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Topics in hyperplane arrangements, by Marcelo Aguiar and Swapneel Mahajan, Mathematical Surveys and Monographs, Vol. 226, American Mathematical Society, Providence, RI, 2017, xxiv+611 pp., US$159.00, ISBN 978-1-4704-3711-4

1. Introduction

Hyperplane arrangements provide a classic example of the deep interactions between space and complexity. Like knots or convex polytopes, they are combinatorial objects that arise from geometry. Also they are quite enjoyable. In just a few moments I can describe to a middle-schooler some of the open questions, and challenge her to try, with pencil and paper, to brute-force an answer to easy cases. A little later, I'd recommend the recent monograph from Aguiar and Mahajan.

A hyperplane arrangement is exactly what it sounds like. Take a collection of distinct codimension-1 linear subspaces, or affine subspaces, of a real, complex, or projective ambient vector space, and arrange them in that ambient space in any way you like. In any of these six options, the hyperplanes will subdivide both the ambient space and, potentially, each other. We often label all the intersections and subdivisions; these are called the *faces* of the arrangement. Top-dimensional faces are the *chambers*, the regions of the subdivided ambient space. If there is a face C that is the intersection of all the hyperplanes, we call the arrangement *central*; if that center face is a point, then the central arrangement is also *essential*.

In the book under review, Mahajan and Aguiar decide to focus on real arrangements, especially the central, essential arrangements, with secondary attention given to the affine, noncentral case. The next decision discussed is how to distinguish between two arrangements. Is the geometry important? Then we might want to only consider two arrangements equivalent if a linear transformation of the ambient space yields a bijection between their respective hyperplanes. If instead combinatorics is key, then we focus on the posets of faces ordered by inclusion: two arrangements are combinatorically equivalent, or *cisomorphic*, when their face posets are isomorphic. There are other options as well, such as defining equivalence of two arrangements if one can be transformed into the other by a series of t-parameterized rigid movements of the hyperplanes in the ambient space, for which at any given t the hyperplane arrangement has the same face poset as originally. This latter parametric equivalence (with reflections) is implied by the geometric equivalence and, in turn, implies the combinatorial equivalence. Relatedly, there is also a rich field of study that allows up-to-homotopy hyperplanes, the pseudo-hyperplane arrangements. For instance, two pseudo-lines may cross each other at most once, just like real lines; and if they are parallel, any third line must cross both if it crosses either. Central pseudo-hyperplane arrangements correspond precisely to oriented matroids, as described in [17].

2. Motivation

Immediately there arises a host of open questions. The hardest are, perhaps, enumerative: How many arrangements are there, up to combinatorial equivalence? —up to parameterized equivalence? Since the affine hyperplane arrangements of

n hyperplanes in real space contain a subset corresponding to convex polytopes with n facets, these questions are clearly hard even in three dimensions (3D). The five combinatorial classes of $n = 3$ planes in 3D are pictured near the beginning of almost every linear algebra text. For $n = 4$ planes in 3D, there are at least 14 classes, but then you are on your own! These enumeration questions are also unsolved in two dimensions (2D), allowing the interested researcher to study an open question just by drawing lines on the plane. The parametric equivalence classes in 2D, up to reflections, are found for small n in sequence A241600 of the OEIS [12]. Many other exciting open problems start in 2D. One problem is to find small examples in which two arrangements are combinatorially but not parametrically equivalent. Peter Shor's paper [11] shows existence of somewhat larger examples, via the fact that there are symmetric pseudo-line arrangements that can be straightened, but only to line arrangements which break the symmetry in two different ways. Another famous question is the Kobon problem, which asks how many triangular chambers are possible in an arrangement of n lines [3]. In

