BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 55, Number 4, October 2018, Pages 405–425
https://doi.org/10.1090/bull/1637
Article electronically published on July 30, 2018

VLADIMIR VOEVODSKY—AN APPRECIATION

MARC LEVINE

ABSTRACT. We give a brief résumé of some of the works of the late Vladimir Voevodsky.

Vladimir Voevodsky is probably best known to the wider mathematical community as a Fields Medalist, awarded for his work on motivic categories, motivic cohomology, and the solution of the Milnor conjecture relating mod 2 étale cohomology and mod 2 Milnor K-theory [56]. He is also very well known for his solution, with Rost and others, of the Bloch–Kato conjecture on the Galois symbol, the extension of the Milnor conjecture from two to arbitrary primes [60]. Since many articles (see for example [21, 32, 33, 55, 69]) have been written on Voevodsky's work on the Milnor conjecture and the Bloch–Kato conjecture, in this brief appreciation I would like to give my perhaps idiosyncratic view on another series of Voevodsky's achievements, all having to do with his development of what is now known as motivic stable homotopy theory.

1. Triangulated categories of motives

The first aspect of Voevodsky's work I want to discuss is his construction of a triangulated category of motives over a perfect field k, presented in the text [65] with Friedlander and Suslin, although many of the ideas appear in Voevodsky's earlier work [67]. Voevodsky identifies the main themes that have become the building blocks of motivic homotopy theory:

i. transfers,
ii. the use of presheaf categories,
iii. localization to force $\mathbb{A}^1$-invariance and to reflect a sheaf theory.

For the triangulated category of motives, the notion of presheaf already incorporates that of transfers, and it is aptly called the category of preheaves with transfers (PST). A PST is a presheaf of abelian groups on the category $\mathbf{Sm}/k$ of smooth varieties over k, endowed with "wrong-way maps" or transfers, corresponding to certain finite surjective morphisms. This is more precisely formulated using the category of finite correspondences Cor_k, where a basic finite correspondence from X to Y is simply a subvariety W of $X \times_k Y$, mapping finitely and surjectively to a connected component of X. These form a basis of the abelian group of all finite correspondences $\mathrm{Cor}_k(X, Y)$. Morally speaking, a basic correspondence may be viewed as a multivalued map from X to Y and this leads to a composition law defining the additive category Cor_k. A PST is simply an additive presheaf on

Received by the editors July 3, 2018.
2010 *Mathematics Subject Classification.* Primary 14F42, 19E15 ; Secondary 55P42.
The author is supported by the DFG through the SFB Transregio 45 and the SPP 1786 "Homotopy theory and algebraic geometry".

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Cor_k with values in abelian groups; we have the functor $\mathbf{Sm}/k \to \mathrm{Cor}_k$ sending a morphism to its graph, so a PST restricts to give a usual presheaf on $\mathbf{Sm}/k$, and we recover the initial notion of a PST as a presheaf with the additional wrong-way maps, or transfers, as described above. Each smooth variety X has its representable PST $\mathbb{Z}^{\mathrm{tr}}(X) := \mathrm{Cor}_k(-, X)$.

To build a triangulated category out of the category of PSTs is now pretty straightforward: first form the category of complexes, then the homotopy category $K(PST)$, and then perform the localization as indicated in (iii). For the $\mathbb{A}^1$-invariance ($\mathbb{A}^1 = \mathrm{Spec}\ k[T]$ the affine line), one can express this by endowing $K(PST)$ with a tensor structure, extending the assignment $\mathbb{Z}^{\mathrm{tr}}(X) \otimes \mathbb{Z}^{\mathrm{tr}}(Y) := \mathbb{Z}^{\mathrm{tr}}(X \times_k Y)$, and one localizes with respect to maps $C \otimes \mathbb{Z}^{\mathrm{tr}}(\mathbb{A}^1) \to C \otimes \mathbb{Z}^{\mathrm{tr}}(\mathrm{Spec}\ k) = C$ induced by the projection $\mathbb{A}^1 \to \mathrm{Spec}\ k$. For the sheaf theory one chooses a suitable topology (the Nisnevich topology is the usual choice) and inverts quasi-isomorphisms $C \to C'$, that is, maps which induce an isomorphism on all cohomology sheaves. This works if the topology has enough points, such as in the Nisnevich or étale topology, if not, one imposes descent via hypercovers. This builds the triangulated category of effective motives over k, $\mathrm{DM}^{\mathrm{eff}}(k)$. Unless we say otherwise, we will always work with the version using the Nisnevich topology.

As it stands, this construction is difficult to use. It is in general nearly impossible to get concrete information on the morphisms in a localization. Here is where the transfer structure plays a crucial role. Voevodsky considers a homotopy invariant PST, that is, an additive presheaf $P : \mathrm{Cor}_k^{\mathrm{op}} \to \mathbf{Ab}$ such that the projection $X \times \mathbb{A}^1 \to X$ induces an isomorphism $P(X) \to P(X \times \mathbb{A}^1)$ for all $X \in \mathbf{Sm}/k$. He considers the associated Nisnevich sheaf P_{Nis} and the cohomology presheaves

$$X \mapsto H^n(X_{\mathrm{Nis}}, P_{\mathrm{Nis}}).$$

Assuming that k is a perfect field, Voevodsky shows that all the cohomology presheaves are again $\mathbb{A}^1$-invariant. This is a remarkable result with remarkable consequences. It shows that $\mathrm{DM}^{\mathrm{eff}}(k)$ may be identified with the full subcategory of the derived category of Nisnevich sheaves on $\mathbf{Sm}/k$, $D(\mathrm{Sh}_{\mathrm{Nis}}(\mathbf{Sm}/k))$ consisting of those complexes C whose cohomology sheaves $\mathcal{H}^n(C)$ are $\mathbb{A}^1$-invariant, and makes some computations in $\mathrm{DM}(k)$ much more tractable.

One such consequence is the identification of Suslin's algebraic singular homology (see [54]) with the homology theory defined within $\mathrm{DM}^{\mathrm{eff}}(k)$, that is

$$H^n(X; \mathbb{Z}) := \mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff}}(k)}(\mathbb{Z}^{\mathrm{tr}}(\mathrm{Spec}\ k), \mathbb{Z}^{\mathrm{tr}}(X)[n]).$$

The Suslin homology $H_n^{\mathrm{Sus}}(X; \mathbb{Z})$ is defined quite concretely: we have the algebraic n-simplex $\Delta_k^n \subset \mathbb{A}^{n+1}$ defined as the hyperplane $\sum_{i=0}^n t_i = 1$. The usual coface and codegeneracy maps for the standard n-simplices in topology make perfect sense algebraically, and one has for an arbitrary PST P the Suslin complex $C_*^{\mathrm{Sus}}(P)(Y)$ with

$$C_n^{\mathrm{Sus}}(P)(Y) := P(Y \times \Delta^n)$$

and differential d_n induced by the alternating sum of the face maps $\Delta^{n-1} \to \Delta^n$. Suslin defines

$$H_n^{\mathrm{Sus}}(X; \mathbb{Z}) := H_n(C_*^{\mathrm{Sus}}(\mathbb{Z}^{\mathrm{tr}}(X))(\mathrm{Spec}\ k)),$$

and the identification of $\mathrm{DM}^{\mathrm{eff}}(k)$ described above leads to an isomorphism $H_n^{\mathrm{Sus}}(X; \mathbb{Z}) \cong \mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff}}(k)}(\mathbb{Z}^{\mathrm{tr}}(\mathrm{Spec}\ k), \mathbb{Z}^{\mathrm{tr}}(X)[n])$.

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Another important consequence of the theory of presheaves with transfer is the Mayer–Vietoris property for the $\mathbb{Z}^{\mathrm{tr}}(X)$ in $\mathrm{DM}^{\mathrm{eff}}(k)$: for $X = U \cup V$ a union of open subschemes, the sequence

$$\mathbb{Z}^{\mathrm{tr}}(U \cap V) \to \mathbb{Z}^{\mathrm{tr}}(U) \oplus \mathbb{Z}^{\mathrm{tr}}(V) \to \mathbb{Z}^{\mathrm{tr}}(X)$$

extends to a distinguished triangle in $\mathrm{DM}^{\mathrm{eff}}(k)$, and thus gives a long exact Mayer–Vietoris sequence for Suslin homology. This is a good point to address the question: Why the Nisnevich topology? One answer is that the Nisnevich topology is the coarsest one for which the sheafification of the presheaf sequence for the cover of X by Zariski open subschemes U, V,

$$0 \to \mathbb{Z}^{\mathrm{tr}}(U \cap V) \to \mathbb{Z}^{\mathrm{tr}}(U) \oplus \mathbb{Z}^{\mathrm{tr}}(V) \to \mathbb{Z}^{\mathrm{tr}}(X) \to 0$$

becomes exact upon sheafification. Remarkably, this property leads to a Mayer–Vietoris property for the associated Suslin complexes, even though one loses the exactness on the right. Suslin defined his algebraic homology in the mid-1980s, and it was not until Voevodsky embedded this construction in his triangulated one that the Mayer–Vietoris property for Suslin homology could be proved.

