups in $I, A, \overline{A}$ and in Z, I, R, respectively. They together lead to the theorem. Combined with the fact that (2) implies the vanishing of higher cohomology groups, due to Serre, Lyndon, Hochschild and the reviewer [cf. Hochschild and Nakayama, Ann. of Math. (2) **55**, 348–366 (1952); MR0047699], it leads to the fact that $H^r(G, A)$ is isomorphic with $H^{r-2}(G, Z)$; the isomorphism is given by cup product with α. As the fundamental (or canonical) Galois 2-cohomology class α in the idèle-class group satisfies (1) [Hochschild and Nakayama, loc. cit.], the statement at the opening of the review follows. The vanishing of $H^3(G, A)$ in this class field theory case, proved independently by Hochschild, is a particular instance. The author promises a subsequent paper in which negative-dimensional cohomology groups will be introduced and the reciprocity law will be regarded as another special case of the result.

T. Nakayama

From MathSciNet, June 2017

MR0086072 (19,119b) 18.0X

Tate, John

Homology of Noetherian rings and local rings.

Illinois J. Math. **1** (1957), 14–27.

In this paper the author makes systematic use of skew-commutative differential graded algebras over a commutative noetherian ring R to study the structure of

$$\mathrm{Tor}^R\,(R/M, R/N).$$

(Such an algebra is called an R-algebra.) He does this by first showing that there always exists a free resolution X of the residue class ring R/M which is an R-algebra. The algebra structure of $\mathrm{Tor}^R(R/M, R/N)$ can then be determined directly from these R-algebra resolutions of R/M and R/N.

To prove that R/M always has a free R-algebra resolution, the author introduces the device of killing cycles in an arbitrary R-algebra. Specifically, he shows that if X is an R-algebra, and t is a $(\rho - 1)$-dimensional cycle ($\rho > 0$), then there is a canonical way of constructing an R-algebra Y containing X such that $Y_\lambda = X_\lambda$ for $\lambda < \rho$ and

$$B_{\rho-1}(Y) = B_{\rho-1}(X) + Rt$$

(where $B_{\rho-1}(X)$ means the boundaries of the R-algebra X of dimension $\rho - 1$). The procedure depends on the parity of ρ. If ρ is odd, Y is essentially the exterior algebra over X generated by an element T (of degree ρ) and $dT = t$. If ρ is even, Y is the twisted polynomial ring in one generator T over X, with $dT = t$. The algebra Y is denoted by the symbols $X\langle T\rangle$, $dT = t$.

The map $i_*\colon H(X) \to H(Y)$ induced by $i\colon X \to Y$ is shown to be a surjection (epimorphism) if the homology class τ of t is a skew non-zero divisor, i.e., if, for $\xi \in H(X)$, $\tau\xi = 0$ implies $\xi = 0$ if ρ is odd and $\xi \in \tau H(X)$ if ρ is even.

Using these methods, a special resolution is obtained which yields an efficient method for computing the homology and cohomology groups of a finitely generated abelian group.

Generalizations of results of Eilenberg (unpublished) and of Serre [see the paper reviewed above] are obtained. In particular, if R is a local ring and K is the residue field of R, denote by $B_q(R)$ the dimension of the vector space $\operatorname{Tor}_q{}^R(K,K)$ over K. Now, if R is not a regular local ring, then $B_r(R) \geq \binom{n}{r} + \binom{n}{r-2} + \cdots$ and hence $\geq 2^{n-1}$ for $r \geq n$, where n is the minimum number of elements required to generate the maximal ideal of R. Therefore, one obtains a new proof of the fact that regular local rings are precisely those of finite global dimension (Serre). Moreover, if $B_r(R) = \binom{n}{r}$ for one single dimension $r \geq 2$, then R is regular. This generalizes the result of Eilenberg, which was proved only for $r = 2$ or 3.

D. Buchsbaum

From MathSciNet, June 2017

MR0206004 (34 #5829) 14.51; 14.40

Tate, John

Endomorphisms of abelian varieties over finite fields.

Invent. Math. **2** (1966), 134–144.

