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COMMENTARY ON "DIFFERENTIABLE DYNAMICAL SYSTEMS" BY STEPHEN SMALE

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Abstract. Immediately following the commentary below, this previously published article is reprinted in its entirety: Stephen Smale, "Differentiable dynamical systems", *Bull. Amer. Math. Soc.* **73** (1967), no. 6, 747–817.

It would be difficult, and impossible for me, to describe the full impact of Smale's article on differentiable dynamical systems. I will concentrate on one topic from Part II of the paper: dynamical zeta functions for flows. There has been significant improvement in our understanding of these zeta functions in the last few years and many advances culminated in Dyatlov and Guillarmou's [DyGu18] resolution of Smale's conjecture about their meromorphic continuation.

To set things up, a flow on a compact smooth manifold $\mathcal{M}$ is a one-parameter group of diffeomorphisms: $\varphi_t : \mathcal{M} \to \mathcal{M}$, $\varphi_0 = \mathrm{id}$, $\varphi_{t+s} = \varphi_t \circ \varphi_s$. It is generated by a smooth vector field $X(x) := d\varphi_t(x)/dt|_{t=0}$. Let Γ denote the set of closed orbits of the flow, and for $\gamma \in \Gamma$ let $\ell(\gamma)$ be the minimal period of γ; that is, the first $t > 0$ such that $\varphi_t(x) = x$ for some $x \in \gamma$.

A dynamical zeta function for flows is defined by Smale in [Sm67, §II.4] as follows:

$$Z(s) := \prod_{\gamma\in\Gamma} \prod_{k=0}^{\infty} (1 - e^{-(s+k)\ell(\gamma)}). \tag{1}$$

This is a generalization of a zeta function defined by Selberg when the flow is the geodesic flow on a Riemann surface, but in this generality Smale considered a zeta function as "a wild idea".

A closely related zeta function was later introduced by Ruelle [Ru76]:

$$\zeta(s) := \prod_{\gamma\in\Gamma} (1 - e^{-s\ell(\gamma)}), \qquad \zeta(s) = \frac{Z(s)}{Z(s+1)}. \tag{2}$$

If convergence of the product in the definition of $Z(s)$ is known for $\operatorname{Re} s \gg 1$, then meromorphic continuation of one zeta function follows from that of the other.

Needless to say, some assumptions are required to make sense of equation (1), discreteness of the set of $\ell(\gamma)$ being the first requirement. Convergence for $\operatorname{Re} s \gg 1$ follows from knowing that $|\{\gamma : \ell(\gamma) \leq T\}| \leq Ce^{CT}$. Refining such estimates to obtain *prime geodesic theorems* is one of the applications of dynamical zeta functions; see [GLP13] and references given therein.

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A class of flows discussed by Smale is given by suspensions of diffeomorphisms, that is, flows obtained from discrete dynamical systems defined by iterating diffeomorphisms. For diffeomorphisms, the dynamical zeta function was first defined by Artin and Mazur, and [Sm67, §I.4] is devoted to that better developed subject. (For a modern analytic approach and for references to the literature, see the recent book [Ba18].)

The zeta function of a diffeomorphism f is rational when f is an *Anosov diffeomorphism.* In Theorem 4.1 Smale and Narasimhan show that for suspensions of Anosov diffeomorphisms the zeta function $Z(s)$ is meromorphic in $\mathbb{C}$. Smale asks if $Z(s)$ is meromorphic for flows which are *Axiom A* suspensions. He then adds "I must admit a positive answer would be a little shocking!"

Axiom A flows, also introduced by Smale, are a generalization of flows which are close to suspensions of Anosov diffeomorphisms. To define them, we need to present several concepts from dynamical systems.

Definition 1 (Smale [Sm67, §II.2])**.** A fixed point $x \in \mathcal{M}$, $X(x) = 0$, is called *hyperbolic* if the linear map $T_x^*\mathcal{M} \ni df(x) \mapsto d(Xf)(x) \in T_x^*\mathcal{M}$, $f \in \mathcal{C}^\infty(\mathcal{M})$, has no eigenvalues on the imaginary axis. Hyperbolic fixed points are nondegenerate and thus are isolated.

