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The theory of $\mathcal{H}(b)$ spaces, Vol. 1, by Emmanuel Fricain and Javad Mashreghi, New Mathematical Monographs, Vol. 20, Cambridge University Press, Cambridge, 2016, xix+681 pp., ISBN 978-1-107-02777-0

The theory of $\mathcal{H}(b)$ spaces, Vol. 2, by Emmanuel Fricain and Javad Mashreghi, New Mathematical Monographs, Vol. 21, Cambridge University Press, Cambridge, 2016, xix+619 pp., ISBN 978-1-107-02778-7

One of the first infinite-dimensional Hilbert spaces appearing in any functional analysis course is

$$\ell^2 := \left\{ a = (a_n) = (a_0, a_1, a_2, \ldots) : \|a\|_{\ell^2} = \sum_{n=0}^{\infty} |a_n|^2 < \infty \right\}.$$

A simple operator one can study on this space is the (right) *shift* S defined by

$$Sa = (0, a_0, a_1, a_2, \ldots),$$

which just adds a zero to the front of the sequence. Clearly, $\|Sa\|_{\ell^2} = \|a\|_{\ell^2}$. One checks that the adjoint of the shift operator is the *left shift*

$$S^*a = (a_1, a_2, a_3, \ldots),$$

which just drops the first element of the sequence. Later on, the space of two-sided sequences $\ell^2(\mathbb{Z})$ will also be of our interest. Analogues of the shift operators can be naturally defined on this space as well (without eliminating entries or concatenating by zeros). Despite the "simplicity" of S and S^*, in the last 50 years there has been a concerted effort to reduce much of operator theory to the behavior of these particular operators and their restrictions to certain subspaces.

Suppose we wanted to know what are the invariant subspaces of the operator S, i.e., nontrivial subspaces $E \subset \ell^2$ such that $SE \subset E$. An issue that makes this problem nontrivial is that S has no eigenvectors, and so we are forced to understand the operator using its lattice of closed invariant subspaces. In terms of spaces of sequences, this turns out to be hard to address, and so we will consider a different incarnation of the problem. To do that, we will venture into the realm of analytic function theory on the unit disc.

Another infinite-dimensional separable Hilbert space that one encounters in first courses in analysis is the Lebesgue space $L^2(\mathbb{T})$ of measurable functions on the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ such that

$$\|f\|^2_{L^2(\mathbb{T})} = \int_{\mathbb{T}} |f(z)|^2 dm(z) < \infty,$$

where $dm(z)$ is the normalized Lebesgue measure on the circle. A basic fact of Fourier analysis is that the set of functions $\{z^n\}_{n\in\mathbb{Z}}$ provides an orthonormal basis for $L^2(\mathbb{T})$, and so we can represent any function $f \in L^2(\mathbb{T})$ via its sequence of

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Fourier coefficients $\hat{f}(n) := \langle f, z^n \rangle_{L^2(\mathbb{T})}$,

$$f(z) = \sum_{n \in \mathbb{Z}} \hat{f}(n) z^n.$$

In particular, this isometrically identifies $L^2(\mathbb{T})$ with $\ell^2(\mathbb{Z})$, since by Plancherel's theorem $\|f\|_{L^2(\mathbb{T})} = \|\{\hat{f}(n)\}\|_{\ell^2(\mathbb{Z})}$. We can now ask how the "two-sided" versions of the operators S and S^*, defined on the sequence $\{\hat{f}(n)\}$, behave on $L^2(\mathbb{T})$. One easily arrives at the formulas

$$Sf(z) = zf(z) \quad \text{and} \quad S^* f(z) = \frac{f(z) - f(0)}{z}.$$

This change of point of view allows for using powerful tools of the function theory.

