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Semi-bounded differential operators, contractive semigroups, and beyond, by Alberto Cialdea and Vladimir Maz'ya, Operator Theory: Advances and Applications, Vol. 243, Birkhäuser/Springer, Cham, Switzerland, 2014, xiv+252 pp., ISBN 978-3-319-04557-3

A (C_0) *semigroup* is a one-parameter strongly continuous semigroup of bounded linear operators on a Banach space X. More precisely,

$$T = \{T(t) : t \in \mathbb{R}^+ = [0, \infty)\} \subset L(X),$$

$$T(t+s) = T(t)T(s),\ T(0) = 1,\ T(\cdot)f \in C(\mathbb{R}^+, X) \text{ for all } t, s \in \mathbb{R}^+,\ f \in X.$$

To an algebraic or topological "semigroupie", T is merely a special kind of representation of the half-line $(\mathbb{R}^+, +)$. Thus, in a sense, T is trivial. But (C_0) semigroups have many subtle properties that make them ubiquitous and surprisingly useful in many areas of analysis and applied mathematics.

The (infinitesimal) *generator* A of T is

$$Af = \lim_{h\to 0} \frac{T(h)f - f}{h},$$

and the domain $D(A)$ consists of all f for which this limit exists in the norm topology of X. Formally, $A = T'(0)$ and $T(t) = e^{tA}$, but unless T is continuous in uniform operator topology, A is unbounded, so one should be a little careful in using the notation e^{tA}.

The associated initial value problem is

$$\frac{du}{dt} = Au,\ u(0) = f, \tag{1}$$

for a function $u : \mathbb{R}^+ \to X$. This problem is *well posed* if $D(A)$ is dense in X and for each $f \in D(A)$ there exists a unique solution u which depends continuously on f. The minimal basic theory of (C_0) semigroup theory consists of two results.

Theorem 1 (Well-posedness)**.** *Well-posedness holds for* (1) *in X iff A is the generator of a (C_0) semigroup T in $L(X)$. Moreover, A is the generator of a (C_0) semigroup T in $L(X)$ iff* (1) *has a unique continuously differentiable solution given by $u(t) = T(t)f$. In this case, T "governs"* (1).

Theorem 2 gives a necessary and sufficient condition for A to generate a (C_0) semigroup T. An easy argument using the uniform boundedness principle shows that

$$M = \sup_{0\le t\le 1} \|T(t)\| < \infty$$

and then

$$\|T(t)\| \le Me^{\omega t} \tag{2}$$

holds for some $M \ge 1$ and some real ω, for instance, $\omega = \log(M)$ above. Then

$$|f| = \sup_{t\ge 0}\ e^{-\omega t}\,\|T(t)f\|$$

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defines a norm equivalent to $\|\cdot\|$, and on $(X, |\cdot|)$, $S(t) = e^{-\omega t}T(t)$ defines a (C_0) contraction semigroup with generator $A - \omega I$. The Laplace transform

$$R(\lambda)f = \int_0^\infty e^{-\lambda t} T(t) f dt$$

for $\lambda > 0$, $f \in X$ is the resolvent of A, $R(\lambda) = (\lambda I - A)^{-1}$. This can be guessed by writing $T(t) = e^{tA}$, regarding A as a number, and evaluating the integral explicitly.

Theorem 2 (Hille–Yosida generation theorem)**.** *An operator A is the generator of a (C_0) contraction semigroup T on X iff A is closed, $D(A)$ is dense in X, and A is m-dissipative, that is,*

$$\left\|(\lambda I - A)^{-1}\right\| \leq \frac{1}{\lambda} \textit{ for all } \lambda > 0 \text{ (dissipative)},$$

$$\mathrm{Range}(\lambda I - A)) = X \textit{ for all } \lambda > 0 \text{ (hypermaximal, or “m”)}.$$