Bloch's version of *motivic cohomology* via his cycle complexes and higher Chow groups [8] also gets embedded in the Voevodsky theory. This is I think more subtle, as the Bloch cycle complexes do not directly rely on correspondences, but rather use for each integer $q \geq 0$ the subgroup $z^q(X, n)$ of the group of codimension q algebraic cycles on $X \times \Delta^n$, which "intersects faces properly". This proper intersection property allows one to define intersection with faces, and allows one to define a differential and resulting chain complex $z^q(X, *)$. Bloch's higher Chow group is the homology

$$\mathrm{CH}^q(X, p) := H_p(z^q(X, *)).$$

Friedlander, Suslin, and Voevodsky rephrase this into a cohomology theory in the setting of $\mathrm{DM}^{\mathrm{eff}}$ by looking at the sheaf of equidimensional cycles $z_{\mathrm{equi}}(W)$ with value $z_{\mathrm{equi}}(W)(Y)$ the cycles on $Y \times W$ which are equidimensional over Y of relative dimension 0. This is a PST, and the Suslin complex $C_*^{\mathrm{Sus}}(z_{\mathrm{equi}}(W))$ gives an object of $\mathrm{DM}^{\mathrm{eff}}(k)$. The connection with the Bloch complex is that $C_*^{\mathrm{Sus}}(z_{\mathrm{equi}}(\mathbb{A}^q))(X)$ is a subcomplex of $z^q(X \times \mathbb{A}^q, *)$ and in the end one shows (see [65, Chap. 6] for the case of characteristic zero, and see [64] for the general case) that the inclusion is a quasi-isomorphism, giving isomorphisms

$$\begin{aligned}\mathrm{CH}^q(X, 2q-p) &\cong \mathrm{CH}^q(X \times \mathbb{A}^q, 2q-p) \cong H_{2q-p}(C_*^{\mathrm{Sus}}(z_{\mathrm{equi}}(\mathbb{A}^q))(X)) \\ &\cong \mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff}}}(\mathbb{Z}^{\mathrm{tr}}(X)[2q-p], \mathbb{Z}(q)[2q]) = \mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff}}}(\mathbb{Z}^{\mathrm{tr}}(X), \mathbb{Z}(q)[p]).\end{aligned}$$

Here $\mathbb{Z}(q) := \mathbb{Z}(1)^{\otimes q}$ and $\mathbb{Z}(1)[2] := \mathbb{Z}^{\mathrm{tr}}(\mathbb{P}^1)/\mathbb{Z}^{\mathrm{tr}}(\mathrm{Spec}\ k)$ is the reduced motive of $\mathbb{P}^1$; $\mathbb{Z}(0) = \mathbb{Z}^{\mathrm{tr}}(\mathrm{Spec}\ k)$. One defines the *motivic cohomology* of X as

$$H^{p,q}(X, \mathbb{Z}) := \mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff}}(k)}(\mathbb{Z}^{\mathrm{tr}}(X), \mathbb{Z}(q)[p])$$

for $q \geq 0$, $p \in \mathbb{Z}^{\mathrm{tr}}$, giving a categorical basis for motivic cohomology together with an identification of this with the Bloch higher Chow groups.

There is also a version of $\mathrm{DM}^{\mathrm{eff}}(k)$ for which tensor product with the Tate motive $\mathbb{Z}(1)$ is inverted, this is best constructed using a category of $\mathbb{Z}(1)[2]$-spectra and is written $\mathrm{DM}(k)$. In general, passing to spectra in this way can have a huge effect on the Hom-sets. In this case, one has the Voevodsky cancellation theorem [58], which says that, at least on compact objects, this functor is fully faithful and, in

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particular, the motivic cohomology does not change, which leads to the vanishing of negative weight motivic cohomology:

$$\mathrm{Hom}_{\mathrm{DM}(k)}(\mathbb{Z}^{\mathrm{tr}}(X), \mathbb{Z}(q)[p]) = 0 \quad \text{for } q < 0.$$

The main reason one needs the category $\mathrm{DM}(k)$ is to allow a duality for certain objects: for X a smooth projective variety of dimension d over a perfect field k, $\mathbb{Z}^{\mathrm{tr}}(X)$ has the dual $\mathbb{Z}^{\mathrm{tr}}(X)(-d)[-2d]$ in $\mathrm{DM}(k)$. This duality extends to smooth X over k if k has characteristic 0 or, in the case of characteristic $p > 0$, if one inverts p. These duals exist in $\mathrm{DM}^{\mathrm{eff}}(k)$ only if X has dimension 0 over k, in general.

After this very successful construction, there has been a good deal of interest in extending the construction of a triangulated category of motives to arbitrary base-schemes. In fact, this is still an open problem; we will discuss this in more detail in the section on Grothendieck six-functor formalism.

To conclude this section, I want to emphasize that Voevodsky's use of Grothendieck topologies was a striking and surprising component of his overall approach to the construction of a triangulated category of motives. His introduction of the h- and *cdh*-topologies, starting with his earlier paper on the subject [67], has had an impact on the field far beyond the direct application to the construction of motivic categories. I should mention at the very least the work of Cortiñas, Haesemeyer, Schlichting, Walker, and Weibel [13–17] and of Kerz, Strunk, and Tamme [34, 35] on using properties of *cdh*-descent to study properties of the K-theory of singular schemes.

2. The slice tower in the motivic stable homotopy category: Atiyah–Hirzebruch spectral sequences, algebraic cobordism, and the Steenrod algebra

I would now like to move to a discussion of the relation of motivic cohomology with algebraic K-theory, leading to aspects of the motivic stable homotopy category. According to the conjectures of Beilinson and Lichtenbaum, motivic cohomology is supposed to be an integral form for the eigenspaces of the Adams operations operating on rational algebraic K-theory, and this relation should be refined to a spectral sequence of the form

$$E_2^{p,q} := H^{p-q,-q}(X, \mathbb{Z}) \Rightarrow K_{-p-q}(X)$$

reminiscent of the Atiyah–Hirzebruch spectral sequence from singular cohomology to topological K-theory. Such a spectral sequence for X the spectrum of a field F was first constructed by Bloch and Lichtenbaum. The basic idea was to hark back to the original idea of Atiyah and Hirzebruch, who construct their spectral sequence

$$E_2^{p,q} = H^p(X, \mathbb{Z}(-q/2)) \Rightarrow KU^{p+q}(X)$$

using the skeletal filtration of the CW complex X. If one tries the naive translation to the filtration of a scheme X by closed subsets of varying codimension, one does get a spectral sequence, but it was well known that rationally the initial terms usually involve more than one Adams eigenspace, so this is not the correct filtration.

Bloch and Lichtenbaum use the the homotopy invariance of K-theory to replace X with the cosimplicial scheme $n \mapsto \Delta^n_F$, which they then filter by codimension "in good position", giving the resulting tower of spectra

$$\cdots \to K^{(q+1)}(F) \to K^{(q)}(F) \to \cdots \to K^{(0)}(F) \sim K(F)$$

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with $K^{(q)}(F)$ the total spectrum of the cosimplicial spectrum $n \mapsto K^{(q)}(X,n)$, and where $K^{(q)}(X,n)$ is the colimit of the K-theory spectra with support $K_W(\Delta^n_F)$ as W runs over all closed subsets of Δ^n_F having codimension $\geq q$ on Δ^n_F and also after intersection with each face of Δ^n_F. This gives a spectral sequence converging to $K_{-p-q}(F)$; the difficulty is identifying the initial term with the higher Chow groups, but by using some rather elementary properties of K-theory together with an ingenious application of the Bloch method of "moving by blowing up" [7], they manage to accomplish this.

Relying on this result, Friedlander and Suslin [22] use a similar "filtration by codimension on a Suslin complex" approach, modified to fit into the framework of presheaves with a suitable transfer, to generalize to the case of a smooth X over a perfect field. I also managed to generalize the Bloch–Lichtenbaum approach more directly [40] with an analogous result. Both proofs require the Bloch–Lichtenbaum result for fields.

Voevodsky had a completely different approach, which has had consequences for motivic homotopy theory far beyond the problem of the spectral sequence from motivic cohomology to K-theory; it is described in [62, 63]. This is modeled on a more homotopy-theoretic construction of the Atiyah–Hirzebruch spectral sequence: One represents a cohomology theory via a spectrum E in the stable homotopy category as

$$E^n(X) := [\Sigma^\infty X_+, \Sigma^n E]_{\mathrm{SH}}.$$

For topological K-theory, one uses the spectrum KU built out of BU and Bott periodicity. Rather than filtering the space X, one filters the spectrum E via its Moore–Postnikov tower

$$\cdots \to \tau_{\geq q+1}E \to \tau_{\geq q}E \to \cdots \to E,$$

with $\tau_{\geq q}E \to E$ characterized by requiring that this map induce an isomorphism on stable homotopy groups π_n for $n \geq q$ and that $\pi_n\tau_{\geq q}E = 0$ for $n < q$. The homotopy cofiber of $\tau_{\geq q+1}E \to \tau_{\geq q}E$ has just one nonvanishing stable homotopy group $\pi_q = \pi_q E$, and so is equal to the Eilenberg–MacLane spectrum $\mathrm{EM}(\pi_q E, q)$, which represents cohomology with coefficients in the group $\pi_q E$, shifted by q. The Postnikov tower thus gives the spectral sequence

$$E_2^{p,q} := H^p(X, \pi_{-q}E) \Rightarrow E^{p+q}(X)$$

by the standard method of the exact couple for a tower, and it is isomorphic to the sequence given by the skeletal filtration of X (after a reindexing). Bott periodicity gives $\pi_q\mathrm{KU} = \mathbb{Z}(q/2)$, with $\mathbb{Z}(q/2)$ meaning $\mathbb{Z}$ for q even and 0 for q odd, and one recovers the classical Atiyah–Hirzebruch spectral sequence

$$E_2^{p,q} := H^p(X, \mathbb{Z}(-q/2)) \Rightarrow \mathrm{KU}^{p+q}(X).$$

Since the K-groups of a scheme X are known often to have more than one Adams eigenspace in a fixed degree, a naive translation of this method, simply using the Moore–Postinkov tower of the algebraic K-theory spectrum $K(X)$, is not the correct one. Voevodsky replaced the notion of topological connectivity, which governs the topological Moore–Postnikov tower, with the notion of "$\mathbb{P}^1$-connectivity", which yields a construction of what is now known as the Voevodsky slice tower. This is set in the motivic stable homotopy category, which I now briefly explain.