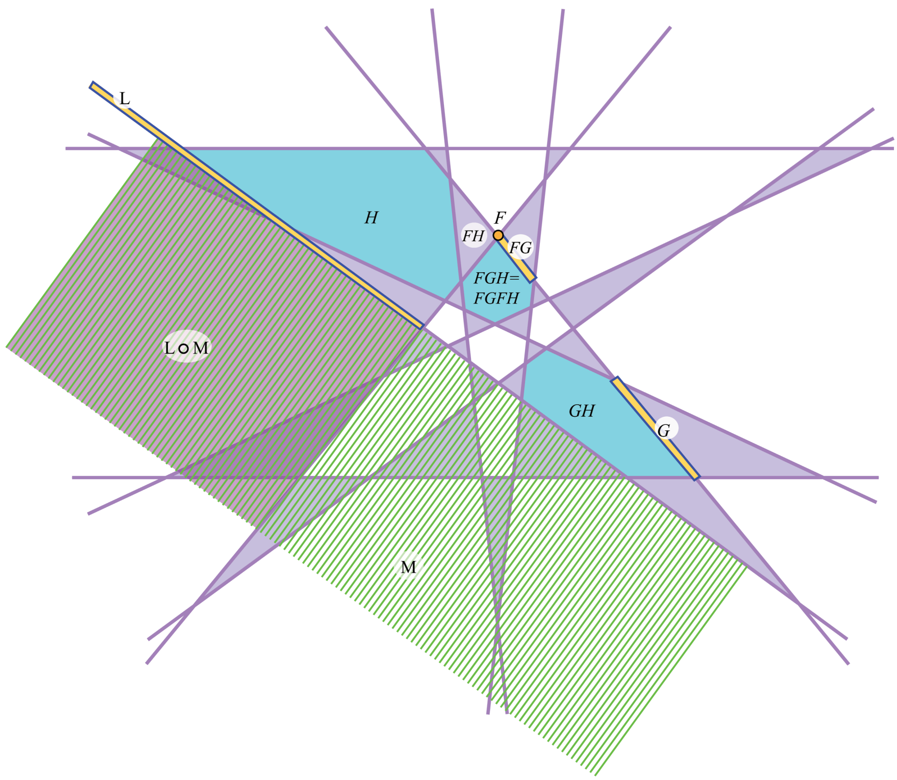

FIGURE 1. A simple affine arrangement of ten lines in $\mathbb{R}^2$ with 25 triangular regions. Illustrated are the Tits product of faces and composition in the category of lunes. For instance FG is the line segment (outlined) which is the product of the vertex F and the line segment G. Lunes L and M are the outlined ray and the hatched half-space, respectively; their composition is the shaded quadrant.

Figure 1 we show a *simple* arrangement (at most two lines cross at each vertex) of ten lines exhibiting 25 triangles. It is a variation on Wajnberg's arrangement as communicated to Eric Weisstein [15]. It is an open question whether ten lines can form 26 triangles, and the problem gets harder after that!

The total number of chambers of all shapes turns out to be easy enough to bound in any dimension: the minimum is when you make all n planes parallel, giving $n + 1$ chambers, while the maximum occurs when the hyperplanes are in general position. The sum $\sum_{i=0}^{k} \binom{n}{i}$ gives the maximum number of chambers in an affine arrangement of n hyperplanes in $\mathbb{R}^k$. (Sequence A008949 in [12].) As well, there is Zaslavsky's famous result that the number of regions of a specific arrangement is found by evaluating the Mobius function (or characteristic polynomial) of the lattice of flats [16]. Many open problems are restricted versions of these questions: we ask about the number of simplicial chambers in the case where all chambers must be simplicial, or about the number of arrangements where certain points in the ambient space must be contained in some of the hyperplanes. A particularly famous set of subproblems starts with asking for the number of combinatorial equivalence classes with n chambers in a given dimension. In fact, since polytopes are bounded by hyperplanes, every question about the former can be extended to one about the latter.