Suppose that A/k is an abelian variety of dimension g which is defined over the field k with algebraic closure $\overline{k}$. Let $A(\overline{k})$ be the abelian variety obtained from A by extending k of $\overline{k}$. Furthermore, let l be a prime distinct from the characteristic of k. Then the groups A_{l^n} of points $a_n \in A_{l^n}$ satisfying $l^n a_n = 0$ determine, by the homomorphisms $a_{n+1} \to l a_{n+1} \in A_{l^n}$, a projective limit which is a free $\mathbf{Z}$-module of rank $2g, T_l(A)$, on which the Galois group G of $\overline{k}|k$ operates in the obvious manner. The author proves, as a first most noteworthy result, that the canonical (injective) map $(*)$ $\mathbf{Z}_l \otimes H_k(A', A'') \to \operatorname{Hom}_G(T_l(A'), T_l(A''))$ is bijective for abelian varieties A'/k and A''/k if k is a finite field. He first reduces this statement (no restriction on k being needed) to the equivalent proposition that the map $(**)$ $\mathbf{Q}_l \otimes \operatorname{End}_k(A) \to \operatorname{End}_G(\mathbf{Q}_l \otimes_{\mathbf{Z}_l} T_l(A))$ be bijective for every abelian variety A/k (Lemma 3). Next, implications of a hypothesis $\operatorname{Hyp}(k, A, d, l)$ which was suggested by Lichtenbaum are discussed. This hypothesis is as follows: there exist (up to k-isomorphism) only a finite number of abelian varieties B defined over k such that (a) there is a polarization ψ of B of degree d^2 defined over k, (b) there is a k-isogeny $B \to A$ of l-power degree. Using a polarization of A to its dual (see, in this connection, D. Mumford [*Geometric invariant theory*, Ergeb. Math. Grenzgeb. (N.F.), Band 34, Academic Press, New York, 1965]) and the associated bilinear form on $\mathbf{Q}_l \otimes_{\mathbf{Z}_l} T_l(A)$ (special care must be taken so that the various polarizations match, pp. 136–137, proof of Proposition 1), it is shown that $\operatorname{Hyp}(k, A, d, l)$, together with the assumption that the algebra which is generated in $\operatorname{End}(\mathbf{Q}_l \otimes_{\mathbf{Z}_l} T_l(A))$ by the elements of G is isomorphic to a product of copies of $\mathbf{Q}_l$, implies that the map $(**)$ is bijective (this subalgebra then turns out to be the commutator algebra of the image of $\mathbf{Q}_l \otimes \operatorname{End}_k(A)$ by $(**)$; see Lemma 4 and Proposition 2, its semi-simplicity being equivalent to the bijectivity of $(**)$). Hence the map $(*)$ is also bijective. Finally, results of Mumford [Invent. Math. **1** (1966), 287–354; MR0204427] imply that $\mathrm{Hyp}(k, A, d, l)$ holds for finite fields. To this end, the author shows that the dimension of $\mathrm{End}_G(\mathbf{Q}_l \otimes_{\mathbf{Z}_l} T_l(A))$ does not depend on l. For the proof, an integer $r(f_A, f_B)$ is associated with a pair of abelian varieties A, B whose Frobenius automorphisms have the characteristic polynomials f_A, f_B; if $f_A = \prod P^{a(P)}$, $f_B = \prod P^{b(P)}$ with irreducible factors in a field $K/\mathbf{Q}$, then $r(f_A, f_B) = \sum_P a(P)b(P) \deg P$. This positive integer is independent of K and is equal to the dimension of $\mathrm{Hom}_G(\mathbf{Q}_l \otimes_{\mathbf{Z}_l} T_l(A), \mathbf{Q}_l \otimes_{\mathbf{Z}_l} T_l(B))$, and the rank of $\mathrm{Hom}_k(A, B)$ equals $r(f_A, f_B)$.

The author's main result has decisive consequences for problems concerning the ζ-functions of abelian varieties and Hasse's sum formula for the invariants of $\mathbf{Q} \otimes \mathrm{End}_k(A)$ [see the author, *Arithmetical algebraic geometry* (Proc. Conf. Purdue Univ., 1963), pp. 93–110, Harper & Row, New York, 1965]. To mention a few: $f_B | f_A$ if and only if B is k-isogeneous to an abelian subvariety of A defined over k; $\mathbf{Q}[\pi]$, π the Frobenius endomorphism of A/k, is the center of $\mathbf{Q} \otimes \mathrm{End}_k(A)$;

$$2g \leq \dim_{\mathbf{Q}}(\mathbf{Q} \otimes \mathrm{End}_k(A)) = r(f_A, f_A) \leq (2g)^2$$