Definition 2 (Smale [Sm67, §II.3])**.** Let $\mathcal{K} \subset \mathcal{M}$ be a compact φ_t-invariant set. We say that $\mathcal{K}$ is *hyperbolic* for the flow φ_t if the generator X of the flow does not vanish on $\mathcal{K}$ and each tangent space $T_x\mathcal{M}$, $x \in \mathcal{K}$, admits a continuous decomposition

$$T_x\mathcal{M} = E_0(x) \oplus E_s(x) \oplus E_u(x), \quad E_0(x) := X(x)\mathbb{R}, \quad x \in \mathcal{K},$$

where $d\varphi_t(E_\bullet(x)) = E_\bullet(\varphi_t(x))$ for all $x \in \mathcal{K}$, $t \in \mathbb{R}$, and for some continuous norm $|\bullet|$ on the fibers of $T\mathcal{M}$, there exist constants $C, \theta > 0$ such that for all $x \in \mathcal{K}$,

$$|d\varphi_t(x)v| \leq Ce^{-\theta|t|}|v| \quad \text{when} \quad \begin{cases} t \geq 0, & v \in E_s(x); \\ t \leq 0, & v \in E_u(x). \end{cases}$$

We say φ_t is an *Anosov flow* if the whole of $\mathcal{M}$ is hyperbolic.

Perhaps the most widely known example of an Anosov flow is the geodesic flow on negatively curved manifolds. In that case $\mathcal{M}$ is the sphere bundle of the manifold.

We also define the *nonwandering set*:

Definition 3 (Smale [Sm67, p. 796])**.** We call $x \in \mathcal{M}$ a *nonwandering point* if for every neighborhood V of x and every $T > 0$ there exists $t \in \mathbb{R}$ such that $|t| \geq T$ and $\varphi_t(V) \cap V \neq \emptyset$. The set of all nonwandering points is called the *nonwandering set.*

We now give the definition of Axiom A flows:

Definition 4 (Smale [Sm67, §II.5, (5.1)])**.** The flow φ_t is *Axiom A* if:

(1) the nonwandering set is the disjoint union of the set $\mathcal{F}$ of fixed points and the closure $\mathcal{K}$ of the union of all closed orbits;
(2) all fixed points of φ_t are hyperbolic;
(3) the set $\mathcal{K}$ is hyperbolic for the flow φ_t.

Smale's conjecture was answered in the affirmative in the generality of Axiom A flows:

Theorem 1 (Dyatlov and Guillarmou [DyGu16], [DyGu18])**.** *Assume that φ_t is an Axiom A flow with orientable stable/unstable foliations $x \mapsto E_\bullet(x)$, $\bullet = s, u$. Then $Z(s)$ and $\zeta(s)$ given in equation* (1) *and equation* (2) *continue meromorphically to $\mathbb{C}$.*

The orientability hypothesis holds in many natural cases (such as geodesic flows on orientable negatively curved manifolds) and can be removed under certain topological assumptions by using twisted zeta functions; see [GLP13, Appendix B].

We also remark that for applications one typically needs meromorphic continuation to a finite strip $\operatorname{Re} s > -a$. However, continuation to large strips does not at the moment appear to be simpler. When instead of C^∞ regularity, C^k regularity, $k \gg 1$, is assumed, the continuation is valid for $\operatorname{Re} s > -k/C$.

Theorem 1 was preceded by earlier results, and I will only mention a few highlights in the case of flows. But I would like to stress that the development of the zeta function for hyperbolic maps was an essential component of the progress on flows; see the book of Baladi [Ba18], herself a key contributor to this story. As Smale stressed on the first page of his paper, "the same phenomena and problems of the qualitative theory of ordinary differential equations are present in their simplest form in the diffeomorphism problem."

Selberg proved the meromorphy of equation (1) for Riemann surfaces using his trace formula. Ruelle [Ru76] represented $\zeta(s)$ in equation (2) using Grothendieck's theory of Fredholm determinants of nuclear operators. That led to meromorphic continuation under the strong assumption that the foliations defined by $x \mapsto E_\bullet(x)$, $\bullet = s, u$, are real analytic. Eventually that turned out to imply that such geodesic flows have to be flows on locally symmetric spaces but the method has had broader applications. Rugh [Ru96] found a new approach to the determinant method for flows in which $\mathcal{M}$ and φ_t, but not the foliations, were real analytic (but not the foliations). That breakthrough allowed the first treatment of a large class of manifolds of non-constant curvature. It was developed further by Fried [Fr95] and led to the formulation of a conjecture I will mention below. One big difference between the results of Fried and Rugh and Theorem 1 is the order estimate on the zeta function in the analytic case.

The case of smooth manifolds and flows was resolved in the Anosov case (see the end of Definition 2) by Giulietti, Liverani, and Pollicott [GLP13]. That paper can also be consulted for more about history and applications. One of the crucial new components were the *anisotropic Sobolev spaces* first developed for maps by Kitaev, Blank-Keller–Liverani, Gouezel–Liverani, and Baladi–Tsujii. Roughly speaking, in these spaces functions are smoother in the stable directions and rougher in the unstable ones. Since the flow moves from stable to unstable, we "move" smoother functions into spaces of rougher functions, and that results in compactness or trace class properties. This notion is fundamentally *microlocal*, that is, it involves phase space (position and momentum) properties of functions.