One first defines the motivic pointed unstable homotopy category $\mathcal{H}_\bullet(S)$ over a reasonable scheme S. This was constructed by Morel and Voevodsky, following to

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some extent a version of Voevodsky's approach to $\mathrm{DM}^{\mathrm{eff}}$; I give here a modification of their construction which is commonly used today. One starts with pointed simplicial sets, $\mathbf{Spc}_\bullet$, as a convenient model for pointed spaces, and defines the category of *pointed spaces over* S, $\mathbf{Spc}_\bullet(S)$, as the category of presheaves on smooth quasi-projective schemes over S, $\mathbf{Sm}/S$, with values in pointed simplicial sets. Sending a smooth S-scheme X to its representable presheaf of sets, with a disjoint base-point, $Y \mapsto \mathrm{Hom}_{\mathbf{Sm}/S}(Y, X)_+$, viewed as a constant simplicial set, gives the functor $\mathbf{Sm}/S \to \mathbf{Spc}_\bullet(S)$. Similarly, sending a simplicial set K to the constant presheaf with value K gives the functor $\mathbf{Spc}_\bullet \to \mathbf{Spc}_\bullet(S)$.

A presheaf category inherits categorical structures from its value category. In this case, this says that $\mathbf{Spc}_\bullet(S)$ admits arbitrary (small) limits and colimits, and the symmetric monoidal structure on $\mathbf{Spc}_\bullet$, $(A, B) \mapsto A \wedge B$, gives $\mathbf{Spc}_\bullet(S)$ a symmetric monoidal structure. For instance, one can speak of $\mathbb{G}_m \wedge S^1$, where S^1 is the pointed circle and $\mathbb{G}_m$ is the presheaf associated to the pointed S-scheme $(\mathbb{A}^1 \setminus \{0\}, \{1\})$, or for $\mathcal{X}, \mathcal{Y} \in \mathbf{Spc}_\bullet(S)$, the suspension $\Sigma_{\mathcal{Y}}\mathcal{X} := \mathcal{X} \wedge \mathcal{Y} \in \mathbf{Spc}_\bullet(k)$.

One then localizes with respect to $\mathbb{A}^1$-invariance: $\mathcal{X} \wedge \mathbb{A}^1_+ \to \mathcal{X}$ for $\mathcal{X} \in \mathbf{Spc}_\bullet(S)$, and with respect to Nisnevich weak equivalences: a map $\mathcal{X} \to \mathcal{Y}$ is a Nisnevich weak equivalence if the associated map on the Nisnevich sheaf of homotopy sets (for π_0) or groups (for π_i, $i > 0$) is an isomorphism. This produces the motivic pointed unstable homotopy category $\mathcal{H}_\bullet(S)$. A rather elementary but exceedingly important fact is that in $\mathcal{H}_\bullet(S)$ there is a canonical isomorphism $\mathbb{P}^1 \cong \mathbb{G}_m \wedge S^1$, here $\mathbb{P}^1$ is the presheaf represented by $\mathbb{P}^1$, pointed by $\infty := (0:1)$.

The stable version is set forth in Voevodsky's 1998 ICM lecture [66]; detailed constructions and further refinements were given by, for example, Jardine [31]. For $\mathcal{X} \in \mathbf{Spc}_\bullet(S)$ and for integers $a \geq b \geq 0$, one has the $\mathbb{A}^1$-homotopy sheaf $\pi^{\mathbb{A}^1}_{a,b}(\mathcal{X})$, this being the Nisnevich sheaf on $\mathbf{Sm}/S$ associated to the presheaf

$$U \mapsto [\Sigma^{a-b}_{S^1}\Sigma^b_{\mathbb{G}_m} U_+, \mathcal{X}]_{\mathcal{H}_\bullet(S)}$$

Note that the isomorphism $\mathbb{P}^1 \cong \mathbb{G}_m \wedge S^1$ gives an isomorphism of presheaves

$$[\Sigma^{a-b}_{S^1}\Sigma^b_{\mathbb{G}_m} U_+, \mathcal{X}]_{\mathcal{H}_\bullet(S)} \cong [\Sigma^{a-2b}_{S^1}\Sigma^b_{\mathbb{P}^1} U_+, \mathcal{X}]_{\mathcal{H}_\bullet(S)}$$

Following the classical definition of spectra, define a $\mathbb{P}^1$-spectrum to be a sequence $\mathcal{E} = (\mathcal{E}_0, \mathcal{E}_1, \ldots)$, $\mathcal{E}_n \in \mathbf{Spc}_\bullet(S)$ together with bonding maps $\epsilon_n : \Sigma_{\mathbb{P}^1}\mathcal{E}_n \to \mathcal{E}_{n+1}$. This gives the category of $\mathbb{P}^1$-spectra over S: $\mathrm{Sp}_{\mathbb{P}^1}(S)$. For $\mathcal{E}$ a $\mathbb{P}^1$-spectrum, $a, b \in \mathbb{Z}$, we have the stable $\mathbb{A}^1$-homotopy sheaf

$$\pi^{\mathbb{A}^1}_{a,b}\mathcal{E} := \mathrm{colim}_N \pi^{\mathbb{A}^1}_{a+2N,b+N}\mathcal{E}_N,$$

where the transition maps are given by

$$\pi^{\mathbb{A}^1}_{a+2N,b+N}\mathcal{E}_N \xrightarrow{\Sigma_{\mathbb{P}^1}} \pi^{\mathbb{A}^1}_{a+2N+2,b+N+1}\mathcal{E}_N \wedge \mathbb{P}^1 \xrightarrow{\epsilon_{n*}} \pi^{\mathbb{A}^1}_{a+2N+2,b+N+1}\mathcal{E}_{N+1}.$$

A map $f : \mathcal{E} \to \mathcal{F}$ in $\mathrm{Sp}_{\mathbb{P}^1}(S)$ is a *stable* $\mathbb{A}^1$*-weak equivalence* if $\pi^{\mathbb{A}^1}_{a,b}(f)$ is an isomorphism for all a, b; we form $\mathrm{SH}(S)$ by inverting all stable $\mathbb{A}^1$-weak equivalences in $\mathrm{Sp}_{\mathbb{P}^1}(S)$:

$$\mathrm{SH}(S) = \mathrm{Sp}_{\mathbb{P}^1}(S)[sWE^{-1}_{\mathbb{A}^1}].$$

$\mathrm{SH}(S)$ is a triangulated symmetric monoidal category admitting arbitrary direct sums, just as the classical stable homotopy category SH. Essentially by construction, the $\mathbb{P}^1$-suspension functor $\Sigma_{\mathbb{P}^1}$ becomes invertible in $\mathrm{SH}(S)$; since $\mathbb{P}^1 \cong \mathbb{G}_m \wedge S^1$, we have $\Sigma_{\mathbb{P}^1} \cong \Sigma_{\mathbb{G}_m} \circ \Sigma_{S^1}$ and so both $\Sigma_{\mathbb{G}_m}$ and Σ_{S^1} are invertible

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on $\mathrm{SH}(S)$. The shift functor in $\mathrm{SH}(S)$ for its triangulated structure is given by Σ_{S^1}, not $\Sigma_{\mathbb{P}^1}$.

This being the case, one might ask: Why invert $\mathbb{P}^1$ and not S^1? One answer is that this fulfills the same goal as inverting the reduced motive $\mathbb{Z}(1)[2]$ of $\mathbb{P}^1$ in $\mathrm{DM}^{\mathrm{eff}}(k)$ to form $\mathrm{DM}(k)$, and one wants to have duals of suitable objects. A better reason can be found in our discussion of the six-functor formalism in §3.

Sending a pointed space $\mathcal{X}$ to its $\mathbb{P}^1$-suspension spectrum

$$\Sigma_{\mathbb{P}^1}\mathcal{X} := (\mathcal{X}, \mathcal{X} \wedge \mathbb{P}^1, \ldots, \Sigma^n_{\mathbb{P}^1}\mathcal{X}, \ldots)$$

defines the infinite $\mathbb{P}^1$-suspension functor $\Sigma^\infty_{\mathbb{P}^1} : \mathcal{H}_\bullet(S) \to \mathrm{SH}(S)$. The objects $\{\Sigma^n_{\mathbb{P}^1}X_+ \mid X \in \mathbf{Sm}/S, n \in \mathbb{Z}\}$ are a set of compact generators for $\mathrm{SH}(S)$.

$\mathrm{SH}(S)$ has the full triangulated subcategory $\mathrm{SH}^{\mathrm{eff}}(S)$, generated by the suspension spectra $\{\Sigma^m_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1}X_+ \mid X \in \mathbf{Sm}/S, m \geq 0\}$ (and closed under arbitrary direct sums). For each $n \in \mathbb{Z}$, we have the full triangulated subcategory (closed under arbitrary direct sums) $\Sigma^n_{\mathbb{P}^1}\mathrm{SH}^{\mathrm{eff}}(S)$, of $\mathrm{SH}(S)$ generated by $\{\Sigma^m_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1}X_+ \mid X \in \mathbf{Sm}/S, m \geq n\}$. One should think of $\Sigma^n_{\mathbb{P}^1}\mathrm{SH}^{\mathrm{eff}}(S)$ as the $(n-1)$-$\mathbb{P}^1$-connected objects in $\mathrm{SH}(S)$.

It follows from results of Ne'eman on compactly generated triangulated categories [52] that the inclusion $i_n : \Sigma^n_{\mathbb{P}^1}\mathrm{SH}^{\mathrm{eff}}(S) \to \mathrm{SH}(S)$ admits a right adjoint $r_n : \mathrm{SH}(S) \to \Sigma^n_{\mathbb{P}^1}\mathrm{SH}^{\mathrm{eff}}(S)$. The *nth slice truncation functor*

$$f_n : \mathrm{SH}(S) \to \mathrm{SH}(S)$$

is simply $f_n := i_n r_n$; the inclusions $\Sigma^{n+1}_{\mathbb{P}^1}\mathrm{SH}^{\mathrm{eff}}(S) \to \Sigma^n_{\mathbb{P}^1}\mathrm{SH}^{\mathrm{eff}}(S)$ induce natural transformations $f_{n+1} \to f_n$ and thereby for each $\mathcal{E} \in \mathrm{SH}(S)$ the natural "slice tower"

$$\cdots \to f_{q+1}\mathcal{E} \to f_q\mathcal{E} \to \cdots \to \mathcal{E}$$

The cofiber of $f_{q+1}\mathcal{E} \to f_q\mathcal{E}$ is $s_q\mathcal{E}$, the *qth slice* of $\mathcal{E}$.

