

SELECTED MATHEMATICAL REVIEWS

related to the paper in the previous section by
JOSÉ SEADE

MR0239612 (39 #969) 57.20; 14.00

Milnor, John

Singular points of complex hypersurfaces. (English)

Annals of Mathematics Studies, 61.

Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo,
1968, iii+122 pp.

The purpose of this work is to study the local behavior of a complex hypersurface V in Euclidean space at a singularity z_0 , principally as application of the following basic fibration theorem: If S is a sphere of sufficiently small radius about z_0 and $K = S \cap V$, then $S - K$ is a smooth fiber bundle ξ over the circle S^1 . In fact if $f(z) = 0$ defines V , f a complex polynomial, then $z \rightarrow f(z)/|f(z)|$ defines the fibration. If z_0 is an isolated singularity, a small enough neighborhood of z_0 admits a homeomorphism which throws V onto the cone over K with vertex z_0 —thus the pair (S, K) determines the topological nature of the singularity at z_0 .

It turns out, assuming z_0 is an isolated singularity, that each fiber F of ξ is the interior of a smooth compact submanifold of S with the common boundary K . Moreover, F is, up to homotopy, a wedge of μ copies of S^n , where μ is the “multiplicity” of z_0 as a zero of f . One of the most interesting invariants of V at z_0 is a polynomial $\Delta(t)$ of degree μ defined as follows: the twist of ξ induces a diffeomorphism of F , and hence an automorphism of $H_n(F)$; $\Delta(t)$ is its characteristic polynomial. For example, K is a topological sphere if and only if $\dim_{\mathbb{C}} V \neq 2$ and $\Delta(1) = \pm 1$ (when $n = 2$, a theorem of Mumford says $\Pi_1(K) \neq 0$). Then V is a topological manifold at z_0 , and the knot type of $K \subset S$ is the measure of local flatness at z_0 (but, in fact, K is always knotted). The question of the differential structure on K is interesting and has been completely settled by Brieskorn and Hirzebruch for the special cases when $f(z) = \sum_i z_i^{a_i}$. The formula for $\Delta(t)$ is worked out for these cases, as well as a more general family.

When $\dim_{\mathbb{C}} V = 1$, an explicit relation is derived between $\Delta(t)$ and the classical Alexander polynomial of the link $K \subset S$ (=3-sphere), and various other local invariants are studied.

In conclusion a weaker fibration theorem is proved for real hypersurfaces (of arbitrary codimension) at an isolated singularity.

Some of these results have appeared in an unpublished work of the author: (“On isolated singularities of hypersurfaces”). There is also some overlap with work of Pham, Brieskorn and Hirzebruch (see the bibliography).

J. P. Levine

From MathSciNet, February 2019

MR0422256 (54 #10247) 14B05; 14D05, 32C40

Teissier, B.

Introduction to equisingularity problems.

Algebraic geometry (Proc. Sympos. Pure Math., Vol. 29, Humboldt State Univ., Arcata, Calif., 1974), 593–632, *Amer. Math. Soc., Providence, R.I.*, 1975.

This is a skillful and imaginative summary of the state of affairs (as of 1975) in the nascent theory of equisingularity. Classical results of Zariski, Thom and Mather on algebraic, differential and topological equisingularity are described. Various desiderata in terms of hyperplane sections, projections and discriminants are discussed. Major emphasis is placed on the behavior of certain numerical characters in a family of hypersurfaces with isolated singularities: the constancy of the Milnor number μ is equivalent with topological equisingularity (except possibly for families of surfaces) (cf. Lê Dũng Tráng and C. P. Ramanujam [*Amer. J. Math.* **98** (1976), no. 1, 67–78; MR0399088]); the constancy of the sequence μ^* of Milnor numbers of sections by hyperplanes of various dimensions is equivalent with differential equisingularity (for this, and other recent developments, cf. the theses of J.-P. Speder [“Équisingularité et conditions de Whitney”, Thèse de doctorat d’état, Univ. Nice, Nice, 1976] and J. Briançon [“Contribution à l’étude des déformations de germes de sous-espaces analytiques de \mathbf{C}^n ”, Thèse de doctorat d’état, Univ. Nice, Nice, 1976]. A more refined sequence of numerical characters—defined via polar curves—is discussed, with applications. (This was treated in detail in a recent preprint of the author [“Invariants polaires. I. Invariants polaires des singularités d’hypersurfaces”, Preprint No. M242–0376, Centre Math., École Polytechnique, Palaiseau, 1976].)

{For the entire collection see MR0360574.}

Joseph Lipman

From MathSciNet, February 2019

MR1215961 (94b:32062) 32S55; 32S15, 32S60, 57R45

Teissier, Bernard

A bouquet of bouquets for a birthday.

Topological methods in modern mathematics (Stony Brook, NY, 1991), 93–122, *Publish or Perish, Houston, TX*, 1993.