(compare with the classical case $k = \mathbf{C}$); $r(f_A, f_A) = 2g$ if and only if $\mathbf{Q} \otimes \mathrm{End}_k(A) = \mathbf{Q}[\pi]$; on the other hand $r(f_A, f_A) = (2g)^2$ if and only if $\mathbf{Q} \otimes \mathrm{End}_k(A)$ is isomorphic to the algebra of all g by g matrices with coefficients in the division algebra which is ramified at p and ∞ (for $g = 1$ compare with the results of Hasse and Deuring on the super-singular invariants). Finally, the author indicates that (i) appealing to results of Ju. I. Manin [Uspehi Mat. Nauk **18** (1963), no. 6 (114), 3–90; MR0157972; translated as Russian Math. Surveys **18** (1963), no. 6, 1–83], the Hasse invariant $\mathrm{inv}_v(\mathbf{Q} \otimes \mathrm{End}_k(A))$ is $\equiv i_v \pmod{\mathbf{Z}}$ for all valuations v of $\mathbf{Q}(\pi)$, where $\|\pi\|_v = q^{-i_v}$, and the Artin-Whaples product formula $\prod_v \|\pi\|_v = 1$ then implies Hasse's sum formula $\sum_v \mathrm{inv}_v(\mathbf{Q} \otimes \mathrm{End}_k(A)) \equiv 0\ (\mathbf{Z})$, and that (ii) for schemes X which are products of curves and abelian varieties with the Néron-Severi group $\mathrm{NS}_k(X)$, the rank of $\mathrm{NS}_k(X)$ equals the order of the pole of the zeta function of X at $s = 1$ [see the author, loc. cit., pp. 108–109].

O. F. G. Schilling

From MathSciNet, June 2017

MR0207680 (34 #7495) 12.40; 10.65

Tate, J.

The cohomology groups of tori in finite Galois extensions of number fields.

Nagoya Math. J. **27** (1966), 709–719.

Let K/L be a galois extension of global fields, with group G. Let S be a (not necessarily finite) G-stable set of places of K containing all archimedean places, all ramified ones, and enough to generate the ideal class group of K. Then there is an exact sequence of G-modules (A): $0 \to E \to J \to C \to 0$, where E is the group of S-units of K, J is the group of S-idèles of K, and C is the idèle class group. Consider also the exact sequence (B): $0 \to X \to Y \xrightarrow{b} Z \to 0$, where Y is the permutation representation afforded by G acting on S, Z is the integers with trivial action, $b(\sum n_P P) = \sum n_P$, and $X = \ker(b)$. The first main result is an isomorphism of the associated long exact sequences of Tate cohomology groups, which takes the form

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & H^r(G,X) & \longrightarrow & H^r(G,Y) & \longrightarrow & H^r(G,Z) & \longrightarrow & H^r+1(G,X) & \longrightarrow \cdots \\
& \big\downarrow{\scriptstyle \alpha_3\cup} & & \big\downarrow{\scriptstyle \alpha_2\cup} & & \big\downarrow{\scriptstyle \alpha_1\cup} & & \big\downarrow{\scriptstyle \alpha_3\cup} & \\
\cdots \longrightarrow & H^{r+2}(G,E) & \longrightarrow & H^{r+2}(G,J) & \longrightarrow & H^{r+2}(G,C) & \longrightarrow & H^{r+3}(G,E) & \longrightarrow \cdots
\end{array}$$

Here $\alpha_3 \in H^2(G, \mathrm{Hom}(X,E))$, $\alpha_2 \in H^2(G, \mathrm{Hom}(Y,J))$, and $\alpha_1 \in H^2(G, \mathrm{Hom}(Z,C))$. The existence of α_1 and α_2, and the fact that $\alpha_1\cup$ and $\alpha_2\cup$ are isomorphisms, are deduced from global and local class field theory, respectively. By some carefully organized homological algebra, the existence and uniqueness of a compatible α_3 is then reduced to a compatibility condition between α_1 and α_2, which again follows from class field theory. Finally, the 5-lemma implies that $\alpha_3\cup$ is an isomorphism.

Abbreviate $\alpha = (\alpha_3, \alpha_2, \alpha_1) \in H^2(G, \mathrm{Hom}((\mathrm{B}), (\mathrm{A})))$. Then if M is any torsion-free G-module, the sequences $(\mathrm{A}) \otimes M$ and $(\mathrm{B}) \otimes M$ are still exact and there is a G-pairing $\mathrm{Hom}((\mathrm{B}), (\mathrm{A})) \times ((\mathrm{B}) \otimes M) \to ((\mathrm{A}) \otimes M)$. Hence $\alpha \cup$ also defines a homomorphism from $H^*(G, (\mathrm{B}) \otimes M) \to H^*(G, (\mathrm{A}) \otimes M)$ with a dimension shift of 2. According to a theorem of T. Nakayama [Ann. of Math. (2) **65** (1957), 255–267; MR0090620] this also is an isomorphism provided it is so, after restriction to all subgroups G' of G, in the special case $M = Z$. The latter is deduced from the fact that the fundamental classes in global (or local) class field theory are compatible under restriction.