Moreover,

$$T(t)f = \lim_{n\to\infty} \left(I - \frac{t}{n}A\right)^{-n} f = \lim_{n\to\infty} \left[\frac{t}{n}\left(\frac{n}{t}I - A\right)\right]^{-n} f.$$

So T is recovered from A by inverting the Laplace transform of A. Some historical and other comments are in order. (C_0) semigroups became part of mainstream mathematics with the appearance of (Carl) Einar Hille’s 1948 book [10] (an American Mathematical Society Colloquium Publication). When this book was accepted for publication, neither Theorem 1 nor Theorem 2 was known. Theorem 2 was proved simultaneously and independently in 1948 (with different proofs) by Hille at Yale (while correcting galley proofs of his book) and by Kosaku Yosida at Tokyo. Theorem 1 was published in the early 1950s by Ralph Phillips and by Hille (independently, using different but equivalent definitions of well-posedness). Theorem 2 was extended, independently, to the case of general (C_0) semigroups by Willy Feller, Phillips and Isao Miyadera in 1951–53. Feller proved it as a consequence of the Hille–Yosida theorem using an equivalent norm trick.

Hille invited Phillips to add his research to Hille’s book. Phillips did, resulting in the 1957 Hille–Phillips book [11], which contained subtractions as well as additions. (Interestingly, Hille and Phillips never coauthored a paper, and Phillips privately admitted he did not read all of Hille’s book.) The Hille book and then the Hille–Phillips book became the main textbook for graduate-level modern analysis, developing measure and integration theory for Banach space valued functions, deeper properties and applications of Laplace transforms, and developing tools of functional analysis for applications involving harmonic analysis, probability theory, differential equations, and other fields. So the Hille and then Hille–Phillips book became a sort of “bible”. Its successor was the book by Nelson Dunford and Jack Schwartz, Part 1 [3].

The notations (C_j) semigroups and (A_j) semigroups were part of a 1948 Hille classification concerning the convergence properties of $T(t)$ as $t \to 0$ and $\lambda(\lambda I - A)^{-1}$ as $\lambda \to \infty$; the “C” (resp. “A”) refers to Cesaro (resp., Abel). Hille’s structural classification stopped being used; its only living remnant is the term (C_0) semigroup.

Well-posedness depends on the norm. For instance, consider the wave equation in $\mathbb{R}^n$, $n \geq 2$,

$$\frac{\partial^2 v}{\partial t^2} = \Delta v,\ v(\cdot,0) = f,\ v_t(\cdot,0) = g \text{ for } x \in \mathbb{R}^n,\ t \in \mathbb{R}.$$

Rewrite this as a system

$$u = \begin{pmatrix} v \\ v_t \end{pmatrix},\ \frac{\partial u}{\partial t} = \begin{pmatrix} 0 & \Delta \\ I & 0 \end{pmatrix} u = Au,\ u(0) = F = \begin{pmatrix} f \\ g \end{pmatrix}.$$

Then A generates a (C_0) semigroup (and even a (C_0) group) T on $X = W^{1,p}(\mathbb{R}^n) \times L^p(\mathbb{R}^n)$ iff $p = 2$. In fact, for $p \neq 2$ and $t \neq 0$, $T(t)$ is an unbounded operator on X (which can be shown using Fourier transforms).

Suppose you are given a linear initial-boundary value problem for an autonomous partial differential equation. If you can, write the problem in the form of (1). Choose an appropriate Banach space X, and define the domain of A in a way that incorporates the boundary condition. Show that A is densely defined and quasi-dissipative in X. That is,

$$\left\| (\lambda I - (A - \omega I))^{-1} \right\| \leq \frac{1}{\lambda}$$

holds for all $\lambda > 0$ and some $\omega \in \mathbb{R}$. Showing this depends on the choice of the norm. Show also that the range of $\lambda I - (A - \omega I)$ is dense in X for some $\lambda > 0$. Then (this is a lemma) this density result holds for all $\lambda > 0$ and $\overline{A}$, the closure of A, generates a quasi-contractive (C_0) semigroup T (satisfying $\|T(t)\| \leq e^{\omega t}$ for $t \geq 0$) on X, which governs (1), and T is given by the exponential formula