There is a natural subspace of $L^2(\mathbb{T})$ that plays a role similar to the elements in ℓ^2. It consists of functions in $L^2(\mathbb{T})$, whose Fourier series contain only nonnegative frequencies, i.e.,

$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

It is typically denoted $H^2(\mathbb{T})$ and consists of functions having an analytic extension to the disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. It can thus be identified with the Hardy space $H^2(\mathbb{D})$. The latter consists of holomorphic functions on $\mathbb{D}$ such that

$$\|f\|^2_{H^2(\mathbb{D})} = \sup_{0<r<1} \int_{\mathbb{T}} |f(re^{i\theta})|^2 dm(\theta) < \infty.$$

It is easy to see that if $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then

$$\|f\|^2_{H^2(\mathbb{D})} = \sum_{n=0}^{\infty} |a_n|^2.$$

One can then identify $H^2(\mathbb{D})$ with a subspace of $L^2(\mathbb{T})$ by taking appropriate boundary values, thus arriving at $H^2(\mathbb{T})$. We then obtain the decomposition

$$L^2(\mathbb{T}) = H^2(\mathbb{D}) \oplus H^2(\mathbb{D})^{\perp}.$$

Our original question about the shift operator on ℓ^2 now addresses the operator of multiplication by z acting on subspaces of $H^2(\mathbb{D})$. When phrased in this language, the answer surprisingly becomes easier (albeit not trivial) to resolve, leading to the beautiful and important result of Beurling [5]:

Theorem 1. *Let $\{0\} \neq E$ be a subspace of $H^2(\mathbb{D})$ such that $zE \subsetneq E$. Then there exists an analytic in $\mathbb{D}$ function Θ, unique up to multiplication by a constant of modulus one, such that $|\Theta| = 1$ almost everywhere on $\mathbb{T}$ and $E = \Theta H^2(\mathbb{D})$.*

Analytic functions with the property that $|\Theta| = 1$ almost everywhere on $\mathbb{T}$ are called *inner functions* and play an important role in complex analysis and function theory. One notices that the space $\Theta H^2(\mathbb{D})$ is closed (since the transformation $f \mapsto \Theta f$ is unitary, due to Θ being unimodular). A related space that will play an important role is $K_\Theta := \left(\Theta H^2(\mathbb{D})\right)^{\perp}$, the orthogonal complement of $\Theta H^2(\mathbb{D})$ in $L^2(\mathbb{T})$, referred to as the *model space*. Just like $\Theta H^2(\mathbb{D})$ is invariant under the operator S, the space K_Θ is invariant under S^*.

The adjective "model" applied to the space K_Θ is justified by the observation that a very general class of Hilbert space contractions can be modeled by (i.e., are unitarily equivalent to) the adjoint of the shift operator S^* acting on K_Θ. The inner function Θ plays the role similar to the one of the characteristic polynomial; for example, the spectrum of T consists of the zeros of Θ. This is the approach developed in the work of B. Sz.-Nagy and C. Foiaş. The interested reader can find many more interesting and beautiful connections between operator theory and function theory in their monograph [30]; see also [18, 21, 31].

There exists an alternative way to arrive at models for contractions, developed by L. de Branges and J. Rovnyak [15]. Here one considers the subspaces (*not necessarily closed*) of $H^2(\mathbb{D})$ that are invariant under S^*. It is shown in their theory that one can model a certain family of contractions as restrictions of S^* to another type of space, denoted $\mathcal{H}(b)$, where b is a nonextremal point of the unit ball of $H^\infty(\mathbb{D})$ (more explanations follow below). The $\mathcal{H}(b)$ spaces are the main theme of the two volumes under review.

These spaces were originally defined using the notion of a *complementary space*, which is a generalization of the orthogonal complement in a Hilbert space. Let us add some details here. Suppose we have two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ with $\mathcal{H}_1 \subset \mathcal{H}_2$, where it is not assumed that they have the same Hilbert space (or even linear space) structure. Given a bijective mapping $A : \mathcal{H}_1 \to \mathcal{H}_2$, we can use this map to push the Hilbert space structure from $\mathcal{H}_1$ to $\mathcal{H}_2$ as follows:

$$\langle Ax, Ay\rangle_{\mathcal{H}_2} := \langle x, y\rangle_{\mathcal{H}_1} \quad \forall x, y \in \mathcal{H}_1.$$

Assuming that $\mathcal{H}_2$ has a linear structure preserved by A, this defines an inner product on $\mathcal{H}_2$ (possibly a different one from the original, if $\mathcal{H}_2$ had one to begin with).

