

*Singular perturbation in the physical sciences*, by John Neu, Graduate Studies in Mathematics, Vol. 167, American Mathematical Society, Providence, RI, 2015, xiv+326 pp., ISBN 978-1-4704-2555-5

Singular perturbations are typically associated with mathematical problems that contain a small parameter such that the solutions do not converge uniformly as this parameter tends to zero. An entertaining example is Michael Berry’s maggot and apple “paradox”: It is not pleasant to observe half a maggot after biting an apple. It is even less pleasant to observe a quarter of a maggot, and it becomes less and less pleasant as the fraction  $\epsilon$  of observed remaining maggot decreases towards zero. However, the no-maggot  $\epsilon = 0$  limit is quite pleasant!

Neu defines singular perturbations as the class of mathematical models that involve highly separated scales, where the solutions of the underlying equations do not converge uniformly as the ratio of the scales becomes very large or very small [19]. This seems to be somewhat narrower than the previous definition; however, as will be shown below, it does include all the main physics problems that are traditionally associated with singular perturbations. Sometimes the separated scales are transparent from the problem’s formulation, and sometimes they are discovered in the process of solving the problem. In the latter case, the mathematical analysis provides the extra benefit of new physical insight.

The birth of singular perturbation theory is often considered to be Prandtl’s paper, presented in 1904 at the 3rd International Congress of Mathematicians. Prandtl tackled the old paradox of approximating fluid flow by Euler equations. He realized the importance of viscosity, and in particular that even negligible viscosity cannot be ignored near boundaries. Prandtl thus showed that viscosity induces a new, small length scale near boundaries, and presented a method to solve such problems. This paved the way to the modern theory of hydrodynamics [25].

At about the same time Sommerfeld and Runge [23] showed that geometrical optics is the singular limit of Maxwell’s equations as the ratio of the wavelength and other length scales in the problem (optical elements, etc.) tends to zero. These two examples, fluid mechanics and optics, demonstrate that singular perturbation problems are ubiquitous in the physical and biological sciences. To quote from Neu’s book, “Though the natural world is an interconnected whole, our models usually treat small pieces of the whole as isolated cases.” Singular perturbation theory enables us to connect these different pieces by identifying the scales and connecting the different phenomena taking place on them.

There are two widely different approaches to singular perturbation problems. On the one extreme stands the purest mathematician whose view is, It’s not what you know, it’s what you can prove. The difficulty with this view is that it is very hard to prove something before you know what you need to prove. By their very nature, singular perturbation solutions are very hard to compute in the first place, and until they are known, nothing can be proved. . . . On the other extreme stands the very applied practitioners of perturbation problems. They are not just technicians—in fact, they are more like magicians, performing amazing mathematical tricks to

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identify the correct scaling and to find, under certain assumptions, approximate solutions. The drawback of this view is that the assumptions do not obviously hold, and moreover, even if some approximation is found, it is often unclear in *what sense* the computed solution approximates the actual solution. As will be now demonstrated, the great success in solving singular perturbation problems in many scientific disciplines is the result of a balanced combination of ingenious techniques to compute correct scaling and to find explicit solutions, and sophisticated rigorous mathematical analysis to elucidate the precise sense in which these solutions approximate the actual behavior of the underlying physical system. Among the many success stories of singular perturbations theory listed below are a number of outstanding examples that had a great scientific and technological impact.

**Short wavelength limit.** The derivation of geometrical optics, including the eikonal equation and transport equation, as the singular limit of Maxwell's equation was already mentioned. The prototype wave equation is of the form

$$\Delta u + \epsilon^{-2} n^2(x) u(x; \epsilon) = 0.$$

Here and elsewhere  $\epsilon$  denotes a small positive parameter. The equation is supplemented by boundary conditions prescribed on a domain of  $O(1)$  size. The existence of a short length scale of size  $O(\epsilon)$  is expected from a dimensional analysis of the equation. Motivated by the plane wave solution in the special case where  $n(x)$  is constant, one seeks an asymptotic expansion of the form

$$u(x; \epsilon) = A(x; \epsilon) e^{is(x)/\epsilon}, \quad A(x; \epsilon) \sim \sum_{m=0}^{\infty} A_m(x) (i\epsilon)^m.$$

This is the essence of the well-known WBK method (for Wentzel, Kramers, and Brillouin, although a more adequate name would be JBKW, to honor both Jeffries who introduced it in his study of water waves and Joseph B. Keller who applied it extensively in his study of acoustic waves and quantum mechanics). The leading order terms in the expansion yield the eikonal equation  $|\nabla s| = n(x)$  and the transport equation  $\nabla \cdot (A_0^2 \nabla s) = 0$ , which together form the foundation of geometrical optics. Along the same lines, Hamilton's equations of classical mechanics emerge as the small Planck's constant singular limit of the Schrödinger equation [3].