$$T(t)f = \lim_{n \to \infty} \left(I - \frac{t}{n} A \right)^{-n} f.$$

If A is not quasi-dissipative, one must check the more complicated conditions

$$\left\| (\lambda I - (A - \omega I))^{-k} \right\| \leq \frac{M}{\lambda^k}$$

for all $\lambda > 0$, $k \in \mathbb{N}$. In practice, this is often too difficult to verify directly. If A is an elliptic operator or a matrix involving an elliptic operator, then showing the density of the range is sometimes independent of the space X; the choice of X is important mainly for accomplishing the quasi-dissipativity calculation.

For the heat equation, the physically "natural norms" are the L^∞ norm (corresponding to maximum temperature for positive solutions) and the L^1 norm (corresponding to total heat content). But, depending on the boundary conditions, the calculations typically also work in L^p, $1 \leq p \leq \infty$, with $p = \infty$ corresponding to some space of continuous functions, not to L^∞ itself.

To indicate applications to nonlinear problems, consider the Navier–Stokes system of fluid dynamics,

$$\begin{aligned} u_t &= \mu \Delta u + u \cdot \nabla u + \nabla p + f_0 \text{ in } \Omega \times \mathbb{R}^+, \\ \operatorname{div}(u) &= 0 \text{ in } \Omega \times \mathbb{R}^+, \\ u(x,0) &= f(x) \text{ in } \Omega, \\ u(x,t) &= 0 \text{ on } \partial\Omega \times \mathbb{R}^+, \end{aligned} \tag{3}$$

where Ω is a bounded domain in $\mathbb{R}^3$, the fluid velocity is $u = u(x,t) : \Omega \times \mathbb{R}^+ \to \mathbb{R}^3$, $\mu > 0$ is the kinematic viscosity, and the pressure $p : \Omega \times \mathbb{R}^+ \to \mathbb{R}$. (One could also consider $n \geq 2$ and/or Ω being unbounded.) The Helmholtz projection P is the orthogonal projection of $H = [L^2(\Omega)]^3$ onto the solenoidal vectors H_σ, given by

$$H = H_\sigma \oplus H_\nabla,$$

where[1]

$$H_\sigma = \mathrm{cl}\{u \in C^1 \left(\bar{\Omega}\right)^3 : u = 0 \text{ on } \partial\Omega,\ \nabla u = 0 \text{ in } \Omega\}$$

and

$$H_\nabla = \mathrm{cl}\{\nabla\varphi : \varphi \in C^1 \left(\bar{\Omega}\right),\ \varphi = 0 \text{ on } \partial\Omega\}.$$

Let $v = Pu$ ($= u$, viewed in H_σ when u satisfies (3)). Then $v(t) \in H_\sigma$ and

$$\frac{dv}{dt} = \mu A v + N(v) + P_0 f,\ v(0) = Pf,\ v = 0 \text{ on } \partial\Omega, \tag{4}$$

where $A = P\Delta$ is the Stokes operator on H_σ and $N(v) = P(v \cdot \nabla v)$ is the nonlinear term. One solves (4) for v and then replaces u by v in (3). This gives ∇p, which in turn gives p up to an additive constant. So the problem is to solve (4). The program for (4) was laid out by Jean Leray in 1934. He recognized that (4) was well posed locally in time (i.e., for $t \in [0, \tau]$ for small $\tau = \tau(\Omega, f, f_0, \mu) > 0$) and well posed globally (i.e., for all $t \geq 0$) for f "small enough" in some sense. The problem of global well-posedness for large (or rather general) initial data remains open and is one of the million dollar Clay prizes.

