

The endoscopic classification of representations—orthogonal and symplectic groups,
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This is not an ordinary mathematics book. It is a research monograph of the highest level on one of the deepest conjectures in number theory. It requires not only much of Arthur’s earlier work, but that of many others; a long and complicated project which has finally been concluded with great success in this book. I start with some history and review.

In a groundbreaking article, *Problems in the theory of automorphic forms*, which appeared in 1970, Robert Langlands proposed a number of questions in the theory of automorphic forms (cf. [L1]). They included a new framework for defining a general class of L -functions which included all those coming from classical theory, and their properties (analytic continuation, functional equations, . . .) were conjectured. The new framework allowed him to formulate one of the most profound and most general conjectures about these L -functions now known as the Langlands Functoriality Principle.

To explain the functoriality principle, one needs to think of automorphic forms as representations of groups over number fields $[BJ, G]$. It is not hard to represent any of the classical or Siegel modular forms by functions in $L^2(\mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A}))$ or $L^2(\mathrm{Sp}_{2n}(\mathbb{Q})\backslash\mathrm{Sp}_{2n}(\mathbb{A}))$, respectively, where \mathbb{A} is the ring of adèles of the field of rational numbers \mathbb{Q} . Decomposing this Hilbert space under the regular action of $G(\mathbb{A})$ to its irreducible constituents, where $G = \mathrm{GL}_2$ or Sp_{2n} , gives a sum of discrete irreducible subspaces and a direct integral, the so-called *automorphic representations* of $G(\mathbb{A})$. Classical cusp forms, which are eigenfunctions for all Hecke operators as well as the Laplacian, will then provide us with *cuspidal automorphic representations*, the most mysterious part of this decomposition. If π is an (irreducible) automorphic representation of $G(\mathbb{A})$, then $\pi = \bigotimes_p \pi_p$, a *restricted* tensor product of irreducible representations of $G(\mathbb{Q}_p)$, with p a rational prime. It is restricted in the sense that, for almost all p , the irreducible representation π_p has a vector invariant under the action of $G(\mathbb{Z}_p)$, a maximal compact subgroup of $G(\mathbb{Q}_p)$, where \mathbb{Z}_p is the ring of p -adic integers. When $G = \mathrm{GL}_2$, a representation π_p that has a vector fixed by $G(\mathbb{Z}_p)$, usually called a *spherical* or *unramified* representation, is parametrized by a semisimple conjugacy class, which is given by a unique diagonal matrix in $\mathrm{GL}_2(\mathbb{C})$. For the group $G = \mathrm{Sp}_{2n}$, one has to consider a diagonal matrix in its L -group, ${}^L G = \mathrm{SO}_{2n+1}(\mathbb{C})$, to parametrize an unramified representation. For either case, each diagonal matrix (semisimple conjugacy class) in ${}^L G$ determines a unique isomorphism class of irreducible unramified representations of $G(\mathbb{Q}_p)$.

The notion of the L -group ${}^L G$ of a general connected reductive group G over \mathbb{Q} , a closed connected subgroup of some $\mathrm{GL}_N(\overline{\mathbb{Q}})$ whose connected center consists of only semisimple elements, was defined by Langlands in the same paper (originally in [L2] for split groups). It is a disconnected complex group ${}^L G = \hat{G} \times \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ with \hat{G} a complex connected reductive group whose root datum (cf. [B]) is dual to that of G over $\overline{\mathbb{Q}}$, examples of which we just saw. The action of the Galois group

$\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is there when the group G is not split over \mathbb{Q} . In the cases discussed earlier, $\tilde{G} = \text{GL}_2(\mathbb{C})$ or $\text{SO}_{2n+1}(\mathbb{C})$ with trivial action of the Galois group, as both GL_2 and Sp_{2n} have their diagonal subgroups as their maximally split tori.