That brings us to listing some more sources containing good summaries of what is known, and discussions of these and many more open problems. A great introduction to the combinatorics of hyperplane arrangements, especially the relevant posets and matroids, is Richard Stanley's open course through MIT: lecture notes are currently available online at [13]. In Branko Grünbaum's book, *Convex Polytopes* [5], chapter 18 on arrangements is focused on open questions about lines in $\mathbb{R}^2$. Especially recommended is the updated edition of [5] edited by Kaibel, Klee, and Ziegler. Speaking of whom, Günter Ziegler's book *Lectures on Polytopes*, [17, Lecture 7], has an excellent chapter covering central arrangements and zonotopes, which are polytopes with point symmetry (generalizing parallelohedra). Goodman and O'Rourke edited a terrific resource, the *Handbook of Discrete and Computational Geometry*, 2nd ed., [4], which contains chapters on arrangements by Goodman and by Halperin. We should mention again Thomas Zaslavsky's groundbreaking book by its main title, *Facing Up to Arrangements* [16]. Some more recent complements to Aguiar and Mahajan's book include the monograph by Orlik and Terao [8], which also covers complex arrangements, and the textbook by Dimca [2], which focuses on algebraic topology and also covers the projective case.

3. Algebraic structures

The book we are currently reviewing really takes off at a point immediately beyond enumerative combinatorics. Algebraic combinatorics, broadly defined, is the study of algebraic structures on combinatorially defined sets. Carlo-Rota famously argued that algebra and combinatorics are each valuable to the other [9]. Combinatorics often affords the algebraist a concrete example of structures to experiment with. Algebra often provides invariants, allowing combinatorialists to prove two structures are distinct. In the extreme case, algebraic invariants can yield contradictions that demonstrate the impossibility of geometric or combinatorial constructions. More typically, advances occur from recognizing that a particular algebraic or combinatorial structure arises in two disparate areas. This might allow the theorems of one theory to be applied in the other. Several kinds of hyperplane arrangements arise in other contexts: for instance the essential central arrangements are precisely the dual fans of zonotopes. Thus an algebraic structure on faces of the arrangements can restrict to a not-so-easily noticed structure on the faces of zonotopes. As we will see, Aguiar and Mahajan add a great deal of new "connective tissue" along these lines.

The first algebraic structures that the authors focus on are monoids, especially *left regular bands*, and their linearizations. *Flats* of a hyperplane arrangement are intersections of any of the hyperplanes. The Birkhoff monoid of flats is simply the join operation on the poset of flats, which is intersection. If the arrangement is central, then intersections are never empty and the unit is the center. The Tits product, pictured in Figure 1, was first defined in [14]. The product of two faces FG is the face encountered after starting at any point interior to F and traveling an infinitesimal distance in a straight line toward any point interior to G. Thus FG might equal F itself, for instance if F is a chamber. Otherwise FG may be a face containing F. If the arrangement is central and C is the center, then $FC = F = CF$. In that case the semigroup is named the *Tits monoid*; in general is is called the *face semigroup.*

The Tits monoid is an associative, left unital structure obeying the equations $F^2 = F$ and $FGF = FG$. The latter equation is equivalent to $FGFH = FGH$, which can be shown in one direction by multiplying on the right with H and in the other by allowing $H = C$. Associative monoids obeying these rules are known as *left regular bands.* Without requiring associativity, these are also known as *left spindles* (with unit) in the context of knot *quandles.* The latter algebraic structures are complete invariants of knots. They were designed to reflect Reidemeister moves, especially the braid relation, on the arcs of a knot diagram in [6]. The spindles and their homology are discussed in [1]. Since many algebraic structures on hyperplane arrangements can be extended to the general setting of left regular bands, this connection to knot theory deserves further exploration!

Aguiar and Mahajan use their book to rapidly expand the list of combinatorial types of subobjects in arrangements, the list of various algebraic structures on those objects, and theorems and connections to other areas of algebra that result. They linearize the monoids we just described by treating the faces as the basis of an algebra. One nice result is that even when the arrangement is affine (or not central), the linearization has a unit. The unit is found as the formal linear combination of all the bounded faces, positive for the even dimensional faces and negative for the odd. (If the arrangement is not essential, we use rank instead of dimension.) Multiplying this combination by any single face gives back the face itself; see Figure 2 for a small example. Of course, for a central essential arrangement, the zero-dimensional center is the only bounded face, so it is itself the unit.