Here are Leray's ideas for local well-posedness, made precise in the 1960s by H. Fujita and T. Kato. Absorb μ into the Stokes operator A, which satisfies $A = A^* \leq -\varepsilon I$ on H_σ for some $\varepsilon > 0$. Also A generates a (C_0) contraction semigroup T which extends to $T = \{e^{tA} : \mathrm{Re}(t) \geq 0\}$, which in turn is analytic in the open right half-plane. By writing (4) as

$$\frac{dv}{dt} = Av + M(v),\ v(0) = g, \tag{5}$$

where $M(v) = N(v) + Pf_0$ and $g = Pf$, we can use successive approximations which lead to the iteration scheme

$$\frac{dv_n}{dt} = Av_n + M(v_{n-1}),\ v_n(0) = g, \tag{6}$$

$$v_n(t) = e^{tA} g_n + \int_0^\infty e^{(t-s)A} M(v_{n-1}(s))ds,\ n \in \mathbb{N},$$

in which we may take $v_0(t) = g$. Then (6) is the standard variation of parameters formula, and a local (mild) solution of (5) is a fixed point of Q_0, where

$$Q_0 w(t) = e^{tA} g + \int_0^\infty e^{(t-s)A} M(w(s))ds \tag{7}$$

on a suitable closed subset in some space $C([0, \tau], Y)$. Since the nonlinear differential operator M is usually not locally Lipschitzian on any of the usual Sobolev or Hölder spaces, we need a factorization to get a Lipschitz function as a composition of Lipschitz functions.

Define the abstract Sobolev space $H_\alpha = D((-A)^\alpha)$ of order 2α with norm $\|f\|_\alpha = \|(-A)^\alpha f\|,\ 0 \leq \alpha < 1$. The spectral theorem of John von Neumann and Marshall Stone says that any self-adjoint operator S on a complex Hilbert space K is unitarily equivalent to a multiplication operator by a real measurable function on

[1]We use the notation "cl" for closure.

some concrete L^2 space, $L^2(\Lambda, \Sigma, \lambda)$. Thus there is a unitary $Q : K \to L^2(\Lambda, \Sigma, \lambda)$ such that $S = Q^{-1} M_m Q$, $M_m h = mh$ for some $m{:}\Lambda \to \mathbb{R}$, and $g \in D(M_m)$ iff $g, mg \in L^2(\Lambda, \Sigma, \lambda)$. The essential range of m is the spectrum of A, and for any Borel function b from $\sigma(S)$ to $\mathbb{C}$, $b(S) = Q^{-1} M_{b(m)} Q$ is a normal operator, self-adjoint if b is real valued, and the mapping $b \to b(S)$ is an algebra homomorphism. This *functional calculus* enables us to treat many infinite-dimensional problems as if they were one dimensional. Semigroup theory on Banach spaces has generators which are much more general than normal operators N on Hilbert space whose spectrum is bounded above: $\sup\{\mathrm{Re}(a) : a \in \sigma(N)\} < \infty\}$. It has a similar functional calculus, but one that is based on just the exponential functions, not on general Borel functions. Still, that is enough for many surprising applications. The Navier–Stokes system is just one example.

Returning to (6), the estimate $\left\|(-A)^{\alpha} e^{sA}\right\| \leq t^{-\alpha}$ holds by the reasoning in the above paragraph. The function M in (7) satisfies $\|M(u) - M(v)\| \leq C_B \|u - v\|_{\alpha}$ for u, v in bounded subsets B of H_{α} for $\frac{3}{4} < \alpha < 1$ because we are in three space dimensions; this estimate depends on Sobolev inequalities and other things. With these tools in hand and additional tools, one can reformulate (7) in a different way so that the Banach fixed point theorem (strict contraction mapping principle) can be applied.

In some ways, this application is typical. Semigroup theory does not by itself solve hard problems in nonlinear partial differential equations, but it provides a key tool in many cases. Another instance of this involves the principle of linearized stability. This is a maddening result, because in many cases the conclusion of the theorem holds but the hypotheses do not.