Now let G and G' be two connected reductive groups over \mathbb{Q} . For simplicity of exposition, we assume $G' = \text{GL}_N$ for some N . Let $\phi : {}^L G \rightarrow \text{GL}_N(\mathbb{C}) \times \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ be a homomorphism. Let $\pi = \bigotimes_p \pi_p$ be a cuspidal representation of $G(\mathbb{A})$. Then for almost all p , π_p is given by a semisimple conjugacy class A_p . Consider $\phi(A_p) \subset \text{GL}_N(\mathbb{C})$. Its conjugacy class will define a spherical representation π'_p of $\text{GL}_N(\mathbb{Q}_p)$. The *Langlands Functoriality Principle* predicts the existence of an irreducible representation π'_p of $\text{GL}_N(\mathbb{Q}_p)$ for every other p , including one for $\text{GL}_N(\mathbb{R})$, such that $\pi' = \bigotimes_p \pi'_p$ appears in $L^2(\text{GL}_N(\mathbb{Q}) \backslash \text{GL}_N(\mathbb{A}))$.

This is an extremely deep conjecture which is still far from being resolved in general, even for $G' = \text{GL}_N$, and with the exception of the book under review (see also [CKPSS1, CKPSS2] for generic cases, among others), it is known only in a few sporadic but important low-rank cases [GJ, KSh, K, Ra].

The great accomplishment of Arthur in this book is to prove functionality for the natural embedding $\phi : {}^L G \hookrightarrow \text{GL}_N(\mathbb{C})$ of the L -group of a classical group G , when $G = \text{Sp}_{2n}$, SO_n^* , and all their *inner forms*. Here SO_n^* is a quasi-split special orthogonal group of rank n which can be nonsplit only if n is even. (We should remark that when cuspidal representations on the classical group are globally generic [Sh2], the conjecture was already proved in [CKPSS1, CKPSS2], using converse theorems, a method disjoint from the tools used by Arthur.)

Arthur uses a major tool, the Arthur–Selberg trace formula and its twisted versions, which he developed throughout his career, with great success.

The trace formula is an equality of distributions on $G(\mathbb{A})$. More precisely, it is an identity of the form

$$\sum_{\chi} J_{\chi}(f) = \sum_{\sigma} J_{\sigma}(f),$$

$f \in C_c^{\infty}(G(\mathbb{A}))$, where the sum on the left is parametrized by spectral data χ consisting of conjugacy classes of parabolic subgroups and cuspidal automorphic representations on their Levi subgroups. This is usually called the *spectral side*. The sum on the right, the *geometric side*, is parametrized by semisimple conjugacy classes in $G(\mathbb{Q})$. We refer to [A1] for an excellent presentation of the trace formula and related topics by Arthur himself. Any use of the trace formula to prove a case of functoriality is a matter of comparing the spectral sides of appropriate trace formulas for G and G' , by proving suitable identities on geometric sides, called *fundamental lemmas* [L3]. The original versions of the trace formula were for connected reductive groups [A2, A3]. Langlands’s proof of the holomorphy of Artin L -functions on all of \mathbb{C} for certain two-dimensional irreducible representations of the Galois group with solvable image (cases of the Artin conjecture [L4]), introduced the notion of the *twisted trace formula* for $\text{GL}(2)$. More precisely, if we consider the case where the Galois group Γ is that of a cyclic extension E/\mathbb{Q} of prime order l , the building blocks of solvable extensions, then one can consider the disconnected group $\tilde{G} = \text{GL}_2(E) \rtimes \Gamma$ and develop a trace formula. This again leads to a case of functoriality, called *base change*. It was generalized by Arthur and Clozel [AC] to GL_n , and was the topic of the “Morning Seminar” at the Institute for Advanced Study during the 1983–84 Special Year, presented by Clozel,

Labesse, and Langlands, which is now the subject matter of a book by Labesse and Waldspurger [LW].