For each $X \in \mathbf{Sm}/S$, the spectral sequence of the slice tower for $\mathcal{E}$ gives the *slice spectral sequence*

$$E_2^{p,q} := s_{-q}^{p+q,n}\mathcal{E}(X) \Rightarrow \mathcal{E}^{p+q,n}(X).$$

In his ground-breaking paper [62] Voevodsky set forth a number of fascinating conjectures about his slice tower. We recall a few of these, in a modified form.

It follows from work of Morel and Voevodsky in [51] and [66], which was refined by Panin, Pimenov, and Röndigs in [43], that for S a regular scheme, algebraic K-theory is represented in $\mathrm{SH}(S)$ by a $\mathbb{P}^1$-spectrum KGL via

$$\mathrm{KGL}^{p,q}(X) \cong K_{2q-p}(X)$$

for $X \in \mathbf{Sm}/S$.

Conjecture 1 ([62, Conjecture 7]). *Let k be a perfect field. Then $s_0\mathrm{KGL}_k$ represents motivic cohomology, and $s_q\mathrm{KGL}_k \cong \Sigma^q_{\mathbb{P}^1}s_0\mathrm{KGL}_k$.*

This is analogous to the fact that $\pi_{2n}\mathrm{KU} = \mathbb{Z}$. Assuming this conjecture holds, we have

$$s_{-q}^{p+q,n}\mathrm{KGL}(X) = H^{p-q,n-q}(X, \mathbb{Z}),$$

giving the spectral sequence

$$E_2^{p,q}(\mathrm{KGL}, X) = H^{p-q,-q}(X, \mathbb{Z}) \Rightarrow \mathrm{KGL}^{p+q,0}(X) = K_{-p-q}(X).$$

that is, one has the expected motivic Atiyah-Hirzebruch spectral sequence. Write $\mathrm{EM}_{\mathrm{mot}}\mathbb{Z}$ for $s_0\mathrm{KGL}$.

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Conjecture 2 ([62, Conjecture 10]). *Let k be a perfect field, and let $\mathbb{S}_k$ be the* motivic sphere spectrum $\Sigma^\infty_{\mathbb{P}^1} \operatorname{Spec} k_+$. *Then* $s_0\mathbb{S}_k \cong \mathrm{EM}_{\mathrm{mot}}\mathbb{Z}$

Again, this has a clear topological analogue, namely the fact that $\pi_0\mathbb{S} = \mathbb{Z}$.

What about the other slices of $\mathbb{S}_k$? In the classical case, the layers in the Postnikov–Moore tower for the sphere spectrum are the Eilenberg–MacLane spectra $\mathrm{EM}(\pi_q\mathbb{S}, q)$. As computations of the stable homotopy groups of spheres are among the most intractable computations in mathematics, it would seem that the situation would be even more complicated in the motivic setting. However, Voevodsky conjectures something different.

Conjecture 3 ([62, Conjecture 9]). *Let $A_{p,q} = \mathrm{Ext}^{p,q}_{\mathrm{MU}_*\mathrm{MU}}(\mathrm{MU}_*, \mathrm{MU}_*)$, the E_2-term in the Adams–Novikov spectral sequence for the sphere spectrum in* SH. *For an abelian group A, let $\mathrm{EM}_{\mathrm{mot}}A \in \mathrm{SH}(k)$ be the spectrum representing motivic cohomology with A-coefficients. Then for $q \geq 0$*

$$s_q\mathbb{S}_k \cong \bigoplus_p \Sigma^q_{\mathbb{P}^1}\mathrm{EM}_{\mathrm{mot}}A_{p,2q}[-p].$$

Since $\mathbb{S}_k$ is in $\mathrm{SH}^{\mathrm{eff}}(k)$, we have $s_q\mathbb{S}_k = 0$ for $q < 0$. The computation of $\mathrm{Ext}^{p,2q}_{\mathrm{MU}_*\mathrm{MU}}(\mathrm{MU}_*, \mathrm{MU}_*)$ is not easy, but it is still a purely algebraic problem, which can be made into a programmable computation.

To recall: the complex cobordism spectrum MU has term $\mathrm{Th}(E_n)$ in degree $2n$ (and $\Sigma\mathrm{Th}(E_n)$ in degree $2n+1$), where $E_n \to \mathrm{BU}_n$ is the universal complex rank n vector bundle and $\mathrm{Th}(E_n) = \mathbb{P}(E_n \oplus \mathbb{C})/\mathbb{P}(E_n)$ is the Thom space. These fit together to a spectrum via the homeomorphism

$$\mathrm{Th}(i_n^*E_{n+1}) = \mathrm{Th}(E_n \oplus \mathbb{C}) = \mathrm{Th}(E_n) \wedge S^2,$$

where $i_n : \mathrm{BU}_n \to \mathrm{BU}_{n+1}$ is the usual inclusion. The Adams–Novikov spectral sequence is the MU-based Adams spectral sequence, where for a ring spectrum E, the E-based Adams spectral sequence is, roughly speaking, built out of the $\pi_*(E^{\wedge n})$ and the maps $\pi_*(E^{\wedge n}) \to \pi_*(E^{\wedge m})$ induced by the multiplication and unit maps for E.

In [66], Voevodsky defines the algebraic cobordism spectrum MGL as a direct analogue of MU, namely, the $\mathbb{P}^1$-spectrum which is $\mathrm{Th}(E_n)$ in degree n, where $E_n \to \mathrm{BGL}_n$ the universal rank n bundle. Bonding maps are defined analogously to those for MU via isomorphisms

$$\mathrm{Th}(i_n^*E_{n+1}) \cong \mathrm{Th}(E_n \oplus \mathbb{A}^1) \cong \mathrm{Th}(E_n) \wedge \mathbb{P}^1.$$

With regard to the slices of MGL, Voevodsky conjectured

Conjecture 4 ([62, Conjecture 5]). $s_0(\mathrm{MGL}) \cong \mathrm{EM}_{\mathrm{mot}}\mathbb{Z}$, $s_q\mathrm{MGL} = 0$ *for $q < 0$ and for $q > 0$*

$$s_q\mathrm{MGL} = \Sigma^n_{\mathbb{P}^1}\mathrm{EM}_{\mathrm{mot}}\mathrm{MU}_{2q}.$$

By work of Quillen, MU_* is isomorphic to the Lazard ring $\mathbb{L}$ which in turn is a polynomial ring $\mathbb{Z}[x_1, x_2, \ldots]$ with x_n in degree $2n$. Thus MU_{2q} is a free finite rank abelian group and $\mathrm{EM}_{\mathrm{mot}}\mathrm{MU}_{2q} = \mathrm{EM}_{\mathrm{mot}}\mathbb{Z} \otimes \mathrm{MU}_{2q}$ is a direct sum of copies of $\mathrm{EM}_{\mathrm{mot}}\mathbb{Z}$.

At this point, the reader will have noticed that all the slices we have looked at so far are just versions of motivic cohomology, just as all the layers in the Moore–Postnikov tower are Eilenberg–MacLane spectra. As $\mathrm{DM}(k)$ is more complicated

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than $D(\mathbf{Ab})$, we cannot expect things to be quite that simple. However, the fact that

$$\Sigma^n_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1}X_+ \wedge \Sigma^m_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1}Y_+ \cong \Sigma^{n+m}_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1}X \times_k Y_+$$

suggests that the slices s_* should have a multiplicative structure. As $\mathbb{S}_S \wedge \mathcal{E} \cong \mathcal{E}$ for every $\mathcal{E} \in \mathrm{SH}(S)$, one might expect that $s_q\mathcal{E}$ is an $s_0\mathbb{S}_S$-module, and, assuming Conjecture 2, this makes $s_q\mathcal{E}$ an $\mathrm{EM}_{\mathrm{mot}}\mathbb{Z}$-module. In classical theory, the homotopy category of $\mathrm{EM}(\mathbb{Z})$-modules is equivalent to $D(\mathbf{Ab})$, so one might expect

Conjecture 5. *For k a perfect field, $\mathrm{DM}(k)$ is equivalent to $\mathrm{Ho}\,\mathrm{EM}_{\mathrm{mot}}$-$\mathbb{Z}$-$\mathbf{Mod}$, and thus for each $\mathcal{E} \in \mathrm{SH}(k)$, there is a canonical object $\pi^\mu_q\mathcal{E}$ of $\mathrm{DM}(k)$ and canonical isomorphism $s_q\mathcal{E} \cong \Sigma^q_{\mathbb{P}^1}\mathrm{EM}(\pi^\mu_q\mathcal{E})$ in $\mathrm{SH}(k)$.*

Here $\mathrm{EM}_{\mathrm{mot}} : \mathrm{DM}(k) \to \mathrm{SH}(k)$ is the composition of the equivalence $\mathrm{DM}(k) \cong \mathrm{Ho}\,\mathrm{EM}_{\mathrm{mot}}$-$\mathbb{Z}$-$\mathbf{Mod}$ with the forgetful functor $\mathrm{Ho}\,\mathrm{EM}_{\mathrm{mot}}$-$\mathbb{Z}$-$\mathbf{Mod} \to \mathrm{SH}(k)$.

Actually, this does not seem to appear explicitly in [62]; however, one does have the following quote from this paper: “One of the implications of Conjecture 10 is that for any spectrum its slices have unique and natural module structures over the Eilenberg–MacLane spectrum which explains that all our conjectures predict that different objects of the form $s_*(-)$ are generalized Eilenberg–MacLane spectra.” Also, Oliver Röndigs relates that Voevodsky raised this question and stressed its importance for motivic stable homotopy theory in private conversations with him and Paul Arne Østvær during the Nordfjordeid Summer School in August of 2002.