Consider a nonlinear partial differential operator N and the associated differential equation $\frac{du}{dt} = N(u)$. We would like to solve it in a Banach space, but looking at N may not suggest a canonical space for this problem. Let h be a fixed point for N, $N(h) = h$. Suppose N is differentiable at h in some sense, so that

$$\lim_{s \to 0} \frac{N(h + sk) - N(h)}{s} = Lk$$

holds for all k in a suitable dense set and L is a linear operator, in fact, a semigroup generator. If the corresponding semigroup satisfies $\|T(t)\| \leq M e^{-\varepsilon t}$ for some $\varepsilon > 0$ and all $t \geq 0$, then there is hope that the conclusion of the principle of linearized stability holds: if v is small enough in some sense, then the solution of

$$\frac{du}{dt} = N(u), \ u(0) = h + v,$$

exists globally and $\lim_{t \to \infty} u(t) = h$. This is an easy result if N is locally Lipschitzian on some Banach space to itself, but this is not normally the case.

For instance, consider

$$u_t = \Delta u + u^p, \tag{8}$$

$t \in \mathbb{R}^+$, $x \in \mathbb{R}^N$, $p > 1$. For each $N \geq 3$, for each $p > \frac{N}{N-2}$, there exists a unique positive radial power equilibrium solution of (8) of the form $u(x, t) = h(r) = Cr^{-a}$, where $a > 0$, $C > 0$, $r = |x|$. The linearization L of the nonlinear operator $N(u) = \Delta u + u^p$ about the fixed point h is

$$L = \Delta + \frac{c}{|x|^2} = \Delta + \frac{c}{r^2}$$

for some constant c. There is no "natural space" for studying the operator N, but L can be viewed as a self-adjoint operator on $L^2(\mathbb{R}^N)$, and using Hardy's inequality one can show that $\sigma(L) = (-\infty, 0]$ or $\mathbb{R}$, according to whether $c \leq C^*(N) = \left(\frac{N-2}{2}\right)^2$ or $c > C^*(N)$. In the latter case, e^{tL} is an unbounded operator on $L^2(\mathbb{R}^N)$ for all $t \neq 0$, but in the former case the semigroup $\{e^{tL} : t \geq 0\}$ is a family of norm 1 operators satisfying $e^{tL} f \to 0$ as $t \to \infty$ for all $f \in L^2(\mathbb{R}^N)$. We do not have exponential linearized stability, but one might wonder whether there exists a Banach space X on which h is asymptotically stable in the nonlinear sense. The surprising positive answer to this question was obtained by C. Gui, W.-M. Ni, and X. Wang [8, 9]. The answer is yes, h is asymptotically stable for the nonlinear equation, provided $N \geq 11$ and

$$\frac{N}{N-2} < p < \frac{(N-2)^2 - 4\left(N - 2\sqrt{(N-1)}\right)}{(N-2)^2 - 8N + 16}$$

in a certain weighted supremum norm space $X \subset C(\mathbb{R}^N)$; and for $f \in X$, $\|f\| = \sup_{x \in \mathbb{R}^N} |f(x) w(x)| < \infty$ and the weight function w is given explicitly. This is a deep and lovely result.

A related problem is connected with Cedric Villani's Fields Medal work [16]. Convergence to equilibrium for solutions of the Boltzmann equation can be thought of as a case of a (yet to be formulated) principle of linearized stability. The equilibria include functions which are Maxwellian in the velocity variables, density functions of the normal distribution with appropriate parameters. The rate or speed of convergence is a problem of enormous technical difficulty. The problem involves suitable normalizations and finding the right norm or norms. The results of Villani and his collaborators involve highly nontrivial calculations with (C_0) semigroups. Is the convergence to equilibrium exponentially fast? Sometimes it is, but sometimes one must replace e^{-ta} for $a > 0$ by $t^{-1/\epsilon}$ for arbitrary $\epsilon > 0$. And Landau damping is involved—the rapid decay of an electric field in a plasma without collisions of particles. Boltzmann's theory was that time irreversibility was caused by collisions. The work of Villani and his collaborators gave us a new and better understanding of entropy, thus satisfactorily explaining a nineteenth century mystery. See [16] and the references therein for a nice introduction to these ideas. Questions of instabilities are also involved in Villani's calculations.