Proof of functoriality from classical groups to GL_N , the main topic of the book under review, is accomplished by comparing trace formulas of classical groups with the twisted trace formula for GL_N under the automorphism $\theta(g) = {}^t g^{-1}$, $g \in \mathrm{GL}_N$, or the trace formula for the disconnected group $\tilde{G} = \mathrm{GL}_N \rtimes \langle \theta \rangle$. The θ -twisted trace formula vanishes on the spectrum $L^2(G(\mathbb{Q}) \backslash G(\mathbb{A}))$, $G = \mathrm{GL}_N$, unless the representation is fixed by θ , i.e., is self-dual. Thus comparing trace formulas of classical groups with the θ -twisted trace formula for GL_N transfers cuspidal automorphic representations of different classical groups $\mathrm{Sp}(\mathbb{A})$ and $\mathrm{SO}(\mathbb{A})$ to self-dual automorphic representations of $\mathrm{GL}_N(\mathbb{A})$ for suitable N . In fact, $N = 2n$ for $G = \mathrm{SO}_{2n+1}$ or SO_{2n} , and $N = 2n + 1$ for $G = \mathrm{Sp}_{2n}$. This proves functoriality for the embedding ${}^L G \hookrightarrow \mathrm{GL}_N(\mathbb{C}) \times \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

The great accomplishment of the book, a consequence of this transfer, is to reduce representation theory of classical groups to that of self-dual representations of GL_N , both locally and globally. This is of course easier said than done. It requires most of Arthur's work on developing his trace formula. One complication, which is quite general and happens for any other group but GL_N , is the presence of *L-packets* and *stability*. In fact, functoriality is not an injection on its domain and there may be more than one cuspidal representation on a classical group that goes to the same self-dual representation of $\mathrm{GL}_N(\mathbb{A})$ with N as above. Roughly speaking, the preimage of a given self-dual representation of $\mathrm{GL}_N(\mathbb{A})$ makes up an *L-packet* on corresponding $G(\mathbb{A})$. Of course, the *L-packets* are intrinsically attached to the group $G(\mathbb{A})$ and come from *stabilization* of its trace formula. Stabilization is the process of writing each side of the trace formula as a sum of *stable* distributions of *shortest length*. A distribution T is *stable* if $T(f^g) = T(f)$, $f \in C_c^\infty(G(\mathbb{A}))$, where $f^g(x) = f(g^{-1}xg)$, $x \in G(\mathbb{A})$, and $g \in G(\overline{\mathbb{A}})$, $\overline{\mathbb{A}} = \mathbb{A}_{\overline{\mathbb{Q}}}$, with $\overline{\mathbb{Q}}$ algebraic closure of \mathbb{Q} , so that conjugation $\mathrm{Int}(g)$ is defined over \mathbb{Q} , i.e., those $g \in G(\overline{\mathbb{A}})$ for which $g^{-1}xg \in G(\mathbb{A})$ for $x \in G(\mathbb{A})$.

The first case of stability was introduced in the work of Labesse and Langlands [LL] for $G = \mathrm{SL}_2$, an idea which has its roots in the theory of Shimura varieties, and here is where the notion of *L-packets* was first defined. Already for $\mathrm{SL}_2(\mathbb{R})$ the stable conjugacy is different from conjugacy. For example,

$$\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

gives a pair in $\mathrm{SL}_2(\mathbb{R})$ which are conjugate by an element in $\mathrm{SL}_2(\mathbb{C})$, but not $\mathrm{SL}_2(\mathbb{R})$, i.e., are stably conjugate but not conjugate. On the other hand, conjugation by $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R})$ will put the pair in the same conjugacy class in $\mathrm{GL}_2(\mathbb{R})$. Generalization of these concepts to an arbitrary connected reductive group already for the regular trace formula has required the efforts of not only Arthur, but many others, notable among them are Langlands, Shelstad, Kottwitz [L3, L7, L8, S, KS].

Stabilization of the trace formula is done inductively through *endoscopic groups*. They are quasi-split reductive groups H whose *L-groups* ${}^L H$ are embedded into ${}^L G$. Their connected *L-groups* ${}^L H^0 = \hat{H}$ are defined as connected centralizers of semisimple elements which, with appropriate action of Γ , give the *L-group* ${}^L H$. Those of interest would be the *elliptic* ones, i.e., those not contained in any parabolic subgroup (cf. [Ha] for their classification). The functoriality coming from the