Another new construction herein is the *category of lunes.* A lune is a *cone* (an intersection of half-spaces) which lies *between* a pair of nested flats: the *base* flat contained in L and the *case* flat containing L. Specifically, a lune L is a chamber in the arrangement formed by considering its case as a new ambient space subdivided by only those hyperplanes which contain its base. For instance, in Figure 2 P is a lune with base x and case the ambient plane. The chambers P and Q together form a lune with base the vertex z, and case again the plane. Lunes that have matching base and case flats can be composed—the base is the target and the case is the source. For example, in Figure 1 the lunes L and M are composed as the

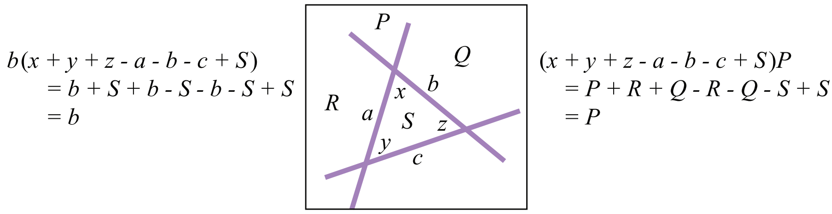

FIGURE 2

lune L$\circ$M; the matching source and target is the flat which is the line containing L and bounding M. The category of lunes supports further structures such as actions of the face semigroup and Birkhoff monoid. On top of that, there is a substitution product of lunes which allows an operad to be described. The authors promise a sequel which will present a theory of Hopf monoids for hyperplane arrangements in which operads will play an even larger role.

Special sequences of arrangements correspond to well-known combinatorial species. Several examples covered in the book under review (see chapter 6) include the braid arrangement, also known as the type A reflection arrangement, and the type B reflection arrangement. The braid arrangement in $\mathbb{R}^n$ is well known as the dual fan of the n-dimensional permutohedron. The latter is the zonotope constructed by taking the convex hull of the permutations of the vector $(1, 2, \ldots, n+1)$. Thus the chambers of the braid arrangement correspond to permutations of $[n+1]$. The rest of the faces, flats, cones, lunes, face-types, lune-types, and so forth, also correspond to well-known combinatorial sequences, and Aguiar and Mahajan provide a comprehensive dictionary including many new entries. For instance, in the braid arrangement each top lune corresponds to a certain linear partition, and the chambers of that lune are shuffles of the partition. Thus lunes can be useful for organizing or generalizing classical constructions based on shuffles. In later chapters the authors demonstrate this principle. For instance, in chapter 10 they use top lunes to characterize Lie elements: the *Ree top-lune criterion*, which generalizes Ree's shuffle-based characterization.

Indeed, Aguiar and Mahajan's book is a gold mine for anyone searching for just the right algebraic structure, possibly an obstruction or invariant tailored to a combinatorial problem. There are over 600 pages, including appendices on the basic algebraic concepts one might need to review.

I will end with mention of a few more highlights. The authors spend a good bit of time on the theory of Eulerian families and Eulerian idempotents in the Tits algebra. These are idempotent elements of the algebra (thus formal sums of faces) indexed by the flats. The idempotents in these families are also mutually orthogonal. Complete systems (collections of elements that sum to the unit element) of primitive orthogonal idempotents of the Tits algebra were constructed previously by Saliola in [10]. However, Aguiar and Mahajan show that every complete system of the Tits algebra arises from the Saliola construction. The authors then show a list of characterizations of these idempotents, including use of noncommutative zeta and Möbius functions.

There is one more important item to point out. Some of the best parts of the book are in the voluminous endnotes to each chapter. Here the authors collect tantalizing potential connections and possibilities, as well as making clear how the results just presented are separated into categories of old, new, and newly simplified. The endnotes provide detailed and carefully researched mathematical history to go along with the top-notch exposition throughout. The result is that not only are all the citable theorems cited, but their historical genesis is revealed. The notes are also packed with very useful information; for instance I just now learned that William Lawvere refers to left regular bands as *graphic monoids,* in his papers on topos theory [7]. These sort of hints can easily lead to projects—an experimental study of categories enriched over monoidal categories of lunes suggests itself.