Let R be the ring of elements of K which are integral outside S, and put $R_0 = R^G$. If N is a free $\mathbf{Z}$-module of finite rank on which G operates, then N can be viewed as the character module of an algebraic torus T over R_0 which is split by the étale extension R, and $T(R) = \mathrm{Hom}(N, E) = E \otimes M$, where $M = \mathrm{Hom}(N, Z)$. Thus $H^*(G, E \otimes M)$ is the galois cohomology of T for the extension R/R_0.

H. Bass

From MathSciNet, June 2017

MR0236190 (38 #4488) 14.51

Serre, Jean-Pierre; Tate, John

Good reduction of abelian varieties.

Ann. of Math. (2) **88** (1968), 492–517.

Let K be a field, v a discrete valuation of K, O_v the valuation ring of v, k its perfect residue field of characteristic p, K_s a separable closure of K, $\overline{v}$ an extension of v to K_s, and $I(\overline{v})$ the inertia group of $\overline{v}$. A set on which the Galois group $\mathrm{Gal}(K_s/K)$ operates is said to be unramified at v if $I(\overline{v})$ acts trivially on it. Let A be an abelian variety over K; A is said to have good reduction at v if A comes from an abelian scheme over $\mathrm{Spec}(O_v)$, and is said to have potential good reduction at v if A has good reduction at a prolongation of v to some finite extension of K.

The first fundamental theorem is the criterion of Néron-Ogg-Šafarevič for good reduction: Let A_m be the group of points of order dividing m in the group of K_s-points of A, and for a prime $l \neq p$, let $T_l(A)$ be the inverse limit of the groups A_{l^n} as $n \to \infty$. Then the following are equivalent: (a) A has good reduction at v. (b) A_m is unramified at v for all m prime to p. (c) $T_l(A)$ is unramified at v for some prime $l \neq p$. Some immediate corollaries of this criterion are: (1) Having good reduction is a property of the isogeny class of A. (2) Given an exact sequence $0 \to A' \to A \to A'' \to 0$ of abelian varieties over K, then A has good reduction if and only if both A' and A'' have. (3) If K' is a finite unramified extension of K, and A has good reduction over K', then A already has good reduction over K (same statement if K' is the completion of K at v).

The proof of this criterion is a beautiful application of Néron's theory of minimum models [A. Néron, Inst. Hautes Études Sci. Publ. Math., No. 21 (1964); MR0179172].

Another immediate consequence of the criterion is that if ρ_l is the l-adic representation of $\mathrm{Gal}(K_s/K)$ on $T_l(A)$, then A has potential good reduction at v if and only if the image of $I(\overline{v})$ under ρ_l is finite. If this is the case, then ρ_l has the same kernel in $I(\overline{v})$ for all $l \neq p$ and its character on $I(\overline{v})$ has integer values independent of l; if moreover k is finite with q elements, and σ is a lift to the decomposition group $D(\overline{v})$ of the Frobenius automorphism over k, then the characteristic polynomial of $\rho_l(\sigma)$ has integral coefficients independent of l, and the absolute values of its roots are equal to $q^{1/2}$.

Suppose O_v is Henselian with algebraically closed residue field. Then Ogg has defined a measure δ_l of wild ramification of A_l; in the case of elliptic curves, he has proved that δ_l is independent of l [A. P. Ogg, Amer. J. Math. **89** (1967), 1–21; MR0207694]. The authors generalize this result to higher dimensional abelian varieties under the assumption of potential good reduction (an assumption which Grothendieck has announced to be unnecessary).

In the rest of the paper, the authors give applications to abelian varieties with complex multiplication, defined over a global field. They first show that such a variety has potential good reduction everywhere (generalizing the fact that the j-invariant of an elliptic curve with complex multiplication is integral). They then show that for any finite set S of places which is "ordinary" in a technical sense, the variety can be twisted so as to have good reduction at S (a result due to Deuring in dimension one, except that he did not point out the necessity of excluding the special case). Finally, they show that over a number field, such a variety has good reduction outside the support of a corresponding Grössencharakter (a result also due to Deuring in the case of elliptic curves).