Table 1 shows parallels arising from Voevodsky’s conjectures. For simplicity, we work over a base-scheme $S = \mathrm{Spec}\,k$, k a perfect field, although some of these analogies are expected to hold more generally.

All these conjectures have been settled positively for k a field of characteristic 0, and also in characteristic $p > 0$ after inverting p. Conjectures 1 and 2 were proven by Voevodsky in characteristic 0 [59, 63], and I gave a different proof [39] that works in arbitrary characteristic, without needing to invert the characteristic. With regard

TABLE 1

Classical	Motivic
SH	$\mathrm{SH}(k)$
$D(\mathbf{Ab})$	$\mathrm{DM}(k)$
$\mathrm{EM} : D(\mathbf{Ab}) \to \mathrm{SH}$	$\mathrm{EM}_{\mathrm{mot}} : \mathrm{DM}(k) \to \mathrm{SH}(k)$
$\mathbb{S}$	$\mathbb{S}_k$
$\mathrm{EM}(\mathbb{Z})$	$\mathrm{EM}_{\mathrm{mot}}(\mathbb{Z}(0))$
KU	KGL
MU	MGL
$\pi_0\mathbb{S} = \mathbb{Z}$	$\pi^\mu_0\mathbb{S}_k = \mathbb{Z}(0)$
$\mathrm{KU}_{2n} = \mathbb{Z}$	$\pi^\mu_n\mathrm{KGL} = \mathbb{Z}(0)$
$\mathrm{MU}_{2n} = \pi_{2n}\mathrm{MU}\mathbb{Z}[x_1, x_2, \ldots]_{2n}$	$\pi^\mu_n\mathrm{MGL} = \mathbb{Z}(0) \otimes MU_{2n}$
Atiyah-Hirzebruch spectral sequence	slice spectral sequence
$E_2^{p,q} = H^p(X, \mathbb{Z}(-q/2)) \Rightarrow \mathrm{KU}^{p+q}(X)$	$E_2^{p,q} = H^{p-q,-q}(X, \mathbb{Z}) \Rightarrow K_{-p-q}(X)$

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to Conjecture 1, both proofs are independent of the Bloch–Lichtenbaum theorem establishing the spectral sequence in the case of fields. Voevodsky indicated how Conjecture 3 would follow from Conjecture 4; the details were given in [38] and [47].

The fact that the slice tower carries a multiplicative structure was proven by Pelaez in [45], and another proof in a more general context was given later by Gutiérrez, Röndigs, Spitzweck, and Østvær in [26]. This endows $s_0\mathbb{S}_k$ with a commutative ring structure and the slices $s_q\mathcal{E}$ with an $s_0\mathbb{S}_k$-module structure, which by Conjecture 2 makes $s_q\mathcal{E}$ an $\mathrm{EM}_{\mathrm{mot}}\mathbb{Z}$-module. The equivalence of $\mathrm{Ho}\,\mathrm{EM}_{\mathrm{mot}}\mathbb{Z}$-**Mod** with $\mathrm{DM}(k)$ was proven in characteristic 0 by Röndigs and Østvær in [48] and extended to positive characteristic p after inverting p by Cisinski and Déglise in [10].

Conjecture 4 turned out to be quite a bit more difficult. An argument in characteristic 0 was presented by Hopkins and Morel, but never published. Hoyois [29] gave a detailed proof, which also covers the case of positive characteristic after inverting the characteristic. I want to say a bit about the ingredients that go into the proof, as these point out another important contribution Voevodsky made to the theory, as well as showing how others have developed the theory further based on Voevodsky's work.

We work over a perfect field k. Just as in the classical case, the very definition of MGL_n gives a cell decomposition, which in the motivic setting rather easily shows that MGL_n lives in $\Sigma^n_{\mathbb{P}^1}\mathrm{SH}^{\mathrm{eff}}$ and has a single lowest weight cell. In other words, the unit map $\mathbb{S}_k \to \mathrm{MGL}$ induces an isomorphism $s_0\mathbb{S}_k \to s_0\mathrm{MGL}$, and $s_q\mathrm{MGL} = 0$ for $q < 0$. From Conjecture 2, we have

$$s_0\mathrm{MGL} \cong \mathrm{EM}_{\mathrm{mot}}\mathbb{Z}.$$

One can construct a natural map $\mathrm{MU}_{2*} \to \pi_{2*,*}\mathrm{MGL}(k)$ (more about this later), so under the isomorphism $\mathrm{MU}_{2*} \cong \mathbb{Z}[x_1, x_2, \ldots]$, we have maps $x_n : \Sigma^n_{\mathbb{P}^1}\mathbb{S}_k \to \mathrm{MGL}$. Using the fact that MGL is a commutative ring spectrum in $\mathrm{SH}(k)$, these give maps $\times x_n : \Sigma^n_{\mathbb{P}^1}\mathrm{MGL} \to \mathrm{MGL}$, which allow one to define a "quotient spectrum" $\mathrm{MGL}/(x_1, x_2, \ldots,)$ in $\mathrm{SH}(k)$. Since $\Sigma^n_{\mathbb{P}^1}\mathrm{MGL}$ is in $\Sigma^n_{\mathbb{P}^1}\mathrm{SH}^{\mathrm{eff}}(k)$, we may factor $\mathrm{MGL} = f_0\mathrm{MGL} \to s_0\mathrm{MGL}$ through the quotient map $\mathrm{MGL} \to \mathrm{MGL}/(x_1, x_2, \ldots,)$, giving us the map $\theta : \mathrm{MGL}/(x_1, x_2, \ldots,) \to s_0\mathrm{MGL}$.

Let M_n be the set of degree n monomials in $\mathbb{Z}[x_1, x_2, \ldots]$, where x_i is given degree i. Spitzweck [50] showed that, if θ is an isomorphism, then one has canonical isomorphisms

$$s_n\mathrm{MGL} \cong \bigoplus_{m\in M_n} \Sigma^n_{\mathbb{P}^1} s_0\mathrm{MGL}.$$

As $\mathrm{MU}_{2n} \cong \mathbb{Z}[x_1, x_2, \ldots]_n$, this yields Conjecture 4 once one shows that θ is an isomorphism. We mention that Spitzweck's result is in many ways formal, and it gives a computation of $s_n\mathcal{E}$ in terms of $s_0\mathcal{E}$ assuming that $\mathcal{E}$ is a commutative ring spectrum in $\mathrm{SH}^{\mathrm{eff}}(k)$, and one has maps $x_n : \Sigma^n_{\mathbb{P}^1}\mathbb{S}_k \to \mathcal{E}$ such that $\mathcal{E}/(x_1, x_2, \ldots) \cong s_0\mathcal{E}$.

To show that θ is an isomorphism, first look at $\theta_{\mathbb{Q}}$ in the $\mathbb{Q}$-localization $\mathrm{SH}(k)_{\mathbb{Q}}$. Quillen's isomorphism $\mathrm{MU}_{2*} \cong \mathbb{L}$ arises from the fact that MU is the universal $\mathbb{C}$-oriented commutative ring spectrum in SH; that is, for $E \in \mathrm{SH}$ a ring spectrum with an orientation $\vartheta_E \in E^2(\mathbb{CP}^\infty)$, there there is a unique multiplicative map $\phi_{(E,\vartheta_E)} : \mathrm{MU} \to E$ sending the canonical orientation ϑ_{can} in $\mathrm{MU}^2(\mathbb{CP}^\infty)$ to ϑ_E.

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Here ϑ_{can} is given by the composition

$$\mathbb{CP}^\infty = \mathrm{BU}_1 \xrightarrow{s} \mathrm{Th}(E_1) = \mathrm{MU}_1,$$

where s is induced by the zero-section of E_1, and the canonical map $\Sigma^\infty \mathrm{MU}_1 \to \Sigma^2\mathrm{MU}$. Moreover, the $\mathbb{C}$-orientation ϑ_E gives rise to a formal group law $F_E \in E_*[[u,v]]$ with $c_1^E(L \otimes M) = F_E(c_1^E(L), c_1^E(M)) \in E^2(X)$ for $L, M \to X$ $\mathbb{C}$-line bundles on some space X. Finally, Quillen's theorem [53, Theorem 6.5.] identifies the formal group law F_{MU} with the universal one, giving the isomorphism $\mathrm{MU}_* \cong \mathbb{L}$.

The theory of *Landweber exact cohomology theories* says that for a formal group law $(R, F \in R[[u,v]])$ with classifying map $\phi_F : \mathbb{L} \to R$, if ϕ is flat over the moduli stack of formal group laws, then there is a $\mathbb{C}$-oriented ring spectrum (E, ϑ_E) representing the functor $X \mapsto \mathrm{MU}^*(X) \otimes_{\mathbb{L}} R$ via the classifying map $\mathrm{MU} \to E$.

This entire theory carries over to the motivic setting, as was shown by Panin, Pimenov, and Röndigs in [44] and Naumann, Spitzweck, and Østvær in [42]. This gives in particular the map $\mathbb{L} \to \mathrm{MGL}^{2*,*}(k)$ mentioned above. To apply this to $\theta_{\mathbb{Q}}$, one notes that the additive formal group law $F(u,v) = u+v \in \mathbb{Q}[[u,v]]$ is Landweber exact, and that the corresponding homomorphism $\mathbb{L} = \mathbb{Z}[x_1, x_2, \ldots] \to \mathbb{Q}$ is the one sending each x_i to zero. This implies that $\mathrm{MGL}_{\mathbb{Q}}/(x_1, x_2, \ldots)$ is the universal oriented theory in $\mathrm{SH}(k)_{\mathbb{Q}}$ with additive group law, and it is not hard to show that $\mathrm{EM}_{\mathrm{mot}}\mathbb{Q} = \mathrm{EM}_{\mathrm{mot}}\mathbb{Z}_{\mathbb{Q}}$ has this property as well.