Finally we come to the book being reviewed. The book is about (C_0) semigroups, but it is also about second-order linear elliptic operators on a bounded domain Ω in $\mathbb{R}^N$. The corresponding semigroup governs a parabolic probem, and it is typically a semigroup of positivity-preserving operators, which is tied to the maximum principle. For systems and for higher-order operators, positivity is not preserved in general. The subject matter has an enormous literature, but the book contains a lot of new and deep results, some of which were developed by the authors in recent journal literature, and some of which are presented in the book for the first time. The authors work at a high level of generality, and the book is very technical and not so easy to read. But the results are interesting, and the necessary effort put into studying them is worth it.

The book consists of seven chapters.

Chapter 1. Preliminary facts on semi-boundedness of forms and operators

Chapter 2. L^p-dissipativity of scalar second order operators with complex coefficients

Of concern are semiboundedness of sesquilinear forms and (quasi-)dissipativity of the corresponding partial differential operator on L^p for $p \neq 2$. Typically, the L^2 theory is relatively easy, but the L^p theory is hard. Instead of being real functions, the coefficients are complex functions and even complex measures. This leads to significant technical complications.

Chapter 1 contains many standard results in semigroup theory, but it is presented from the authors' perspective. We point out one such result now. The duality set of $f \in X$, X being a complex Banach space, is

$$i(f) = \{\varphi \in X^* : \langle f, \varphi \rangle = \|f\|^2 \|\varphi\|^2\}.$$

The Lumer–Phillips theorem says that the linear operator $A : D(A) \subset X \to X$ is dissipative (meaning $\left\|(\lambda I - A)^{-1}\right\| \leq \frac{1}{\lambda}$ for all $\lambda > 0$) iff for all $f \in D(A)$ there is a $\varphi \in i(f)$ such that $\operatorname{Re}\langle Af, \varphi\rangle \leq 0$. Let D, D' be dense subspaces of X, X^*, and let

$$L(\cdot,\cdot) : D \times D' \to \mathbb{C}$$

be a sesquilinear form ($L(u,v)$ is linear in u and conjugate linear in v). L is called *semibounded above* or *quasi-dissipative* if there is a real c such that for all $u \in D$ with $i(u) \cap D' \neq \phi$, there exists $\varphi \in i(u) \cap D'$ such that

$$\operatorname{Re} L(u, \bar{\varphi}) \leq c \|u\|^2 .$$

Call L *dissipative* if $c = 0$, and $-L$ is called *accretive* in this case. The authors are concerned with when sesquilinear forms and operators are dissipative on L^p and when the operator is a semigroup generator. Since the word dissipative is used in two different contexts, some clarification is necessary for resolving exactly what is meant. In some but not all cases they are equivalent.

There are operators of the form $Au = \Delta u + \frac{c}{|x|^2}$, defined on on $\mathbb{R}^N$, that generate (C_0) semigroups on $L^p(\mathbb{R}^N)$ but are not quasi-dissipative on $L^p(\mathbb{R}^N)$. In cases such as this, that A generates a (C_0) semigroup on $L^p(\mathbb{R}^N)$ follows from the fact that A generates an analytic semigroup; a direct proof that A generates a (C_0) semigroup on $L^p(\mathbb{R}^N)$ is a very hard problem for these choices of p, N. The authors want to know which concrete elliptic operators E generate a (C_0) semigroup on $L^p(\mathbb{R}^N)$. It is prudent and sensible to forget the maximal generality and restrict one's attention and show E is quasi-dissipative on $L^p(\mathbb{R}^N)$ or simply dissipative by subtracting a term of the form cI in the operator where c is a real constant. This is a main theme of the book starting in Chapter 2.