The geometrical optics limit fails near boundaries and near singularities, such as caustics. However, Keller extended the method in his *Geometrical theory of diffraction* [12], an effective way to compute the singular limit of the interaction of waves with boundaries. Moreover, Berry and others showed how to calculate the fine details of the singular limit near caustics. Following Keller's work, and in particular Luneburg's discovery of the connection between the eikonal equation and the propagation of singularities in the full Maxwell's equation, Hörmander, Taylor and others provided a rigorous foundation of these techniques for general linear wave equations.

**Boundary and internal layers.** A prototype model is the diffusion of ions under an external electric field. The flux  $c(x; \epsilon)$  satisfies the Nernst–Planck equation

$$\epsilon c_x + \phi_x c = b, \quad c(0; \epsilon) = 0, \quad 0 < x < 1.$$

Here  $\phi$  is the electric potential,  $b$  is a constant, and the condition  $c(0; \epsilon) = 0$  indicates that the ions are absorbed as they reach  $x = 0$ . This canonical model appears in many applications, including semiconductors and ionic channels. The

naive power series expansion  $c(x; \epsilon) = c^0(x) + \epsilon c^1(x) + \dots$  implies that to leading order  $c^0(x) = b/\phi_x(x)$ . However, such an expansion cannot satisfy the boundary condition at  $x = 0$ . Writing down the exact solution

$$c(x; \epsilon) = \frac{b}{\epsilon} \int_0^x \exp\left(\frac{\phi(s) - \phi(x)}{\epsilon}\right) ds$$

and in particular drawing this function (Figure (1.5) in [19]) reveals that the function  $c^0$  above provides a very good approximation of the exact solution, except at a narrow layer of size  $\epsilon$  near  $x = 0$ , where  $c(x; \epsilon)$  develops a large gradient to adjust to the boundary condition at  $x = 0$ .

A main difference between the Nernst–Planck model and the wave equation in the preceding example is that in the former case the small scale where rapid variations occur is localized near the boundary, while in the latter case the fast and slow scales interact everywhere. The most important example of a localized boundary layer is the viscous layer established by fluid flow near solid boundaries. The existence, shape, and properties of this layer are key features in fluid mechanics. Following Prandtl's work, a number of practical methods for constructing uniform expansions for boundary layer problems were developed by Van Dyke and others [25], [9].

Localized sharp transition layers need not occur only at boundaries. The solution of the Burgers equation

$$u_t + uu_x = \epsilon u_{xx}, \quad u(x, 0; \epsilon) = f(x),$$

might develop a shock, that is, a moving sharp transition (in fact a jump discontinuity in the limit case  $\epsilon = 0$ ). Here, again, initial progress in formal solutions near the shock, including the equation of motion of the shock itself, was followed by deep mathematical analysis by Friedrichs, Lax, and others, and eventually led to the development of viscous solution theory [5].

Another canonical example of an internal transition layer is provided by the reaction diffusion model

$$(1) \quad u_t = \epsilon \Delta u + \epsilon^{-1} u(1 - u^2), \quad x \in \Omega, \quad \partial_n u(x \in \partial\Omega, t; \epsilon) = 0,$$

where  $\Omega$  is a domain in  $R^3$ . Formally applying the method of matched asymptotic expansion, developed for boundary layer problems, it can be shown that under proper initial conditions the solution  $u(x, t; \epsilon)$  is nearly constant  $\pm 1$  except at a neighborhood of a sharp transition zone centered at a surface  $\Gamma(t)$ . Moreover,  $\Gamma$  evolves by mean curvature flow. A rigorous justification was then provided by de Mottoni and Schatzman [6].