One then needs to show that θ/ℓ is an isomorphism for each prime ℓ. This relies on an entirely different foundation, the mod ℓ motivic Steenrod algebra; more precisely, one needs a description of $\mathrm{EM}_{\mathrm{mot}}\mathbb{Z}/\ell \wedge \mathrm{EM}_{\mathrm{mot}}\mathbb{Z}/\ell$. For k of characteristic zero, Voevodsky [57] computes $\mathrm{EM}_{\mathrm{mot}}\mathbb{Z}/\ell \wedge \mathrm{EM}_{\mathrm{mot}}\mathbb{Z}/\ell$ as a sum of (bigraded) suspensions of $\mathrm{EM}_{\mathrm{mot}}\mathbb{Z}/\ell$, with generators in the topological degrees identical to those in the classical version, that is, for $\mathcal{A}^\ell_* := \pi_*\mathrm{EM}\mathbb{Z}/\ell \wedge \mathrm{EM}\mathbb{Z}/\ell$. Voevodsky's computation of $\mathrm{EM}_{\mathrm{mot}}\mathbb{Z}/\ell \wedge \mathrm{EM}_{\mathrm{mot}}\mathbb{Z}/\ell$ plays a central role in his proof of the Milnor conjecture and his contribution to the proof of the Bloch–Kato conjectures; his computation in characteristic 0 was extended to positive characteristics by Hoyois, Kelly, and Østvær in [30] after inverting the characteristic. Filling in this missing piece (the motivic mod p Steenrod algebra in characteristic p) remains an important open problem in motivic homotopy theory. In any case, the motivic Steenrod algebra is an essential part of the arguments of Hopkins-Morel and Hoyois to show that θ/ℓ is an isomorphism; for details we refer the interested reader to [29].

Voevodsky's slice tower has become an important tool in a continuing study of the basic structure of the motivic stable homotopy category. Besides the work inspired by his conjectures, the slice tower has been used by Röndigs and Østvær to compute aspects of hermitian algebraic K-theory in [46] and by Röndigs, Spitzweck, and Østvær to make computations of the motivic 1-stem in [47]: the groups $\pi_{n+1,n}(\mathbb{S}_k)(k)$ (the 0-stem $\pi_{n,n}(\mathbb{S}_k)$ has been computed as a sheaf by Morel [41] as his sheaf of *Milnor–Witt K-theory*, using quite different methods). I have used the slice tower to show that the constant functor $\mathrm{SH} \to \mathrm{SH}(k)$ is a fully faithful embedding for k an algebraically closed field of characteristic 0 [38].

3. Cross functors and Grothendieck's six-functor formalism

Grothendieck's *six-functor formalism* refers to constructions in the derived category of étale sheaves as developed in [25], namely, the two pairs of adjoint (derived) functors (f^*, f_*) and $(f_!, f^!)$ as well as the adjoint (derived) bifunctors $(\otimes, \mathcal{H}om)$.

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In a somewhat restricted form, this formalism appears earlier in Grothendieck's theory of duality for quasi-coherent sheaves, as detailed by Hartshorne in [27]. In a series of lectures [19], Voevodsky described an axiomatization of this theory in the setting of a functor

$$H : \mathbf{Sch}/B^{\mathrm{op}} \to \mathbf{Tr}$$

where $\mathbf{Sch}/B$ is some reasonable category of schemes over a reasonable base-scheme B (for example, B is noetherian and separated of finite Krull dimension and $\mathbf{Sch}/B$ is the category of quasi-projective B-schemes). For $f : T \to S$ a morphism in $\mathbf{Sch}/B$, one writes $f^* : H(S) \to H(T)$ for $H(f)$. Voevodsky's axioms are the following.

1. $H(\emptyset) = 0$.
2. For $f : T \to S$ a morphism in $\mathbf{Sch}/B$, f^* admits a right adjoint $f_* : H(T) \to H(S)$, and for $i : T \to S$ a closed immersion the counit $i^*i_* \to \mathrm{Id}$ is an isomorphism.
3. For $f : T \to S$ a smooth morphism in $\mathbf{Sch}/B$, f^* admits a left adjoint $f_\#$ and for each pull-back square

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "p'"'] & X \arrow[d, "p"] \\ Y' \arrow[r, "f"'] & Y \end{tikzcd}$$

the exchange morphism (see below) $p'_\# f'^* \to f^* p_\#$ is an isomorphism
4. Let $i : T \to S$ be a closed immersion with open complement $j : U \to S$. Then the pair $(i^*, j^*) : H(S) \to H(T) \times H(U)$ is conservative.
5. Let $p : \mathbb{A}^1 \times_B S \to S$ be the projection. Then the unit of the adjunction $\mathrm{Id} \to p_*p^*$ is an isomorphism.
6. Let $s : S \to \mathbb{A}^1 \times_B S$ be the zero-section. Then $p_\# \circ s_* : H(S) \to H(S)$ is an autoequivalence.

The "exchange morphism" mentioned in (3) is part of a general theory of natural transformations between compositions of adjoint pairs of functors arising from a commutative diagram, and the (essentially formal) construction of exchange morphisms is discussed at length in Voevodsky's lectures.

Let $i : T \to S$ be a closed immersion in $\mathbf{Sch}/B$. It follows from the axioms that one has a functor $i^! : H(S) \to H(T)$ uniquely defined by requiring that $i_*i^!$ fits into a distinguished triangle

$$i_*i^! \to \mathrm{Id}_{H(T)} \to j_*j^*,$$

where the right-hand map is the unit of the adjunction. It follows as well that $i^!$ is right adjoint to i_*.

This gives us two pairs of adjoint functors: $(i_*, i^!)$ for $i : T \to S$ a closed immersion, and $(p_\#, p^*)$ for $p : T \to S$ a smooth morphism. The main problem is to find a modification of the second pair by an automorphism Ω_f of $H(T)$: $p_! := p_\# \circ \Omega_f^{-1}$, $p^! := \Omega_f \circ p^*$, so that these two pairs are compatible in the following sense. Let $f : T \to S$ be a morphism in $\mathbf{Sch}/B$, and factor f as a composition

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$f = p \circ i$ with i a closed immersion and p a smooth morphism. Then defining

$$f_! := p_! \circ i_*, \qquad f^! := i^! \circ p^!,$$

the morphisms $f_!$, $f^!$ are independent of the choice of factorization of f.

I am not sure to what extent Voevodsky discussed the details of the solution of this problem in his lectures; in the notes now available, there are no details given. Ayoub has given a complete treatment of the solution, as well as a detailed discussion of such topics as exchange morphisms and other foundational material necessary for this program in [5, 6]. This covers four of the six Grothendieck operations. The remaining two, tensor and internal Hom, along with related questions, such as constructibility and duality, are also handled in Ayoub's treatment. One can safely say that, through Ayoub's treatment of the subject, Voevodsky's formulation leads to a completely axiomatic treatment of the Grothendieck six-functor formalism.

In addition, Ayoub uses the six-functor formalism to define an abstract nearby cycles functor in the setting of a scheme over the fraction field of a discrete valuation ring.

Crucially, Ayoub verifies the axioms in the case of the functor $\mathrm{SH}(-) : \mathbf{Sch}/B \to \mathbf{Tr}$, that is, he makes available the six-functor formalism for the motivic stable homotopy category. I want to mention at this point that inverting the $\mathbb{P}^1$-suspension functor $\Sigma_{\mathbb{P}^1}$ is absolutely necessary for verifying axiom (6), giving a convincing structural reason for passing to the category of $\mathbb{P}^1$-spectra.

The six-functor formalism has pervaded the entire motivic theory, in that it has become the "gold standard" for testing proposals for constructions of various types of motivic categories. As mentioned above, the motivic stable homotopy category $\mathrm{SH}(-)$ has passed the test. Hoyois [28] has extended the construction of $\mathrm{SH}(-)$ to the G-equivariant setting, satisfying the six-functor formalism, for the so-called "tame" groups G, and Cisinski and Khan in [9] and Khan in [36] have extended the theory, complete with six operations, in a different direction, to the setting of derived schemes. These last two constructions are in the framework of infinity categories.

The six-functor formalism is extremely useful. To mention a few applications, Ayoub and Zucker [4] rely on this to give a motivic structure to the cohomology of the (nonalgebraic) reductive Borel–Serre compactification of a hermitian locally symmetric variety. Constructions of motivic fundamental classes (Déglise, Jin, and Khan [18]) and virtual fundamental classes (Levine [37]) rely heavily on the six-functor formalism.

One can thus say that for $\mathrm{SH}(-)$ and its generalizations, the theory is in very good shape. However, for the triangulated category of motives $\mathrm{DM}(-)$, the situation is still, in general, not resolved.

If one works with rational coefficients, there are a number of equivalent constructions. One, due to Cisinski and Déglise [12], relies on identifying rational motivic cohomology with the Adams eigenspaces in algebraic K-theory. Representing algebraic K-theory in the motivic stable homotopy category $\mathrm{SH}(S)$ over S via the $\mathbb{P}^1$-spectrum KGL, the authors decompose $\mathrm{KGL}_{\mathbb{Q}}$ in $\mathrm{SH}(S)_{\mathbb{Q}}$ via the Adams operations, define the rational motivic cohomology spectrum $\mathrm{EM}_{\mathrm{mot}}\mathbb{Q}$ as weight 0 part, $\mathrm{KGL}^{(0)}_{\mathbb{Q}}$, and define $\mathrm{DM}(S)_{\mathbb{Q}}$ as the homotopy category of $\mathrm{KGL}^{(0)}_{\mathbb{Q}}$-modules.