M. J. Greenberg
From MathSciNet, June 2017

MR0422212 (54 #10204) 12A60
Tate, John
Symbols in arithmetic.
Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 1, 201–211, *Gauthier-Villars, Paris*, 1971.

Let F be a field, G an abelian group. A symbol on F with values in G is defined to be a bimultiplicative map $\sigma\colon F' \times F' \to G$ with the property that $\sigma(a, 1-a) = 0$. For global fields F, the norm residue symbol with values in μ_v, the group of roots of unity in the completion F_v of F at a non-complex prime v, is an important example of a symbol that has been studied in classical number theory. The author notes that for all fields the group K_2F has been shown to be the target group for a universal symbol, so that $\mathrm{Symb}(F,G) \simeq \mathrm{Hom}(K_2F, G)$. In the late 1960s, K_2F came under study, particularly with respect to how much of it could be accounted for by norm residue symbols.

If $\lambda_v \colon K_2F \to \mu_v$ is the homomorphism corresponding to the norm residue symbol at v, one can obtain a map $\lambda\colon K_2F \to \oplus\mu_v$, the sum taken over all non-complex primes v of F. The author summarizes recent study of λ, giving the result of C. C. Moore that Coker$\,(\lambda)$ is isomorphic to μ_F, induced by reciprocity, and mentioning that Ker(λ) was first shown by Bass and the author to be finitely generated, and then by Garland to be finite in the case of number fields.

The remainder of the paper gives a preview of much of the work in this area in the succeeding few years, relating K_2 of global fields to Galois cohomology. In particular, the author describes a symbol, and hence a homomorphism, $h\colon K_2F \to H^2(F, T^{(2)})$. By studying the rank of the Z_l-module $H^1(F, T^{(2)})$, in connection with h, one obtains enormous information about K_2F and $\operatorname{Ker}\lambda$. The author enunciates his "main conjecture" that this rank is $r_2(F)$ and derives the following consequences (among others) when $\mu_l \subset F$ for a prime l: (1) $K_2F(l)$ is isomorphic to the torsion of $H^2(F, T^{(2)})$. (2) The map $\alpha\colon \mu_l \otimes F \to (K_2F)_1$ is surjective and $|\operatorname{Ker}\alpha| = l^{1+r_2}$.

The main conjecture was known for function fields at the time the paper was written and was proved shortly thereafter for number fields by the author [Invent. Math. **36** (1976), 257–274]. Other questions raised in this survey paper have been fully or partially settled. The question whether Ker(λ_v) is a uniquely divisible subgroup of K_2F_v has been settled affirmatively for primes different from the residue characteristic of F_v [J. Carroll, *Algebraic K-theory, II: "Classical" algebraic K-theory and connections with arithmetic* (Proc. Conf., Seattle Res. Center, Battelle Memorial Inst., 1972), pp. 464–473, Lecture Notes in Math., Vol. 342, Springer, Berlin, 1973; MR0399052] and negatively for the residue characteristic by many authors. The possible relation given between $|\operatorname{Ker}\lambda|$ and the values of $\zeta_F(-1)$ has been confirmed in some cases by J. Coates and S. Lichtenbaum [Ann. of Math. (2) **98** (1973), 498–550; MR0330107]. The author's 1976 paper [op. cit.] provides a good list of references for work done in this area in the early 1970s.

Alan Candiotti

From MathSciNet, June 2017

MR0442061 (56 #449) 18F25; 12F05, 12A65

Bass, H.; Tate, John

The Milnor ring of a global field.

Algebraic K-theory, II: "Classical" algebraic K-theory and connections with arithmetic (Proc. Conf., Seattle, Wash., Battelle Memorial Inst., 1972), 349–446, *Lecture Notes in Math.*, 342, *Springer, Berlin*, 1973.

For an arbitrary field F, J. W. Milnor defined a graded ring (now called the Milnor ring) $K_*F = \coprod_{n\geq 0} K_nF$, generated by $l(a)\,(a \in \dot{F})$ with relations $l(ab) = l(a)+l(b)$, $l(a)l(1-a) = 0$. For $i \leq 2$, K_iF agrees with the corresponding K-groups in algebraic K-theory, namely, $K_0F \cong \mathbf{Z}$, $K_1F = \{l(a)\colon a \in \dot{F}\} \cong \dot{F}$, and K_2F is the recipient group of a universal Steinberg symbol. In the paper under review, the Milnor ring K_*F is investigated in great detail, and applications are made to the computation of K_*F for global fields. For these fields, the authors have succeeded in determining K_iF completely, except for $i = 2$.