Similar singular perturbation calculations were used for the difficult problem of magnetic vortices in nonlinear field theory. In particular, Neu [17], [18] resolved the *paradox of infinite mass* of magnetic vortices. Again, just as in boundary layer theory, the key idea was to identify correctly a small scale near the zeros of the order parameter wave function, compute the solution near the vortex and far from it, and match the two solutions to obtain a uniform expansion. Neu's formulas were rigorously justified via elliptic and geometric estimates by Bethuel, Brezis, and Helein [4]. Similarly, formal equations for the geometric flow of the vortex line, such as normal curvature flow of superconducting vortex lines and binormal curvature flow of superfluid vortex lines [21], were later proved, generalized, and carefully analyzed by Jerrard, Soner, and others [10].

**Homogenization.** Boundary or internal layer problems are characterized by the appearance of a small scale only in a small part of the global domain. On the other hand, problems with rapidly varying coefficients, such as transport in composites, fluid flow in porous media, and waves propagation in a domain with many small scatterers, involve a complex network of fast and slow scales. The prototype problem is a diffusion equation with the rapidly varying (periodic or random) diffusion coefficient

$$(2) \quad \nabla \cdot (a(x, x/\epsilon)\nabla u) = f(x), \quad x \in \Omega,$$

together with boundary conditions at  $\partial\Omega$ , where  $\Omega$  is a domain of  $O(1)$  size. Problems of this type are analyzed and solved under the general framework of homogenization, where the underlying PDE for the microscopic structure is replaced by a PDE with smooth coefficients for the macro geometry; namely, one seeks an approximation  $\bar{u}(x)$  that is close to  $u(x; \epsilon)$  in a proper norm, such that  $\bar{u}(x)$  solves an appropriate PDE, say  $\nabla \cdot (\bar{a}(x)\nabla \bar{u}) = \bar{f}$ , characterized by a smooth effective conductivity  $\bar{a}$ . The homogenization multiple scale technique was initially developed to compute effective equations for diffusion in composites and underground water flow (Darcy equation). The formal method was given a first theoretical foundation in [1]. Later, a complete rigorous justification in the form of H-convergence was proved by Oleinik, Tartar, and others [24], showing that the solutions of the homogenized PDE are a physically proper approximation of the solutions of the original PDE in the sense that they capture the correct global transport. Homogenization theory has met with enormous success over a wide range of disciplines, ranging from oil reservoirs [22], through elasticity [15] and metamaterials, to cardiac defibrillation [11], and even dentistry [7].

Due to the huge breadth of singular perturbation theory and practice, it is not surprising that there are many books dealing with it. In light of the specific nature of Neu's book, it is useful to compare it to those books that specialize in the technical and practical aspects of singular perturbations. Some of the earlier books, such as [9] and [25], concentrate on boundary layers, particularly with an eye towards applications to fluid mechanics. These books contain essential "recipes" for finding the location and scaling of the boundary layer, constructing the solutions in the layer and out of it, and matching these solutions. Another classic book [13] deals mostly with the multiple scale method, in particular in the context of dynamical systems.

Singular perturbation courses are typically taught at the graduate level. The teacher of such a course has at his or her disposal a rich body of literature, such as [8], [9], [13], and more. On the other hand, the books of Bender and Orszag [2] and by Lin and Segal [14] focus on advanced examples and case studies, which make them suitable to serve as supplemental materials for the course. A somewhat exceptional case is Murdock's book [16] which provides some recipes too but also provides convergence proofs for some of the classical solution methods. The book of Sanchez-Palencia [22] on homogenization can also be included in the present list, since it provides solution techniques and occasionally convergence proofs.

Neu's book is quite different from all of these books. It consists of a collection of problems that mainly arise from fluid mechanics or wave phenomena, and it presents various singular perturbation methods for solving these problems. The problems are typically at an advanced level, and a basic knowledge in perturbation theory is assumed. For example, while a "standard" perturbation theory textbook

introduces boundary layer theory via various two-point boundary value problems with boundary layers, high-order matching conditions, and then some examples of internal layers, Neu's book focuses on corner layers, internal layers, and derivative layers.