A clear microlocal description of anisotropic spaces was given by Faure and Sjöstrand [FaSj11], and that was the starting point of my joint work with Dyatlov in which we gave a micorlocal proof of the theorem of Giulietti–Liverani–Pollicott [DyZw16]. As in previous works the connection with spectral methods is given by the Atiyah–Bott–Guillemin trace formula:

$$\text{(3)}\quad \operatorname{tr}\varphi_t^* = \sum_{\gamma\in\Gamma}\sum_{k=1}^{\infty}\frac{\ell(\gamma)\delta(t-k\ell(\gamma))}{|\det(I-P_\gamma^k)|},\quad t>0,\quad P_\gamma := d\varphi_{\ell(\gamma)}(x)|_{E_u(x)\oplus E_s(x)},\quad x\in\gamma.$$

Here $\varphi_t^* f(x) := f(\varphi_t(x))$ is the pull-back operator and the trace is defined by integrating the operator kernel (roughly, $\delta(\varphi_t(x)-y)$) on the diagonal. That operation is allowed under some analytic *wave front set* conditions and makes the above formula valid in the sense of distributions.

Thinking of X as a differential operator acting on $C^\infty(\mathcal{M})$, we have $\varphi_t^* f = \exp(tX)f$. One can then easily imagine that by various manipulations, one of them, the Laplace transform, equation (3) can be expressed using zeta functions related to equation (2). It then turns out that the meromorphy of the zeta function is related to the properties of $\operatorname{tr}(X-s)^{-1}$. Here the trace is meant in the above distributional sense, and the inverse is defined on the anisotropic spaces. To obtain the needed wave front set conditions, the new component was an adaptation of Melrose's *radial estimates* [Me94] developed for scattering theory on asymptotically Euclidean spaces. These estimates have been extensively used in scattering theory and mathematical general relativity (where they give a microlocal explanation of the red shift effect at the event horizon), notably in the work of Vasy [Va13]. Here, the stable and unstable bundles in phase space play a role of radial sources and sinks, and, unlike the sources and sinks of [Me94], [Va13], they are highly irregular. The estimates are however robust enough to handle that.

The situation becomes more complicated when one moves from Anosov flows to Axiom A flows. Construction of the resolvent $(X-s)^{-1}$ requires new estimates as now "escape" is possible both in space (away from hyperbolic sets) and momentum (to infinite frequencies). For a class of open systems related to Axiom A flows, Dyatlov and Guillarmou [DyGu16] constructed the resolvent and applied it to prove meromorphy of the corresponding zeta function. Using ideas from [Sm67, §II.5], the work of Conley–Easton and more recent contributions by Guillarmou–Mazzucchelli–Tzou, [DyGu18] used the results of [DyGu16] to give Theorem 1.

One of the interesting questions about zeta functions is the *Fried conjecture* [Fr95, p. 181]. Consider a "twisted" generalization of equation (2) in the case of (Σ, g), a negatively curved oriented compact Riemannian manifold of odd dimension $n\geq 3$,

$$\zeta_\alpha(\lambda) := \prod_{\gamma\in\Gamma}\det(I-\alpha(\gamma)e^{i\lambda\ell_\gamma}), \tag{4}$$

for a representation $\alpha : \pi_1(S^*\Sigma)\to GL(N,\mathbb{C})$. (The meromorphy of ζ_α for smooth Anosov flows follows from the proof in [DyZw16].) Referring to [Fr95] for precise definitions, Fried conjectured that when α is an acyclic unitary representation of $\pi_1(\Sigma)$, then

$$|\zeta_\alpha(0)|^{(-1)^{n-1}} = T_\alpha(\Sigma)^2, \tag{5}$$

where $T_\alpha(\Sigma)$ is the analytic torsion of Σ. The analytic torsion $T_\alpha(\Sigma)$ was defined by Ray and Singer using eigenvalues of an α-twisted Hodge Laplacian. Their conjecture that $T_\alpha(\Sigma)$ is equal to the Reidemeister torsion, a topological invariant, was proved independently by Cheeger and Müller. Hence, equation (5) would link dynamical, spectral, and topological quantities. In the case of locally symmetric manifolds a more precise version of the conjecture was recently proved by Shen [Sh18] following earlier contributions by Fried and Moskovici–Stanton.

In the case of smooth manifolds of variable negative curvature, equation (5) remains completely open. Here however is a small, and easy to understand, contribution toward it:

Theorem 2 (Dyatlov and Zworski [DyZw17]). *Suppose that* (Σ, g) *is an orientable Riemannian surface of genus* g. *Then,*

$$\zeta(s) = cs^{2g-2}(1 + \mathcal{O}(s)), \quad c \neq 0.$$

In particular, the length spectrum, $\{\ell(\gamma)\}_{\gamma\in\Gamma}$, *determines the* genus.