Another approach is closer to the construction of DM sketched above. Suslin and Voevodsky define a general theory of equidimensional cycles over an arbitrary base

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[65, Chapter 2]. This leads to a category of finite correspondences over S, Cor_S, and the categories $\mathrm{DM}^{\mathrm{eff}}(S)$ and $\mathrm{DM}(S)$, this program is carried out by Cisinski and Déglise [12]. These constructions agree rationally with the ones using $\mathrm{KGL}^{(0)}_{\mathbb{Q}}$ if S is a unibranch scheme over a field.

Yet another approach is to use an "$\mathbb{A}^1$-derived category in the étale topology". Following ideas of Morel, Ayoub has given a treatment of this approach in [3]; see also [2]. This approach has also been taken by Cisinski and Déglise in [11]. Here the basic idea is to form a triangulated category of "motives without transfer" by replacing the category of correspondences over the base-scheme S with the $\mathbb{Z}$-linear extension of the category of smooth (possibly also quasi-projective) S-schemes. Equivalently, one uses the category of presheaves of abelian groups on $\mathbf{Sm}/S$ which takes disjoint union to products. One then follows the same script as for $\mathrm{DM}^{\mathrm{eff}}(k)$ and $\mathrm{DM}(k)$ only in the étale topology, namely, localize with respect to $\mathbb{A}^1$-invariance and étale descent, giving the category $DA^{\mathrm{eff}}_{\text{ét}}(S)$; a category of $\mathbb{Z}_{\text{ét}}(1)[2]$ spectra gives the category $DA_{\text{ét}}(S)$. Remarkably, if S is, for instance, a smooth scheme over a perfect field, the $\mathbb{Q}$-localizations of $DA_{\text{ét}}(S)$ and $\mathrm{DM}(S)$ are equivalent. Somehow the étale topology endows $DA_{\text{ét}}(S)_{\mathbb{Q}}$ with transfers without imposing them from the start.

One can use other topologies, such as Voevodsky's h-topology, and follow the same line. This method is also considered in [11] along with the étale version.

In any case, the functor $DA_{\text{ét}}(-) : \mathbf{Sch}/B^{\mathrm{op}} \to \mathbf{Tr}$ does satisfy the six-functor formalism. However, even though $DA_{\text{ét}}(k)_{\mathbb{Q}}$ is equivalent to $\mathrm{DM}(k)_{\mathbb{Q}}$ for k a perfect field, the étale version does not agree with $\mathrm{DM}(k)$ integrally.

The problem in defining a good integral theory satisfying the six-functor formalism can be rephrased as the problem of defining for each base-scheme S a "motivic cohomology spectrum" $\mathrm{EM}_{\mathrm{mot}}\mathbb{Z}_S$ with the cartesian property, that is, for each morphism $f : T \to S$, one has a natural isomorphism $f^*\mathrm{EM}_{\mathrm{mot}}\mathbb{Z}_S \to \mathrm{EM}_{\mathrm{mot}}\mathbb{Z}_T$. There is a good model for this motivic cohomology spectrum over a field k of characteristic 0, relying on the theorem of Röndigs and Østvær identifying $\mathrm{DM}(k)$ with the homotopy category of $\mathrm{EM}_{\mathrm{mot}}\mathbb{Z}_k$-modules in this case [48].

There are currently a number of candidates for $\mathrm{EM}_{\mathrm{mot}}\mathbb{Z}_S$, all of which have some difficulties. One relies on Voevodsky's slice tower and the isomorphism $\mathrm{EM}_{\mathrm{mot}}\mathbb{Z}_k \cong s_0(\mathbb{S}_k)$ for k a perfect field. There is a slice tower in $\mathrm{SH}(S)$ for arbitrary S, and one could simply define $\mathrm{EM}_{\mathrm{mot}}\mathbb{Z}_S$ as $s_0(\mathbb{S}_S)$, where $\mathbb{S}_S$ is the sphere spectrum in $\mathrm{SH}(S)$. This works for S smooth over a perfect field, but for arbitrary S, T and an arbitrary morphism of schemes $f : T \to S$, one does not know that the canonical map $f^*s_0\mathbb{S}_S \to s_0\mathbb{S}_T$ is an isomorphism.

Spitzweck [49] has defined a motivic cohomology spectrum in mixed characteristic with a long list of good properties, however, the cartesian property seems to still be open.

In short, the path forged by Voevodsky in his construction of a triangulated category of motives continues to be well traveled to this day.

4. Framed correspondences and recognition principles

One conjecture from [62] we have not yet mentioned involves the motivic S^1-stable homotopy category. This is in a sense the homotopy-theoretic version of the category of effective motives $\mathrm{DM}^{\mathrm{eff}}(k)$. Just a classical spectrum is a sequence of pointed spaces $(E_0, E_1, \ldots)$ with bonding maps $E_n \wedge S^1 \to E_{n+1}$, the category of

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S^1-spectra over a scheme S, $\mathrm{Sp}_{S^1}(S)$, is the category of spectrum objects $E := (E_0, E_1, \ldots)$, $E_n \in \mathbf{Spc}_\bullet(S)$, with bonding maps $\epsilon_n : E_n \wedge S^1 \to E_{n+1}$. This is the same as the category of presheaves of classical spectra on $\mathbf{Sm}/S$. One has a notion of stable homotopy sheaves for $a \in \mathbb{Z}$,

$$\pi_a^{\mathbb{A}^1}(E) = \mathrm{colim}_N \pi_{a+N}^{\mathbb{A}^1}(E_N),$$

and one forms the homotopy category $\mathrm{SH}_{S^1}(S)$ by inverting maps $f : E \to F$ that induce isomorphisms on $\pi_a^{\mathbb{A}^1}(-)$ for $a \in \mathbb{Z}$, $b \geq 0$.

One can form a $\mathbb{P}^1$-spectrum $\Sigma^\infty_{\mathbb{G}_m} E$ out of an S^1 spectrum E by (roughly speaking) replacing E_n with $\Sigma^n_{\mathbb{G}_m} E_n$ and defining

$$\tilde{\epsilon}_n : \Sigma^n_{\mathbb{G}_m} E_n \wedge \mathbb{P}^1 \to \Sigma^{n+1}_{\mathbb{G}_m} E_{n+1}$$

as the composition

$$\Sigma^n_{\mathbb{G}_m} E_n \wedge \mathbb{P}^1 \cong \Sigma^n_{\mathbb{G}_m} E_n \wedge \mathbb{G}_m \wedge S^1 \cong \Sigma^{n+1}_{\mathbb{G}_m}(E_n \wedge S^1) \xrightarrow{\Sigma^{n+1}_{\mathbb{G}_m} \epsilon_n} \Sigma^{n+1}_{\mathbb{G}_m} E_{n+1}.$$

One should rather invoke an intermediate category of S^1-$\mathbb{G}_m$-spectra to make this work, but we avoid these technicalities here.

In any case, one has two commutative triangles of infinite suspension functors and their right adjoints, the infinite loop space functors

$$\begin{tikzcd}
\mathcal{H}_\bullet(S) \arrow[rr, shift left, "\Sigma^\infty_{\mathbb{P}^1}"] \arrow[dr, shift right, "\Sigma^\infty_{S^1}"'] & & \mathrm{SH}(S) \arrow[ll, shift left, "\Omega^\infty_{\mathbb{P}^1}"] \arrow[dl, shift right, "\Omega^\infty_{\mathbb{G}_m}"'] \\
& \mathrm{SH}_{S^1}(S). \arrow[ul, shift right, "\Omega^\infty_{S^1}"'] \arrow[ur, shift right, "\Sigma^\infty_{\mathbb{G}_m}"'] &
\end{tikzcd}$$

We have as well the $\mathbb{G}_m$-loop space functor $\Omega_{\mathbb{G}_m}$ on $\mathcal{H}_\bullet(S)$ and on $\mathrm{SH}_{S^1}(S)$.

Replacing $\mathrm{SH}^{\mathrm{eff}}(S)$ with the entire category $\mathrm{SH}_{S^1}(S)$, we have the sequence of full triangulated subcategories (closed under arbitrary direct sums)

$$\cdots \subset \Sigma^{q+1}_{\mathbb{P}^1}\mathrm{SH}_{S^1}(S) \subset \Sigma^{q}_{\mathbb{P}^1}\mathrm{SH}_{S^1}(S) \subset \cdots \subset \Sigma^{1}_{\mathbb{P}^1}\mathrm{SH}_{S^1}(S) \subset \mathrm{SH}_{S^1}(S)$$

and the corresponding slice tower

$$\cdots \to f^{q+1}_{S^1} E \to f^{q}_{S^1} E \to \cdots \to f^{1}_{S^1} E \to f^{0}_{S^1} E = E.$$

In topology, one has the Freudenthal suspension theorem which implies that, for X a path connected and $(q-1)$-connected pointed space, the loop space of the suspension, $\Omega\Sigma X$ is also $(q-1)$-connected. Voevodsky conjectured the analogue for $\mathbb{P}^1$-connectedness in $\mathrm{SH}_{S^1}(k)$.

Conjecture 6 ([63, Conjecture 5], [62, Conjecture 16])**.** *Let k be a perfect field. For $E \in \Sigma^q_{\mathbb{P}^1}\mathrm{SH}_{S^1}(k)$, $q \geq 1$, $\Omega_{\mathbb{P}^1} E$ is in $\Sigma^{q-1}_{\mathbb{P}^1}\mathrm{SH}_{S^1}(k)$.*

I proved in this in [39], but the reason I mention this conjecture here is to recall a comment that Voevodsky makes concerning this conjecture.

We recall the problem of the *recognition principle*: How does one tell if a space is an n-fold loop space? There are solutions to this in topology, relying on the existence of a suitable operad. In [63], Voevodsky suggests that there should be a

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$\mathbb{P}^1$-version of the James model:

> $\Omega\Sigma(X)$ has a model (James construction) possessing a filtration whose quotients are $X^{\wedge i}$. This is the starting point of the operadic theory of loop spaces and it appears that any such theory for $\mathbb{P}^1$-loop spaces in $\mathrm{SH}_{S^1}(k)$ would provide a proof of Conjectures 4 and 5.