When $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert spaces and A is a bounded operator between them, this allows one to assign a Hilbert space structure to $\mathcal{R}(A)$, the range of A (the books under review calls this new Hilbert space $\mathcal{M}(A)$, and we will follow along). We now further assume that A is a Hilbert space contraction between $\mathcal{H}_1$ and $\mathcal{H}_2$, i.e., $\|A\| \le 1$, which can be phrased as $AA^* \le I$. From elementary operator theory one has that the operator $(I - AA^*)^{\frac{1}{2}}$ is well-defined, and we can consider as a subspace its range $\mathcal{R}((I - AA^*)^{\frac{1}{2}})$, with the corresponding Hilbert space structure using the construction alluded to above, and we have $\mathcal{M}((I-AA^*)^{\frac{1}{2}})$. The resulting space will be denoted by $\mathcal{H}(A)$ and is the *complementary subspace* to the Hilbert space $\mathcal{R}(A)$. The relationship between $\mathcal{R}(A)$ and $\mathcal{H}(A)$ could be viewed as an analogue of orthogonal complements in a Hilbert space.

Let us define $\mathbb{P}_{H^2} : L^2(\mathbb{T}) \to H^2(\mathbb{D})$ as the orthogonal projection onto $H^2(\mathbb{D})$. To finally arrive at the space $\mathcal{H}(b)$, we specialize to the case when $\mathcal{H}_1 = \mathcal{H}_2$ is the Hilbert space $H^2(\mathbb{D})$ and A is a particular Toeplitz operator. Here the *Toeplitz operator* with symbol φ acts as follows:

$$T_\varphi f = \mathbb{P}_{H^2}(\varphi f), \quad f \in H^2(\mathbb{D}).$$

Here $\varphi \in L^\infty(\mathbb{T})$, and so $T_\varphi : H^2(\mathbb{D}) \to H^2(\mathbb{D})$ is a bounded operator with $\|T\| \le \|\varphi\|_{L^\infty}$. In fact, the Brown–Halmos theorem [8] claims the equality here. If we assume further that $\|\varphi\|_{L^\infty(\mathbb{T})} \le 1$, then T_φ is a contraction. We then use the Toeplitz operator T_φ in the above construction of complementary subspaces.

To define the space $\mathcal{H}(b)$, one needs to consider nonconstant analytic functions in the closed unit ball of $H^\infty(\mathbb{D})$, the space of bounded analytic functions on $\mathbb{D}$. Namely, one considers the functions b such that

(i) $b \in H^\infty(\mathbb{D})$;
(ii) b is not a constant;
(iii) $\|b\|_{L^\infty(\mathbb{T})} = \sup_{z\in\mathbb{D}} |b(z)| \leq 1$.

Then the space $\mathcal{H}(b)$ is the Hilbert space associated as above with the operator $(I - T_b T_{\overline{b}})^{\frac{1}{2}}$, with the norm defined as

$$\|(I - T_b T_{\overline{b}})^{\frac{1}{2}} f\|_{\mathcal{H}(b)} = \|f\|_{H^2(\mathbb{D})} \quad \forall f \in H^2(\mathbb{D}) \perp \ker((I - T_b T_{\overline{b}})^{\frac{1}{2}}).$$

Here the $\perp$ sign means standard orthogonality of vector in $H^2(\mathbb{D})$, so we are looking at functions which are orthogonal to $\ker((I - T_b T_{\overline{b}})^{\frac{1}{2}})$. There are some nice similarities here with the model spaces K_Θ encountered earlier. In particular, the following version of Beurling's theorem holds in this context:

Theorem 2. *Let $\mathcal{M}$ be a Hilbert space contained contractively in $H^2(\mathbb{D})$ such that $S\mathcal{M} \subset \mathcal{M}$, i.e., $\mathcal{M}$ is a linear subspace of $H^2(\mathbb{D})$ and $\|x\|_{H^2(\mathbb{D})} \leq \|x\|_{\mathcal{M}}$ for all $x \in \mathcal{M}$, and that S acts as an isometry on $\mathcal{M}$. Then there exists a function b in the closed unit ball of $H^\infty(\mathbb{D})$, unique up to a unimodular constant, such that $\mathcal{M} = \mathcal{M}(T_b)$, where $\mathcal{M}(T_b)$ is the Hilbert space generated by the operator T_b as discussed above.*