This is a beautiful expository paper which is part of a collection dedicated to John Milnor on the occasion of his 60th birthday. The author first describes Milnor’s classical work on what is now known as the Milnor fibration of a hypersurface singularity. He then proceeds to explore the stream of ideas which began with Milnor’s investigations and continues to the present day. This stream contains results and questions concerning exotic spheres, knots, multiplicities, polar varieties, stratified spaces, equisingularity, and many other topics which are central to the modern study of complex analytic singularities.

For those seeking an introduction and overview of the subject matter, or for those seeking to find how their own work fits into the big picture, this paper is an excellent place to begin.

David B. Massey

From MathSciNet, February 2019

MR1441075 (98h:32061) 32S55; 32S15, 32S25, 32S60

Massey, David B.

Lê cycles and hypersurface singularities. (English)

Lecture Notes in Mathematics, 1615.

Springer-Verlag, Berlin, 1995, xii+131 pp., \$29.95, ISBN 3-540-60395-6

In the beginning of the preface the author explains motives, in which he showed his interest in the study of non-isolated singularities when he was a graduate student. He then enumerates new results which have not yet been published elsewhere, describes how the terms Lê cycles and Lê numbers came into existence, and thanks personally 15 specialists and 4 institutions for their help during his study from 1987–1995 of the Lê cycles and numbers.

In the introduction the notions of Milnor fibration, Milnor fibre, and Milnor numbers of an analytic function with a critical locus of arbitrary dimension and some well-known important results concerning these objects are discussed.

In the first chapter basic definitions and main properties of the Lê cycles and Lê numbers are recalled. In particular, the behavior of the Lê numbers under the taking of hyperplane sections is described. The key role in this study is played by the notions of polar variety and polar cycles due to Lê Dũng Tráng and B. Teissier [*Ann. of Math. (2)* **114** (1981), no. 3, 457–491; MR0634426] and modified by the author [*Math. Ann.* **282** (1988), no. 1, 33–49; MR0960832]. Several good examples given in the next chapter serve to illustrate all the objects introduced above.

Chapter 3 deals with a handle decomposition of the Milnor fibre. Its study is based essentially on the argument from Lê Dũng Tráng and B. Perron [*C. R. Acad. Sci. Paris Sér. A-B* **289** (1979), no. 2, A115–A118; MR0549082]. Thus, an explicit formula for the reduced Euler characteristic of the Milnor fibre as well as Morse inequalities which satisfy the reduced Betti numbers with respect to the Lê numbers are obtained.

The next chapter is devoted to a generalization of the Lê-Iomdin formula which describes relations between the Lê numbers of a hyperplane section and a certain sequence of hypersurface singularities whose critical loci have dimensions one lower than the approached original singularity. One of its applications is an elegant variant of the Plücker formula for the Lê numbers of a homogeneous polynomial with a critical locus of arbitrary dimension.

In Chapter 5, using the lattice of flats of the arrangement from [P. Orlik and H. Terao, *Arrangements of hyperplanes*, Springer, Berlin, 1992; MR1217488], the author calculates the Lê numbers for a central hyperplane arrangement by means of pure combinatorial techniques. Moreover, a relationship between the Lê numbers of the arrangement and the Möbius function is proved.

In Chapter 6 conditions under which a submanifold of affine space satisfies the Thom a_f -condition [J. Mather, “Notes on topological stability”, unpublished notes, Harvard Univ., 1970; per bibl.] with respect to the ambient stratum are described. More precisely, making use of five statements, the author classifies situations when the constancy of the Lê numbers implies Thom’s a_f -condition. As a result a multi-parameter generalization of the theorem of Lê Dũng Tráng and K. Saito [*C. R. Acad. Sci. Paris Sér. A-B* **277** (1973), A793–A795; MR0350063] for an analytic function with a critical locus of arbitrary dimension is obtained. In fact, it is proved that

if one has a family of hypersurface singularities with constant Lê numbers parameterized along a submanifold then that submanifold satisfies Thom's a_f -condition with respect to the ambient space.

The main objective of the next two chapters is to compute the Lê numbers for aligned and suspended singularities. The former can be thought of as a natural generalization of isolated singularities, non-isolated line singularities, and the singularities occurring in hyperplane arrangements, while the latter present a well-known important class of singularities which have already been studied in many papers from various points of view [M. Sebastiani and R. Thom, *Invent. Math.* **13** (1971), 90–96; MR0293122].

Chapter 9 is devoted to the proof of the most significant result of the author: if the Lê numbers are constant in a one-parameter family then the Milnor fibrations are constant in the family, regardless of the dimension of the critical loci. Hence, it gives a generalization of the famous result of Lê Dũng Tráng and C. P. Ramanujam [*Amer. J. Math.* **98** (1976), no. 1, 67–78; MR0399088].

In the final chapter the author discusses some alternative characterizations of the Lê cycles and Lê numbers of a hypersurface singularity which lead to a generalization of the Lê numbers and can be applied to any perverse sheaf. Such a more general viewpoint enables him to state that the case of the Lê numbers of a function f is just the case where the underlying perverse sheaf is the sheaf of vanishing cycles along f (see also his later paper [D. B. Massey, *Topology* **35** (1996), no. 4, 969–1003; MR1404920]).