He repeats very much the same thing following [62, Conjecture 16]:

> This conjecture says that for a space (X, x), the space $\Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1}\Sigma^n_{\mathbb{P}^1}(X, x)$ can be built, at least S^1-stably, from n-fold $\mathbb{P}^1$-suspensions. It connects the theme of this paper to another bunch of conjectures describing the hypothetical theory of operadic description of $\mathbb{P}^1$-loop spaces. Any such theory should provide a model for $\Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1}$ which could then be used to prove Conjecture 16.

In unpublished notes [68] Voevodsky gives an approach to provide a model for the *infinite* loop space functor $\Omega^\infty_{\mathbb{G}_m} : \mathrm{SH}^{\mathrm{eff}}(k) \to \mathrm{SH}_{S^1}(k)$, via a refinement of his theory of transfers. This goes back to Pontryagin's description of the stable homotopy groups of spheres via framed bordisms: If we have a C^∞-map of spheres $f : S^{m+n} \to S^m$ that gives a nontrivial element of $\pi_n(\mathbb{S})$, then f must be generically a submersion, and for a general point $x \in S^m$, the fiber $M := f^{-1}(x) \subset S^{m+n}$ will be an n-dimensional manifold; as a fiber of a C^∞-map, a trivialization of $T_x(S^m)$ will induce a trivialization of the normal bundle of M in S^{n+m}. A suitably differentiable homotopy of maps, $f \sim f'$ will give a bordism of the fibers M and M', which is compatible with the trivialization of the normal bundles. And in fact one ends up with an isomorphism of $\pi_n(\mathbb{S})$ with this framed bordism group.

In the simplest case, we are talking about zero-dimensional manifolds, i.e., points. If we have a spectrum E and a collection of m points $\{p_1, \ldots, p_m\}$, as SH is an additive category, we get a map $\bigoplus_{p_u} E \to E$. This will lift to a map of spectra but will only satisfy the axioms of addition in an abelian group up to homotopies and all higher homotopies, and this type of structure will pass to the infinite loop space $\Omega^\infty E$. There are a number of ways way of encoding this structure, one is via the notion of a Segal space.

Segal's category Γ has as objects finite sets and morphisms $\phi : S \to T$ given by associating to each $s \in S$ a subset $\phi(s) \subset T$, such that $\phi(s)$ and $\phi(s')$ are disjoint for $s \neq s'$. Such a ϕ is defined by its graph $\Gamma_\phi \subset S \times T$ consisting of pairs (s, t) with $s \in S$ and $t \in \phi(s)$. Composition is $(\psi \circ \phi)(s) := \bigcup_{t \in \phi(s)} \psi(t)$, and one has for $\phi : S \to T$, $\psi : T \to U$,

$$\Gamma_{\psi\circ\phi} = p_{\mathrm{SU}}(\Gamma_\phi \times U \cap S \times \Gamma_\psi).$$

Thus, one may think of Γ as a category of correspondences on finite sets.

A Γ-space is a functor $X : \Gamma^{\mathrm{op}} \to \mathbf{Spc}$, that is, an $\mathbf{Spc}$-valued presheaf on Γ. X is called *special* if the map $X(\{1\})^n \to X(\{1, \ldots, n\})$ induced by the correspondences

$$\phi_i(j) := \begin{cases} \emptyset & \text{for } i \neq j, \\ 1 & \text{for } i = j, \end{cases}$$

is a weak equivalence of spaces and $X(\emptyset)$ is contractible. One can interpret the special Γ-space condition as a locality condition with respect to a certain topology

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on Γ. For a special Γ-space, the maps $X(\{1,\dots,n\}) \to X(1)$, defined by the correspondence $\phi_n(1) = \{1,\dots,n\}$ and the map $X(\emptyset) \to X(1)$ given by the unique correspondence $\{1\} \to \emptyset$, make $\pi_0(X(1))$ a monoid. Call X *group-like* if $\pi_0(X(1))$ is a group. Segal shows that the homotopy category of group-like special Γ-spaces is equivalent to the homotopy category of connective spectra. One beautiful application is a proof of the theorem of Barratt, Priddy, and Quillen identifying the group completion of $\amalg_n B\Sigma_n$ with the infinite loop space of the sphere spectrum.

Voevodsky modified Segal's category Γ in view of Pontryagin's framed bordisms to give a category of *framed correspondences* $\mathrm{Fr}_*(k)$ over a field k. Roughly speaking, a framed correspondence from a smooth k-scheme X to a smooth k-scheme Y consists of the following data:

i. A closed subset Z of $X \times \mathbb{A}^n$, with each component of Z finite over X.
ii. An étale morphism $\pi : U \to X \times \mathbb{A}^n$ with a section $s : Z \to U$ over Z and a morphism $g : U \to Y$.
iii. A "framing", that is, a morphism $\phi : U \to \mathbb{A}^n$ such that $s(Z) = \phi^{-1}(0)$ as closed subsets. There is an equivalence relation among such tuples (Z, π, g, ϕ) and a composition law, reminiscent of the composition law in Cor_k.

This is perhaps a bit complicated. Recently, Elmanto, Hoyois, Khan, Sosnilo, and Yakerson [20] defined a *coordinate free* version of $\mathrm{Fr}_*(k)$, $\mathrm{Corr}^{fr}(k)$ (as an ∞-category) for which a framed correspondence $X \to Y$, $X, Y \in \mathbf{Sm}/k$, is a syntomic morphism $f : Z \to X$, a morphism $g : Z \to Y$, and a framing $\psi : [L_{Z/X}] \rightsquigarrow 0$. To explain, a finite morphism $f : Z \to X$ is syntomic if f is flat and locally over X, f is a closed immersion $Z \to \mathbb{A}^n \times \mathrm{Spec}\, A$, with ideal I_Z generated by n elements in $A[X_1,\dots,X_n][1/g]$ for some g a unit modulo I_Z. The framing ψ involves the relative cotangent complex $L_{Z/X} \in D^b(X)$, and $[L_{Z/X}] \rightsquigarrow 0$ means a path in the K-theory space $K(Z)$ connecting 0 with the point $[L_{Z/X}] \in K(X)$ corresponding to $L_{Z/X}$.

This last is a bit technical, but one can make this somewhat more explicit in the situation of $Z \subset \mathbb{A}^n \times X$ a closed regularly embedded subscheme, finite and surjective over X. In this case $L_{Z/X}$ is the two-term complex

$$L_{Z/X} : \mathcal{I}_Z/\mathcal{I}_Z^2 \xrightarrow{d} p_1^*\Omega_{\mathbb{A}^n}.$$

A framing $\phi : U \to \mathbb{A}^n$ in the previous sense, $\phi = (\phi_1,\dots,\phi_n)$ will give an isomorphism $d\phi : \mathcal{I}_Z/\mathcal{I}_Z^2 \to p_1^*\Omega_{\mathbb{A}^n}$ sending $\phi_i \mod \mathcal{I}_Z^2$ to dX_i. Then $\tilde{\phi}$ will define a trivialization of the image of $L_{Z/X}$ in $K_0(Z) = \pi_0(K(Z))$, giving the path $[L_{Z/X}] \rightsquigarrow 0$ in $K(Z)$.

Based on Voevodsky's ideas and notes, Garkusha and Panin [24], relying on work by Garkusha, Neshitov, and Panin [23] and by Ananyevskiy, Garkusha, and Panin [1], give an explicit expression for the S^1-spectrum $\Omega^\infty_{\mathbb{G}_m}\Sigma^\infty_{\mathbb{P}^1}X_+$, $X \in \mathbf{Sm}/k$; for $X = \mathrm{Spec}\, k$, this may be viewed as a motivic version of the Barratt, Priddy, and Quillen theorem. In the work of Elmanto, Hoyois, Khan, Sosnilo, and Yakerson, this is refined to give a true parallel of Segal's description of connective spectra as special group-like Γ-spaces, describing the homotopy category $\mathrm{SH}^{\mathrm{eff}}(k)$ of effective $\mathbb{P}^1$-spectra as the group-like presheaves of (usual) spectra on $\mathrm{Corr}^{fr}(k)$ that are $\mathbb{A}^1$-invariant and satisfy descent for the Nisnevich topology. This leads to a description of the motivic sphere spectrum in terms of an $\mathbb{A}^1$-Nisnevich-localization of a Hilbert scheme of framed points in $\mathbb{A}^\infty$, the latter being the colimit over n of the Hilbert

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scheme of dimension 0 lci closed subschemes Z of $\mathbb{A}^n$ with trivialization of the conormal bundle $\mathcal{I}_Z/\mathcal{I}_Z^2$. This identification is described by the authors as "an algebro-geometric analogue of the description of the topological sphere spectrum in terms of framed 0-dimensional manifolds and cobordisms".

5. Conclusion

I hope I have given the reader unfamiliar with the scope of Voevodsky's work in motivic homotopy theory a hint of the enormous contribution Voevodsky has made to this field. As I am sure has become apparent in this article, I personally owe Voevodsky a huge and continuing debt in terms of his influence on my own work, and I am heartened to see the ever-increasing numbers of young mathematicians who use the tools and ideas Voevodsky has fashioned to answer questions that fascinate us all.

Acknowledgments

The author thanks Maria Yakerson for her careful reading and helpful suggestions and Oliver Röndigs for his interesting comments on conversations with Voevodsky.

About the author

Marc Levine has been professor of algebraic geometry at the Universität Duisburg-Essen since 2009. Before that he had been a member of the department of mathematics at Northeastern University since 1984. He has written research papers on topics in algebraic geometry, algebraic K-theory, the theory of algebraic cycles and motivic cohomology, and most recently in motivic homotopy theory.

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Fakultät Mathematik, Universität Duisburg-Essen, Thea-Leymann-Str. 9, 45127 Essen, Germany

Email address: `marc.levine@uni-due.de`

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