References

[1] A. S. Crans, J. H. Przytycki, and K. K. Putyra, *Torsion in one-term distributive homology*, Fund. Math. **225** (2014), no. 1, 75–94, DOI 10.4064/fm225-1-5. MR3205566

[2] A. Dimca, *Hyperplane arrangements*: *An introduction*, Universitext, Springer, Cham, 2017. MR3618796

[3] M. Gardner, *Wheels, life and other mathematical amusements*, W. H. Freeman and Co., San Francisco, CA, 1983. MR717760

[4] J. E. Goodman and J. O'Rourke (eds.), *Handbook of discrete and computational geometry*, CRC Press Series on Discrete Mathematics and its Applications, CRC Press, Boca Raton, FL, 1997. MR1730156

[5] B. Grünbaum, *Convex polytopes*, 2nd ed., Graduate Texts in Mathematics, vol. 221, Springer-Verlag, New York, 2003. Prepared and with a preface by Volker Kaibel, Victor Klee and Günter M. Ziegler. MR1976856

[6] D. Joyce, *A classifying invariant of knots, the knot quandle*, J. Pure Appl. Algebra **23** (1982), no. 1, 37–65, DOI 10.1016/0022-4049(82)90077-9. MR638121

[7] F. W. Lawvere, *Linearization of graphic toposes via Coxeter groups*, J. Pure Appl. Algebra **168** (2002), no. 2-3, 425–436, DOI 10.1016/S0022-4049(01)00107-4. Category theory 1999 (Coimbra). MR1887168

[8] P. Orlik and H. Terao, *Arrangements of hyperplanes*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 300, Springer-Verlag, Berlin, 1992. MR1217488

[9] G.-C. Rota, *Report on the present state of combinatorics*, Proceedings of the 5th Conference on Formal Power Series and Algebraic Combinatorics (Florence, 1993), Discrete Math. **153** (1996), no. 1-3, 289–303, DOI 10.1016/0012-365X(95)00143-K. MR1394961

[10] F. V. Saliola, *The face semigroup algebra of a hyperplane arrangement*, Canad. J. Math. **61** (2009), no. 4, 904–929, DOI 10.4153/CJM-2009-046-2. MR2541389

[11] P. W. Shor, *Stretchability of pseudolines is NP-hard*, Applied geometry and discrete mathematics, DIMACS Ser. Discrete Math. Theoret. Comput. Sci., vol. 4, Amer. Math. Soc., Providence, RI, 1991, pp. 531–554. MR1116375

[12] N. J. A. Sloane, *The on-line encyclopedia of integer sequences*, Notices Amer. Math. Soc. **65** (2018), no. 9, 1062–1074. MR3822822

[13] R. P. Stanley, *Hyperplane arrangements, interval orders, and trees*, Proc. Nat. Acad. Sci. U.S.A. **93** (1996), no. 6, 2620–2625, DOI 10.1073/pnas.93.6.2620. MR1379568

[14] J. Tits, *Buildings of spherical type and finite BN-pairs*, Lecture Notes in Mathematics, Vol. 386, Springer-Verlag, Berlin-New York, 1974. MR0470099

[15] E. W. Weisstein, *Kobon triangle*, MathWorld—A Wolfram Web Resource, `mathworld.wolfram.com/KobonTriangle.html` (2018).

[16] T. Zaslavsky, *Facing up to arrangements: face-count formulas for partitions of space by hyperplanes*, Mem. Amer. Math. Soc. **1** (1975), no. issue 1, 154, vii+102. MR0357135

[17] G. M. Ziegler, *Lectures on polytopes*, Graduate Texts in Mathematics, vol. 152, Springer-Verlag, New York, 1995. MR1311028

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