The concept of singular perturbations is introduced in Chapter 1 via a number of examples, including algebraic and differential equations. This is followed by a chapter that formally defines the notion of asymptotic expansion, with an emphasis on asymptotic expansion of integrals with applications to probability, and in particular to waves. The classical method of matched asymptotic expansion for boundary layers in ordinary differential equations is considered in Chapter 3, with applications to fluid mechanics and to ion diffusion. Neu correctly warns the reader not to accept the matched asymptotic method as an automatic machine, but rather as a trial-and-error process.

The far more difficult subject of boundary or internal layers in partial differential equations is considered in Chapter 4. The idea of a transition layer around a moving interface, including the Chapman–Enskog paradigm, is presented for the canonical equation

$$\epsilon^2 \theta_t = \epsilon \Delta \theta - \sin 2\theta$$

that models the response of the liquid crystal direction field to a unidirectional applied electric field. The difficult problem of deriving the geometric flow of the interface is analyzed in great detail. An even more impressive calculation is provided for the solution of the nonlinear wave equation

$$\epsilon^2 \theta_{tt} = \epsilon \Delta \theta - \sin 2\theta$$

that is related to deep results in nonlinear field theory. The chapter ends with the problem of finding the shift of the Laplacian spectrum when the domain is perforated by a single small hole, and later by a collection of many small holes. This problem has many important applications in optics (scattering by many tiny objects), fluid mechanics (flow in bubbly liquids), chemical engineering (oil refinery), and more. A simple formal calculation shows that as the number of holes tends to infinity at a rate inversely proportional to the holes' radii, the spectrum is shifted (up to a constant) by the density distribution function of the holes [20].

Prandtl's theory of viscous boundary layers is discussed in Chapter 5. The analysis is limited to two-dimensional problems, and therefore the stream function formulation

$$(u \cdot \nabla) \Delta \Psi = \epsilon^2 \Delta^2 \Psi,$$

where  $u$  is the velocity field,  $\Psi$  is the stream function, and  $\epsilon^2$  is the inverse Reynolds number, can be used. Since  $u$  can be expressed as a function of  $\Psi$ , this forms a closed equation. Although this is an equation of high order, the boundary layer calculations are simplified since the equation is scalar, and since the level sets of  $\Psi$  are naturally related to the problem's geometry.

The method of multiple scales is introduced in Chapter 6. The prototype model is the Duffing equation

$$x_{tt} + x + \epsilon k x^3 = \epsilon a \cos((1 + \epsilon \Omega)t).$$

The formal two scales expansion consists of expressing the solution as  $x = x(t, \tau = \epsilon t; \epsilon)$ , and treating  $t$  and  $\tau$  as two independent variables. Even on the formal level such expansions are highly nontrivial, since identifying the correct long time scale is often not obvious. An alternative approach, invented by Whitham [26],

that has been very successful in studying modulations of both linear and nonlinear waves is to exploit the Hamiltonian nature of the wave equation, and to average the slowly varying wave action over the fast time scale. This method is presented by Neu via a number of detailed canonical examples. While the multiple scale expansion method was originally developed to compute wave modulations, it was used very successfully in solving the problem of homogenizing transport in a composite medium. Chapter 6 thus ends with two alternative ways to apply this method for diffusion in rapidly varying periodic media. The last two chapters of the book contain examples of applying the averaging principle to Hamiltonian systems, again exploiting the Lagrangian structure and the action-phase variables coordinates.

The book is written in a somewhat unusual way. As stated earlier, the theory is presented only via examples that are worked out in great detail. Often the examples are presented as problems and then a full solution is given. This is helpful to the reader who wants to solve the problem first, and then to compare the solution with Neu's. The numbering of the equations in each chapter is a little bit confusing since different numbering systems are used for the main text in the chapter and for the examples there. It would also have been useful to add captions to the figures in order to relate them to the main text. Each chapter includes a guide to the literature where reference to classic books and papers in perturbation theory is provided.

Neu writes (p. 85): "Singular perturbation is 'practice' first and 'field of knowledge' second." This statement succinctly illustrates his philosophy and thus the philosophy and style of the entire book. Solving a singular perturbation problem is half art and half math. While this book is weak on the rigorous *mathematical* theory, it opens a rare window into the *art* component. In particular the reader can enjoy the privilege of observing John Neu, one of the great masters of this art, in real action.

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