The paper is divided into two long chapters, the first of which, entitled "Some general remarks on the Milnor ring", is addressed to completely arbitrary fields. Part of the chapter is a review and retreatment of the Milnor theory; for instance, the nil radical of K_*F is determined, and the orderings of the field F are related to the ring structure of K_*F. If F is equipped with a (rank 1) discrete valuation v with residue class field $k(v)$, one gets two maps $\partial_\pi{}^0$, $\partial_v \colon K_*F \to K_*k(v)$, of degrees 0 and -1, respectively. The second depends only on v, but the first depends on the choice of a uniformizer π. In case F is a rational function field $k(x)$, the residue maps $\partial = (\partial_v)$ lead to the Milnor exact sequence $0 \to K_*k \to K_*F \xrightarrow{\partial} \coprod K_*k(v) \to 0$, where v ranges over the (discrete) k-valuations on F, except the "infinity" ($(1/x)$-adic) valuation. To take this valuation into account as well, one is led to an exact sequence with one more term: $0 \to K_*k \to K_*F \to \coprod_{\text{all } v} K_*k(v) \xrightarrow{(N_v)} K_*k \to 0$, which essentially "defines" the transfer maps N_v (with $N_\infty = \text{Id}$). The authors give an inductive formula for N_v, and explain the behavior of N_v under a change of the constant field. However, it seems to be unknown whether the transfer map of K-groups under a simple extension is independent of the choice of a primitive element. Consequently, a transitivity formula for the transfer map remains lacking. The only known case is in dimension 1, where $K_1k(v) \to K_1k$ is shown to be identical with the field norm, so everything is well behaved. For general dimensions, it is only known that, if α and β generate the same (algebraic) extension, then $N_{\alpha/k}$ and $N_{\beta/k}$ agree modulo torsion in K_ik. As an application of the transfer maps, various divisibility properties of the K-groups are derived. It is also shown that, if $1 \le n \le \delta(F)$, then $\operatorname{rank} K_nF = \operatorname{Card} F$. Here, $\delta(F)$ denotes the transcendence degree of F over its prime field if $\operatorname{char} F > 0$, and denotes $1 + \operatorname{tr}\deg F/\mathbf{Q}$ otherwise.

In the second chapter, entitled "The Milnor ring of a global field", the authors consider the map $K_*F \xrightarrow{(\partial_v)} \coprod_{v \notin S} K_*k(v)$, where F is a global field and S is a finite set of places on F containing all the Archimedean places. This map vanishes on $K_*{}^SF$, the subring of K_*F generated by $l(a)$, where a ranges over the units in the ring of S-integers. If one lists the finite places $v_1, v_2, \cdots$ so that Card $k(v_i)$ are non-decreasing, the following result is obtained. Finiteness theorem: Let $S_m = \{\text{arch. places}\} \cup \{v_1, \cdots, v_m\}$; then, for all sufficiently large m, $K_*F/K_*{}^{S_m}F \to \coprod_{v \notin S_m} K_*k(v)$ is an isomorphism. It follows from this and the Dirichlet unit theorem that, for $n > 0$, the kernel H_n of $K_nF \to \coprod_i K_{n-1}k(v_i)$ is a finitely generated abelian group. This leads to a complete determination of K_nF for $n \ge 3$, viz., $K_nF \cong (\mathbf{Z}/2\mathbf{Z})^r$, where r is the number of real places of F (if any). This determination is achieved via the consideration of the quotients K_nF/pK_nF (p = prime), and via the use of the transfer homomorphisms. For $n = 2$, the results on $H_2 = \ker(K_2F \to \coprod_i K_1k(v_i))$ have been announced on numerous occasions by the authors under the title of "K_2 of global fields" [cf. the first author, Seminar on Modern Methods in Number Theory (Inst. Statist. Math., Tokyo, 1971), Paper No. 1, Inst. Statist. Math., Tokyo, 1971; MR0429838]. If $\operatorname{char} F = p > 0$, H_2 is a finite group of order prime to p, and was completely determined by Tate. If, on the other hand, $\operatorname{char} F = 0$ (i.e., F is a number field), it is also known that H_2 is finite, by results of Dennis and Garland. The order and the exact structure of H_2 are, however, unknown except in special cases. Some conjectures in this direction have been formulated by Birch and Tate; further conjectures on the arithmetic of the higher Quillen K-groups for global fields have been formulated by Lichtenbaum.