embeddings ${}^L H \subset {}^L G$ can be established by transferring functions $f \in C_c^\infty(G(\mathbb{A}))$, $f = \otimes_p f_p$, to functions $f^H \in C_c^\infty(H(\mathbb{A}))$, $f^H = \otimes_p f_p^H$ with *matching* orbital integrals over *strongly regular* (regular semisimple elements with connected centralizers) stable conjugacy classes in $H(\mathbb{Q}_p)$ and those in $G(\mathbb{Q}_p)$ for each p , by means of certain *transfer factors* defined by Langlands and Shelstad [L7, L8] for orbital integrals on $G(\mathbb{Q}_p)$. Roughly speaking, “matching” means equality of the sum of orbital integrals of f_p^H over conjugacy classes within the stable conjugacy class of a strongly regular element γ_H in $H(\mathbb{Q}_p)$ with a linear combination of orbital integrals of f_p over conjugacy classes within the stable conjugacy class of a strongly regular element γ in $G(\mathbb{Q}_p)$, whose stable conjugacy class determines that of γ_H through a *norm* or *image* map, for every such γ . The coefficients of the linear combination are the transfer factors. These are objects whose products over all primes appear in the geometric side of the stable trace formula. Dual to this transfer of functions is a matching of stable characters of irreducible representations of $H(\mathbb{Q}_p)$ with a linear combination of characters of irreducible representations of $G(\mathbb{Q}_p)$ for each p ; and putting them together, a matching for adelic points. The L -packets and their nontempered counterparts, A -packets [A4, ABV], which are essential to stabilization, on $G(\mathbb{Q}_p)$ and $G(\mathbb{A})$ are those representations whose characters appear in the matching, locally and globally. Assuming inductively that the trace formula on each $H(\mathbb{A})$ is stabilized, the *endoscopic transfer* allows one to remove the transfer of its spectral side from the spectral side of $G(\mathbb{A})$. When done for all elliptic H , what remains will be stable on $G(\mathbb{A})$, and stabilization on $G(\mathbb{A})$ is achieved.

This is a massive undertaking. But in a series of papers [A5, A6, A7], Arthur has achieved the stabilization for the regular trace formula discussed earlier, conditionally based on validity of the *fundamental lemma* for endoscopy which requires the matching of orbital integrals discussed earlier for characteristic functions of $H(\mathbb{Z}_p)$ and $G(\mathbb{Z}_p)$ for all p . This “lemma”, a term coined by Langlands as such [L3], despite the efforts of many experts, had to wait until Bao Châu Ngô proved it using algebraic geometry in [N], a result which earned him the Fields Medal in 2010.

One still has to do what Arthur did in the regular case for the twisted case: stabilize the twisted trace formula for GL_N . This is now accomplished by Mœglin and Waldspurger for any disconnected reductive group in two volumes [MW2, MW3].

With this background in hand, let me now give an overview of the book.

The classification of representations of classical groups is done by means of *parametrization* introduced in Chapter 1. To appreciate this, one has to think in terms of generalizations of the Artin reciprocity law. The idea is to parametrize irreducible representations of $G(\mathbb{Q}_p)$ and $G(\mathbb{A})$ in terms of arithmetic objects, or more precisely, equivalence classes of homomorphism from the Weil–Deligne group $W'_{\mathbb{Q}_p}$ into ${}^L G$, when we consider $G(\mathbb{Q}_p)$, or its conjectured replacement $L_{\mathbb{Q}}$, when dealing with $G(\mathbb{A})$. Note that when $G = \mathrm{GL}_N$, parameters would be N -dimensional representations of $W'_{\mathbb{Q}_p}$, and (conjectural) $L_{\mathbb{Q}}$, an exact generalization of Artin’s reciprocity, where $G = \mathrm{GL}_1$.

Arthur uses representation theory of GL_N over local and global fields, which is now rather well understood, to define parameters for GL_N and eventually for the classical groups, using the transfer (functoriality) that he proves by means of the trace formula.