Consider an operator A of the form

$$Au = \operatorname{div}(M\nabla u) + b \cdot \nabla u + \operatorname{div}(cu) + au$$

with complex coefficients such that M is an $n \times n$ matrix of complex measures, b, c are n-vectors of complex measures, a is a complex measure, and $\operatorname{Im}(M)$ is symmetric. The homogeneous Dirichlet boundary condition is always assumed. Associated with A is the sesquilinear form L defined by

$$L(u,v) = -\int_\Omega \left(\langle M\nabla u, \nabla v\rangle + \langle b\nabla u, v\rangle + \langle u, \bar{c}\nabla v\rangle - a\langle u, v\rangle\right) dx$$

defined on $\left[C_0^1(\Omega)\right]^2$; the subscript 0 refers to compact support. Using self-adjoint operator theory as well as dissipative operator theory on Hilbert space, the question of whether or not A is quasi-dissipative on $L^2(\Omega)$ is pretty well understood. The question is much more difficult when L^2 is replaced by L^p. Define L to be L^p-*dissipative* if

$$\operatorname{Re} L(u, |u|^{p-2}\, u) \leq 0 \text{ for } p \geq 2,$$
$$\operatorname{Re} L(|u|^{p'-2}\, u, u) \leq 0 \text{ for } 1 < p < 2.$$

Let the matrix $\operatorname{Im}(M)$ be symmetric: $\operatorname{Im} M^t = \operatorname{Im} M$. A nice clean result is the following theorem.

The form $L(u,v) = -\int_\Omega \langle M\nabla u, \nabla v\rangle dx$ is L^p-dissipative on $L^p(\Omega), 1 < p < \infty$, iff

$$|p-2|\, |\langle \operatorname{Im} M\xi, \xi\rangle|_{TV} \leq 2\, |p-1|\, \langle \operatorname{Re} M\xi, \xi\rangle$$

for all $\xi \in \mathbb{R}^n$, where TV refers to the total variation norm.

The cases $1 < p < 2$ and $p \geq 2$ are different because $u \in C_0^1(\Omega)$ implies $|u|^{p-2}\, u \in C_0^1(\Omega)$ for $p \geq 2$ but not for $p < 2$. Characterizations for L^p-dissipativity of the operator A are also given. Suppose $c = 0$, Ω is bounded and the coefficients satisfy $a^{jk}, b^k \in C^1(\bar{\Omega})$, $a \in C^0(\bar{\Omega})$. Then $D(A) = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ when a modest regularity assumption is imposed on $\partial\Omega$. Then, in this case, the form L is L^p-dissipative iff the corresponding operator A is L^p-dissipative.

Here are some further results. For $Au = \nabla^t(M\nabla u) + \mu u$, with μ a nonnegative finite measure and $\operatorname{Re}\langle M\xi, \xi\rangle\rangle \geq 0$ for all $\xi \in \mathbb{R}^n$, A is L^p-dissipative if

$$\int_\Omega |u|^2\, d\mu \leq \frac{4}{pp'} \int_\Omega \langle M\nabla u, \nabla u\rangle dx$$

for all $u \in C_0^\infty(\Omega)$. Now let

$$\lambda = \sup_S \frac{\langle \operatorname{Re} M(x)\xi, \xi\rangle}{|\langle \operatorname{Im} M(x)\xi, \xi\rangle|}$$

where

$$S = \{(x,\xi) \in \Omega \times \mathbb{R}^n : \langle \operatorname{Im} M(x)\xi, \xi\rangle \neq 0\}.$$

If $\langle \operatorname{Im} M(x)\xi, \xi\rangle = 0$ for some $x \in \Omega$, then A is L^p-dissipative for all p. If $\langle \operatorname{Im} M(x)\xi, \xi\rangle$ never vanishes, then A is L^p-dissipative iff p satisfies

$$2 + 2\lambda(\lambda - \sqrt{\lambda^2+1}) \leq p \leq 2 + 2\lambda(\lambda + \sqrt{\lambda^2+1}).$$