In an appendix to the paper, Tate computes the group H_2 for the first six imaginary quadratic fields F, i.e., those with discriminants $d = -3, -4, -7, -8, -11$, and

-15. For these d's, the result of Tate's computation is that $H_2 = 0$ for $d \not\equiv 1 \pmod 8$, and that $H_2 \cong \mathbf{Z}/2\mathbf{Z}$, generated by $l(-1)^2$, for $d \equiv 1 \pmod 8$.

T. Y. Lam

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MR0899413 (88k:11039) 11G40; 11G05, 14G25, 14K15

Mazur, B; Tate, J.

Refined conjectures of the "Birch and Swinnerton-Dyer type".

Duke Math. J. **54** (1987), *no.* 2, 711–750.

From the introduction: "The idea behind the present article is that, in certain instances, arithmetic conjectures concerning the special values of derivatives of p-adic L-functions can be 'refined' to obtain formulations of stronger conjectures. These stronger conjectures avoid any mention of p-adic L-functions and therefore obviate the necessity of constructing the p-adic L-functions for the statement of the conjectures. Moreover, they avoid any reference to a prime number p, and require no p-adic limiting process; they should ultimately be phrased, perhaps, in adelic language. In this paper, however, we state our conjectures 'at a finite layer M', where M is a possibly composite number (somewhat restricted). Even when $M = p$, however, our conjecture 'at layer p' is not implied by the analogous conjecture for the p-adic L-function. Indeed, our conjecture predicts congruence formulas modulo divisors of $p - 1$, in this case. When M is a product of distinct primes, our conjectured congruence formulas involve what seems to us to be a thoroughgoing mixture of phenomena related to those prime divisors.

"Let $A_{/\mathbf{Q}}$ be an elliptic curve admitting a modular parametrization. We define, for any integer $M \geq 1$, the modular element $\theta_{A,M} \in \mathbf{Q}[(\mathbf{Z}/M\mathbf{Z})^*/(\pm 1)]$ {whose coefficients are given by modular symbols}. We view $\theta_{A,M}$ as our analogue 'at layer M' of the L-function of A. Let R be a subring of $\mathbf{Q}$ containing the coefficients of $\theta_{A,M}$. Let $I \subset R[(\mathbf{Z}/M\mathbf{Z})^*/(\pm 1)]$ be the augmentation ideal. The analogue of saying that the 'L-function vanishes to order $\geq r$ at $s = 1$' is simply to say that $\theta_{A,M}$ is contained in the rth power of the augmentation ideal I.

"If $\theta_{A,M}$ lies in I^r (i.e., 'vanishes to order $\geq r$ at $s = 1$'), the analogue of the 'rth coefficient of the Taylor expansion of the L-function at $s = 1$' is simply the image $\tilde{\theta}_{A,M} \in I^r/I^{r+1}$ of $\theta_{A,M}$.

"Our 'refined Birch and Swinnerton-Dyer conjectures' at layer M will then be statements about (a) the 'order of vanishing' of the element $\theta_{A,M}$ and (b) the image of its 'leading coefficient' in I^r/I^{r+1}, if the order of the torsion subgroup of $A(\mathbf{Q})$ is invertible in R.

"As for 'order of vanishing', let s denote the number of primes of split multiplicative reduction for A which divide M. Let $r = \operatorname{rank} A(\mathbf{Q}) + s$. We conjecture that $\theta_{A,M}$ lies in I^r.

"To produce a conjectural formula for the 'rth Taylor coefficient' of $\theta_{A,M}$ is significantly more difficult. In this paper we do this only under the hypothesis that if p is not split multiplicative for A then p^2 does not divide M.

"We define a 'regulator term' by a circuitous process using 'tame height pairings'. The conjectural formula is obtained by multiplying this by a scalar term (which involves, among other things, the order of the Shafarevich group Ш).

"At the end of Chapter 3 we provide a certain amount of numerical evidence in support of these conjectures."

Karl Rubin

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Artin, M.; Tate, J.; Van den Bergh, M.

Some algebras associated to automorphisms of elliptic curves.

The Grothendieck Festschrift, Vol. 1, 33–85, *Progr. Math.*, 86, *Birkhäuser Boston, Boston, MA*, 1990.