We now present a different way to think about $\mathcal{H}(b)$ spaces by using the theory of reproducing kernels. The space $H^2(\mathbb{D})$ is a *reproducing kernel Hilbert space*, meaning that for each point $\lambda \in \mathbb{D}$ there exists a function $K_\lambda \in H^2(\mathbb{D})$ such that

$$f(\lambda) = \langle f, K_\lambda \rangle_{H^2(\mathbb{D})}.$$

This equality just rephrases the Cauchy integral formula from complex analysis. It also says that point evaluation $f \mapsto f(\lambda)$ is a bounded functional on $H^2(\mathbb{D})$. One computes easily (either via Cauchy's theorem or using Fourier series) the explicit form of this reproducing kernel

$$K_\lambda(z) = \frac{1}{1 - \overline{\lambda} z}.$$

The model spaces K_Θ are also reproducing kernel Hilbert spaces, with the kernels given by

$$K_\lambda^\Theta(z) = \frac{1 - \overline{\Theta(\lambda)}\Theta(z)}{1 - \overline{\lambda} z}.$$

As Θ is an inner function, it is natural to wonder what happens if we replace Θ with a function b from the unit ball of $H^\infty(\mathbb{D})$:

$$K_\lambda^b(z) = \frac{1 - \overline{b(\lambda)}b(z)}{1 - \overline{\lambda} z}.$$

This enables one to define an inner product of the finite sums of reproducing kernels K_λ^b as follows:

$$\left\langle \sum_{\lambda\in\Lambda} a_\lambda K_\lambda^b, \sum_{\mu\in\Lambda} b_\mu K_\mu^b \right\rangle = \sum_{\lambda,\mu\in\Lambda} a_\lambda \overline{b_\mu} K_\lambda^b(\mu).$$

This inner product defines a norm. Taking the closure of the span of the functions K^b_λ relative to this new norm miraculously gives the same space $\mathcal{H}(b)$ as the one constructed above.

The function theoretic properties of $\mathcal{H}(b)$ are strongly tied to a particular property of the function b. Namely, there is a major dichotomy, depending on whether b is an extremal or nonextremal point of the unit ball of $H^\infty(\mathbb{D})$. Recall that, in a normed linear space X and points $a \neq b \in X$, the set of convex combinations $ta + (1-t)b$, $t \in [0,1]$ is called the *interval* $[a,b]$. For a convex set $\Omega \subset X$, a point $p \in \Omega$ is an *extremal point* if it is not an interior point of any interval contained in Ω; equivalently, p is an extremal point for Ω if and only if $\Omega \setminus \{p\}$ is convex. For the closed unit ball Ω in $H^\infty(\mathbb{D})$, there is a nice test for extremal points [16]: a function $f \in H^\infty(\mathbb{D})$ with $\|f\|_{H^\infty} \leq 1$ is an extremal point of Ω if and only if

$$\int_0^{2\pi} \log(1 - |f(e^{it})|)dt = -\infty.$$

When b is not extremal, then roughly speaking the space $\mathcal{H}(b)$ "is like" $H^2(\mathbb{D})$. In this case one can define a related function a that plays an important role in this analysis:

$$a(z) = \exp\left(\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \log(1 - |b(\zeta)|^2)^{\frac{1}{2}} dm(\zeta)\right).$$

This function belongs to $H^\infty(\mathbb{D})$, $\|a\|_{H^\infty(\mathbb{D})} \leq 1$, and $|a|^2 + |b|^2 = 1$ almost everywhere on $\mathbb{T}$. The function a is affectionately called the *Pythagorean mate* of b. It allows one to define a norm on $\mathcal{H}(b)$ that is similar to that on $H^2(\mathbb{D})$. Indeed, for $f \in \mathcal{H}(b)$, one can show that there is a unique element $f^+ \in H^2(\mathbb{D})$, the so-called *twin* of f, such that

$$T_{\overline{b}} f = T_{\overline{a}} f^+.$$

Then one can also show that

$$\|f\|^2_{\mathcal{H}(b)} = \|f\|^2_{H^2(\mathbb{D})} + \|f^+\|^2_{H^2(\mathbb{D})},$$

giving rise to a useful representation of the norm on $\mathcal{H}(b)$, and by polarization, of the inner product.