For this A, if also $\operatorname{Im} M = \operatorname{Im} M^t$, then A is m-dissipative on L^p iff

$$|p-2|\, |\langle \operatorname{Im} M(x)\xi, \xi\rangle| \leq 2\sqrt{p-1}\langle \operatorname{Re} M(x)\xi, \rangle$$

for all x, ξ. The proofs of these recent and new results are quite technical and intricate. It is nice to see them presented in a unified fashion.

Chapter 3 is concerned with the Lamé operator

$$Eu = \Delta u + \left(\frac{1}{1-2\mu}\right) \nabla(\operatorname{div} u)$$

on $[L^p(\Omega)]^n$, where the constant μ satisfies $\mu > 1$ or $\mu < 1/2$. For dimension $n = 2$, the authors prove that E is L^p-dissipative iff

$$\left|\frac{1}{2} - \frac{1}{p}\right| \leq \frac{2(\mu-1)(2\mu-1)}{(3-4\mu)^2}.$$

While this nontrivial result is clean and elegant, it gives new information about the parabolic problem $u_t = Eu$. But it does not give information about the elastic wave equation which, like the acoustic wave equation, is ill posed when $n > 1$ unless $p = 2$.

Chapter 5 deals with the angle of dissipativity. When A (or L) is L^p-dissipative, it typically generates a semigroup which is analytic in a sector

$$\Sigma(\theta) = \{z \in \mathbb{C} : \mathrm{Re}(z) > 0, \ \ |\arg(z)| < \theta\}$$

for some $\theta \in (0, \pi/2]$. The angle of dissipativity of A is the supremum of these values of θ. It is also

$$\sup\{\alpha > 0 : e^{i\alpha A} \text{ is } L^p\text{-dissipative}\}.$$

Using complicated calculations, the authors compute the exact angle of dissipativity in certain cases. This is an extremely tough problem in general. For some related complementary results, see [6].

Chapter 6 deals with partial differential operators of order higher than two. Consider the order k (> 2) partial differential operator $A = \sum_{|\alpha| \le k} a_\alpha \partial^\alpha$, where each a_α is an $n \times n$ matrix with entries in $L^1_{\mathrm{loc}}(\Omega)$ for some $n \in \mathbb{N}$. Let $1 \le p < \infty$ with $p \ne 2$, and let Ω be a domain in $\mathbb{R}^n$. If $[C_0^\infty(\Omega)]^n \subset D(A)$, then A cannot generate a (C_0) contraction semigroup on $[L^p(\Omega)]^n$. The idea behind this theorem was known to Feller in the 1950s, but this form of the theorem is both general and elegant.

In Chapter 7, the sesquilinear form L satisfying $\mathrm{Re} \int_\Omega \langle Lu, u\rangle dx \le 0$ is instead required to satisfy $\mathrm{Re} \int_\Omega \langle Lu, u\rangle \Psi(x) dx \le 0$ for $u \in C_0^\infty(\Omega)$ for some weight function Ψ. Single operators and systems are treated, again in considerable generality.

This book is valuable; it contains a lot of new information and deep, complicated proofs. It has some minor flaws; for instance, there are typos, and while there is an author index, there is no subject index. The book does not emphasize heuristics and motivation. It is a research monograph aimed at active scholars; I think it would be difficult for many graduate students to master. But it is a very good book, and every serious research university library should get it. I expect it to inspire new research.

There are many good books devoted to operator semigroup theory and its applications. Nine of these [1, 2, 4–7, 10, 11, 14] are cited as a representative example of the books on the subject. All nine have the property that each covers some aspect of the theory or applications better than any of the others. In addition four other books by outstanding authors [3, 12, 14, 17] are cited. They cover analysis broadly, and each presents semigroup theory from a special perspective.

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