For $\mathrm{GL}_N(\mathbb{Q}_p)$, the parametrization is called the *Local Langlands Correspondence*. It attaches to N -dimensional continuous representations of $W'_{\mathbb{Q}_p}$, irreducible admissible representations of $\mathrm{GL}_N(\mathbb{Q}_p)$, preserving certain root numbers and L -functions. This has already been proved for any local field [HT, He, Sch], including $\mathrm{GL}_N(\mathbb{R})$ (cf. [L5], where it is proved for any reductive group over \mathbb{R} by Langlands). For $\mathrm{GL}_N(\mathbb{A})$, this is still conjectured since $L_{\mathbb{Q}}$ itself is still conjecturally defined. But the work of Jacquet, Piatetski-Shapiro, and Shalika [JS, JPSS] as well as Mœglin and Waldspurger [MW1], which classifies automorphic representations of $\mathrm{GL}_N(\mathbb{A})$ in terms of cuspidal representation of $\mathrm{GL}_m(\mathbb{A})$, $m \leq N$, allows a substitute for global parameters as explained in section 1.4 of the book.

The global parameters for $\mathrm{GL}_n(\mathbb{A})$, when restricted to self-dual ones, produce candidates for parameters for classical groups. An important fact about the parameters is that they fit well into the theory of endoscopy, and more generally, *twisted* endoscopy, that of disconnected groups. In fact, the groups SO_m , $m = 2n, 2n + 1$, and Sp_{2n} are twisted endoscopic groups for $\tilde{G} = \mathrm{GL}_N \rtimes \langle \theta \rangle$ for a suitable N .

Now, comparisons of the stable trace formulas of classical groups and the stable twisted ones on \tilde{G} verify that these self-dual parameters on GL_N account for all the representations of the classical groups. This is a long and delicate inductive argument which requires much of Arthur's earlier work and his *intertwining relations* concerning *normalized intertwining operators* on induced representations, proved in this book and elsewhere [LL, A4, A8, A9, KeSh]. Many of the local character identities, which are used in implementing endoscopic transfer, are proved by local-global arguments, i.e., using the trace formula and enough known character identities at other places to get the one for the desired place.

Arthur has capsulated two of his important global results as Theorems 1.5.2 and 1.5.3. Theorem 1.5.2 gives a decomposition of the discrete spectrum $L^2_{\mathrm{disc}}(G(F)\backslash G(\mathbb{A}_F))$, with F any number field, $G = \mathrm{SO}$ or Sp , in terms of the parameters, showing in particular that the irreducible constituents appear with multiplicities either 1 or 2. Theorem 1.5.3 shows that a self-dual cuspidal representation π of $\mathrm{GL}_N(\mathbb{A}_F)$ comes through twisted endoscopic transfer from either $G = \mathrm{SO}_{2n}$ or Sp_{2n} if and only if the exterior square L -function $L(s, \pi, \Lambda^2)$ has a pole at $s = 1$. If the L -function does not have a pole at $s = 1$, then π comes from $G = \mathrm{SO}_{2n+1}$ and the symmetric square L -function $L(s, \pi, \mathrm{sym}^2)$ will have a pole at $s = 1$. These results are expected from the behavior of the parameters, but proving them requires a long and complicated argument.

A result as deep as endoscopic transfer proved here has many important consequences, including local Langlands correspondence for classical groups, analytic properties of Rankin product L -functions for two cusp forms on classical groups, as well as analytic properties of exterior and symmetric square L -functions for automorphic forms on $\mathrm{GL}_N(\mathbb{A}_F)$. Arthur has also proved the existence of a representation with a Whittaker model in every tempered L -packet of a classical group, conjectured in [Sh1], by combining the results of [CKPSS2] and [GRS] with his own.

One expects many deep arithmetic/geometric results, such as on the Hasse–Weil zeta functions of Shimura varieties, to follow.

The progress made in this book in establishing this important case of functoriality, while of great significance, is still only an example of an endoscopic transfer. In fact, with the exception of the cases proved in [K] and [KSh], every other transfer

established so far is an endoscopic one. It is for this reason that Langlands [L6] has now proposed an approach to general functoriality, which he called *beyond endoscopy*. The technique he proposed was to use the trace formula to determine the poles of automorphic L -functions at $s = 1$, and his idea has already been taken up by Arthur and others. It is expected that Arthur's earlier work on the trace formula, including this book, will play a central role in this approach. Any progress will be most welcome.

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