The present paper follows a paper by Artin and W. F. Schelter [Adv. Math. **66** (1987), no. 2, 171–216; MR0917738] which attempts to classify the "3-dimensional regular algebras". They showed that a 3-dimensional regular algebra must be defined by generators and relations of a very special form. Although they showed that a generic algebra with relations of the prescribed form was regular, they were unable to show that particular algebras were regular. The present paper overcomes this problem, thus giving a complete classification of the 3-dimensional regular algebras. A 3-dimensional regular algebra is, by definition, a connected $\mathbf{N}$-graded k-algebra A, which is generated in degree 1, has polynomial growth, is Gorenstein, and has global dimension 3. By Artin and Schelter such an algebra is defined either by two generators and two cubic relations, or three generators and three quadratic relations. We discuss the latter case, although analogous statements apply to the former case.

This paper shows that such algebras determine and are determined by a cubic divisor E in $\mathbf{P}^2$, an invertible sheaf $\mathcal{L}$ giving the embedding in $\mathbf{P}^2$, and an automorphism σ of E. The divisor E arises as the parameter space of the point modules for A; a graded A-module M is a point module if it is cyclic, generated in degree 0, and $\dim(M_n) = 1$ for all $n \geq 0$. The relations of A are given by a 3-dimensional subspace of $A_1 \otimes A_1$ and the zero locus in $\mathbf{P}^2 \times \mathbf{P}^2$ of this subspace is the graph of σ. It does not seem possible to directly analyse the algebra A. However, it is shown that A has a quotient algebra B (by an element g of degree 3, which is often central, and always normal), which is more amenable. The algebra B can be explicitly described as a "twisted" homogeneous coordinate ring of the divisor E [Artin and Van den Bergh, J. Algebra **133** (1990), no. 2, 249–271; MR1067406]. All the point modules for A are actually supported by B, and B is the largest such quotient of A. One of the key steps in showing that A has good homological properties and has the same Hilbert series as the polynomial ring in 3 indeterminates is to show that the Koszul complex for A is acyclic. An ingenious argument involving B shows this and simultaneously proves that g is regular. It is also proved that A is a Noetherian domain. The most interesting of the 3-dimensional regular algebras are those where E is an elliptic curve. These belong to a larger class of algebras defined by A. V. Odesskiĭ and B. L. Feĭgin [Funktsional. Anal. i Prilozhen. **23** (1989), no. 3, 45–54; MR1026987], which also includes some 4-dimensional algebras which had previously been defined by E. K. Sklyanin [Functional Anal. Appl. **16** (1982), no. 4, 263–270; MR0684124] in connection with the Yang-Baxter equation. This paper bubbles over with new methods and ideas which will no doubt be very influential.

S. Paul Smith

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Tate, John

Conjectures on algebraic cycles in l-adic cohomology.

Motives (Seattle, WA, 1991), 71–83, *Proc. Sympos. Pure Math.*, 55, Part 1, *Amer. Math. Soc., Providence, RI*, 1994.

The author discusses his famous conjecture that, for a smooth projective variety over a finitely generated field k, the image of the algebraic cycles in l-adic cohomology spans the $\mathbf{Q}_l$-subspace of $H^{2j}(X, \mathbf{Q}_l)(j)$ fixed by the Galois group $\mathrm{Gal}(\bar{k}/k)$ [in *Arithmetical algebraic geometry (West Lafayette, IN, 1963)*, 93–110, Harper & Row, New York, 1965; MR0225778]. In Sections 1 and 4, he describes the intriguing arithmetic route by which he was led to the conjecture: It is a geometric analogue of the conjectured finiteness of the Tate-Shafarevich group of an abelian variety, which is a necessity if we want the group of rational points to be easily computable. In Sections 2 and 3 he explains the implications his conjecture would have for Grothendieck's standard conjectures: The Tate conjecture implies that the Künneth components of the diagonal are algebraic, and the Tate conjecture together with semisimplicity of the Galois action on l-adic cohomology would imply that numerical and homological equivalence for algebraic cycles are the same. The last section summarizes the special cases of the conjecture which have been proved: divisors on abelian varieties (Tate, Zarkhin in characteristic p; Faltings in characteristic 0, as part of his proof of the Mordell conjecture), divisors on $K3$ surfaces (with some restrictions in characteristic p), and divisors on various modular surfaces and threefolds. In higher codimension, the conjecture has been proved for many Fermat hypersurfaces (Tate, Shioda) and many classes of abelian varieties (Tate, Tankeev, Murty, Shioda).

Burt Totaro

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