In the appendix the author proves a number of important although very technical results which allow him to use certain types of “nice” neighborhoods in order to define the Milnor fibre, and to give conditions under which Milnor fibrations remain constant in a parameterized family.

The work under review is written in a very clear style, and each topic is followed by carefully chosen examples. Above all it contains 13 beautiful pictures which may be considered as real visualizations of the author's ideas. The bibliography includes 90 selected references. Moreover, the author remarks in the preface that this work summarizes all results known to him in 1995 about the Lê cycles and the Lê numbers for non-isolated hypersurface singularities. That is why the present work is not only accessible and very useful to beginning graduate students but also quite interesting to a wide range of active researchers in topology, algebraic geometry, complex analysis and other related fields of modern pure mathematics.

Aleksandr G. Aleksandrov

From MathSciNet, February 2019

MR2904577 14B05; 05B35; 14C17

Huh, June

Milnor numbers of projective hypersurfaces and the chromatic polynomial of graphs.

Journal of the American Mathematical Society **25** (2012), no. 3, 907–927.

Let G be a graph. The function $\chi_G(q)$ assigning to every positive integer q the number of proper colorings of G with q colors, is a polynomial with integer coefficients, called the *chromatic polynomial* of G . In this remarkable paper, the author proves that the coefficients of the chromatic polynomial, and more generally the coefficients of the characteristic polynomial of a matroid M representable over a

field of characteristic zero, form a ‘sign-alternating log-concave sequence of integers with no internal zeros’: if $\chi_M(q) = a_n q^n - a_{n-1} q^{n-1} + \cdots + (-1)^n a_0$ is the characteristic polynomial of M , then $a_{i-1} a_{i+1} \leq a_i^2$ for $0 < i < n$, and the indices of nonzero coefficients are consecutive integers.

This settles long-standing conjectures of Rota, Heron, and Welsh. It follows that the sequence a_i is *unimodal*, i.e., $a_0 \leq \cdots \leq a_{i-1} \leq a_i \geq a_{i+1} \geq \cdots \geq a_n$ for some i ; this had been conjectured by Read in 1968.

The proof involves many interesting ingredients, and the reader wishing to appreciate the broader context is warmly advised to peruse the introduction of the paper, which is written with exceptional clarity. One key technical fact is the following theorem on correspondences: write a homology class ξ in $\mathbb{P}^n \times \mathbb{P}^m$ as $\xi = \sum_i e_i [\mathbb{P}^{k-i} \times \mathbb{P}^i]$; then some positive integer multiple of ξ may be represented by an irreducible subvariety of $\mathbb{P}^n \times \mathbb{P}^m$ if and only if the coefficients e_i form a nonzero log-concave sequence of nonnegative integers with no internal zeros. The main ingredient in the proof of this fact is the Hodge-Teissier-Khovanskii inequality [A. G. Khovanskii, Addendum 3, “Algebra and mixed volumes”, in Yu. D. Burago and V. A. Zalgaller, *Geometric inequalities*, translated from the Russian by A. B. Sosinskiĭ, Grundlehren Math. Wiss., 285, Springer, Berlin, 1988; MR0936419; B. Teissier, C. R. Acad. Sci. Paris Sér. A-B **288** (1979), no. 4, A287–A289; MR0524795]. The author has sharpened this result further for $n = m = 2$ in a recent preprint [“Correspondences between projective planes”, arXiv:1303.4113].

The result on characteristic polynomials follows from this theorem because the coefficients of the polynomial for a representable matroid M may be interpreted as the coefficients of the class of the graph of the *gradient map* $p \mapsto (\partial h / \partial z_0(p) : \cdots : \partial h / \partial z_n(p))$, for h the polynomial defining a hyperplane arrangement corresponding to M . The author studies these coefficients for a general homogeneous polynomial h , in terms of the *mixed multiplicities* $\mu^i(h)$ of h and the irrelevant ideal. He is able to relate these numbers to the topology of the complement $D(h)$ of the hypersurface X defined by h (via a theorem of A. Dimca and Ş. Papadima [Ann. of Math. (2) **158** (2003), no. 2, 473–507; MR2018927]); to the Chern-Schwartz-MacPherson class of X (recovering a formula due to the reviewer [J. Symbolic Comput. **35** (2003), no. 1, 3–19; MR1956868]); and he establishes formulas bounding $\mu^i(h)$ by mixed volumes of convex polytopes, in analogy with a result of A. G. Kushnirenko [Invent. Math. **32** (1976), no. 1, 1–31; MR0419433] computing the Milnor number of a hypersurface in terms of the corresponding Newton polytope.

The application to the log-concavity of the coefficients of chromatic polynomials, while striking, is but a very particular consequence of the results obtained in this article, whose content is much richer than this quick summary can suggest.

Paolo Aluffi

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