Recently Hadfield [Ha18] generalized this result to Riemannian surfaces with boundary. This required replacing the tools from [DyZw16] by the more subtle tools of [DyGu16].

There have been other interesting developments, and I would like to mention two of them. Faure and Tsujii [FaTs17] proved meromorphic continuation of "semiclassical" Gutzwiller–Voros zeta functions and provided information about the distribution of their singularities. Dang and Rivière [DaRi18], as part of their programme of microlocal understanding of Morse–Smale flows (a special case of Axiom A flows), related the singularities of zeta functions in that case to limits of eigenvalues of the Witten Laplacian.

I believe that every section of Smale's seminal paper would merit a commentary of even greater length. It remains readable and interesting after 50 years.

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References

[Ba18] Viviane Baladi, *Dynamical Zeta Functions and Dynamical Determinants for Hyperbolic Maps,* Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge, **68**, Springer Verlag, 2018.

[DaRi18] Nguyen Viet Dang and Gabriel Rivière, *Pollicott-Ruelle spectrum and Witten Laplacians,* arXiv:1709.04265.

[DyGu16] Semyon Dyatlov and Colin Guillarmou, *Pollicott-Ruelle resonances for open systems*, Ann. Henri Poincaré **17** (2016), no. 11, 3089–3146, DOI 10.1007/s00023-016-0491-8. MR3556517

[DyGu18] Semyon Dyatlov and Colin Guillarmou, *Dynamical zeta functions for Axiom A flows*, Bull. Amer. Math. Soc. (N.S.) **55** (2018), no. 3, 337–342.

[DyZw16] Semyon Dyatlov and Maciej Zworski, *Dynamical zeta functions for Anosov flows via microlocal analysis* (English, with English and French summaries), Ann. Sci. Éc. Norm. Supér. (4) **49** (2016), no. 3, 543–577, DOI 10.24033/asens.2290. MR3503826

[DyZw17] Semyon Dyatlov and Maciej Zworski, *Ruelle zeta function at zero for surfaces*, Invent. Math. **210** (2017), no. 1, 211–229, DOI 10.1007/s00222-017-0727-3. MR3698342

[FaSj11] Frédéric Faure and Johannes Sjöstrand, *Upper bound on the density of Ruelle resonances for Anosov flows* (English, with English and French summaries), Comm. Math. Phys. **308** (2011), no. 2, 325–364, DOI 10.1007/s00220-011-1349-z. MR2851145

[FaTs17] Frédéric Faure and Masato Tsujii, *The semiclassical zeta function for geodesic flows on negatively curved manifolds*, Invent. Math. **208** (2017), no. 3, 851–998, DOI 10.1007/s00222-016-0701-5. MR3648976

[Fr95] David Fried, *Meromorphic zeta functions for analytic flows*, Comm. Math. Phys. **174** (1995), no. 1, 161–190. MR1372805

[GLP13] P. Giulietti, C. Liverani, and M. Pollicott, *Anosov flows and dynamical zeta functions*, Ann. of Math. (2) **178** (2013), no. 2, 687–773, DOI 10.4007/annals.2013.178.2.6. MR3071508

[Ha18] Charles Hadfield, *Zeta function at zero for surfaces with boundary*, arXiv:1803.10982.

[Me94] Richard B. Melrose, *Spectral and scattering theory for the Laplacian on asymptotically Euclidian spaces*, Spectral and scattering theory (Sanda, 1992), Lecture Notes in Pure and Appl. Math., vol. 161, Dekker, New York, 1994, pp. 85–130. MR1291640

[Ru76] David Ruelle, *Zeta-functions for expanding maps and Anosov flows*, Invent. Math. **34** (1976), no. 3, 231–242, DOI 10.1007/BF01403069. MR0420720

[Ru96] Hans Henrik Rugh, *Generalized Fredholm determinants and Selberg zeta functions for Axiom A dynamical systems*, Ergodic Theory Dynam. Systems **16** (1996), no. 4, 805–819, DOI 10.1017/S0143385700009111. MR1406435

[Sh18] Shu Shen, *Analytic torsion, dynamical zeta functions, and the Fried conjecture*, Anal. PDE **11** (2018), no. 1, 1–74, DOI 10.2140/apde.2018.11.1. MR3707290

[Sm67] Stephen Smale, *Differentiable dynamical curves*, Bull. Amer. Math. Soc., **73** (1967), no. 6, 747–817. MR0228014

[Va13] András Vasy, *Microlocal analysis of asymptotically hyperbolic and Kerr-de Sitter spaces (with an appendix by Semyon Dyatlov)*, Invent. Math. **194** (2013), no. 2, 381–513, DOI 10.1007/s00222-012-0446-8. MR3117526

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