When b is extremal, then roughly speaking $\mathcal{H}(b)$ is the model space K_Θ. This should not be so surprising since a particular example of an extremal function is when Θ is inner (since $\log(1 - |\Theta|) = -\infty$). In this situation, one can talk about analytic continuation of $f \in \mathcal{H}(b)$ across $\mathbb{T}$ and provide conditions under which this happens (e.g., when f is rational). One can also decide whether some specific functions belong to $\mathcal{H}(b)$. For instance, $K_\lambda \in \mathcal{H}(b)$ if and only if $b(\lambda) = 0$; $z^m \in \mathcal{H}(b)$ if and only if b has a zero of order $m+1$ at the origin.

The theory of $\mathcal{H}(b)$ spaces has played an important role in function theory, e.g., appearing in the solution of the Bieberbach conjecture by de Branges [12–14] and in the theory of rigid functions in $H^1(\mathbb{D})$ [25, 26]. Research on these spaces has since become an active area all on its own. I will attempt now to provide some pointers to work done in this area, which are definitely not exhaustive, but which we hope will direct the interested reader to other sources. (My apologies to those whose work was not cited explicitly!)

A deeper analysis of the functional model for contractions and its connections to $\mathcal{H}(b)$ spaces can be found in [3, 24].

A characterization of the multipliers on $\mathcal{H}(b)$, i.e., the functions m such that $mf \in \mathcal{H}(b)$ whenever $f \in \mathcal{H}(b)$, was carried out in [17, 20]. The multipliers that carry $\mathcal{H}(b_1)$ to $\mathcal{H}(b_2)$ were studied in [11].

The Carleson and reverse Carleson measures were studied in [6].

Comparison of $\mathcal{H}(b)$ and the Dirichlet space of analytic functions was carried out in [9, 10, 28].

A study of kernels of Toeplitz operators and nearly invariant subspaces appears in [29], and the results on surjectivity of Toeplitz operators can be found in [19].

Generalized Schwarz–Pick estimates and analogues of the Carathéodory–Julia theorem were obtained in [1, 7].

A study of weighted norm inequalities and their applications to $\mathcal{H}(b)$ spaces can be found in [4].

Some remarks about the books

We want to end with some general comments about the two volumes under review. They are of a monographic nature and are designed for a person who wants to learn the theory of these spaces and understand the state of the art in the area. All major results are included. In some situations the original proofs are provided, while in other cases they provide the "better" proofs that have become available since. The books are designed to be accessible to both experts and newcomers to the area. Comments at the end of each section are very helpful, and the numerous exercises were clearly chosen to help master some of the techniques and tools used. Combined, the two volumes total more than 1200 pages, and so they are probably more suitable for a year-long sequence in a topics course on complex analysis and operator theory. They are also appropriate for helping a novice learn the techniques and undertake research in the area.

Volume 1 contains a comprehensive treatment of the preliminaries needed to understand $\mathcal{H}(b)$ spaces. It is aimed more at the newcomer and serves as a great introduction to the necessary topics for this area of analysis. The topics include building appropriate foundations in operator theory, such as basic functional analysis, bases in Banach spaces, and some simple and more advanced operator theory facts. It also addresses the tools from function theory that will be needed, including some Fourier analysis and the theory of Hardy spaces. Some of them are then applied to study of Hankel operators, Toeplitz operators, model spaces K_Θ, reproducing kernel Hilbert spaces, Cauchy transforms, and connection between interpolation of sequences and bases of reproducing kernels in $H^2(\mathbb{D})$. As additional resources, the interested reader could also benefit from consulting the relevant chapters in the books [21–23], which provide complementary expositions of the topics.

In Volume 2, the authors turn to more details concerning the theory of $\mathcal{H}(b)$ spaces. In particular, they provide the precise definitions of these spaces via complementary spaces and via reproducing kernels. They then go on to provide geometric representations of the spaces and integral representations of functions belonging to them, and they discuss the existence of angular derivatives of $\mathcal{H}(b)$ functions. The dichotomy between b being extremal versus nonextremal and how this influences the function theory associated to $\mathcal{H}(b)$ forms a major theme of the volume. It is designed more for experts in the field and for those who are interested in detailed properties of the spaces $\mathcal{H}(b)$. The interested reader can compare the volume with some of the results in Sarason's text [27], although Volume 2 goes beyond what is contained there and provides many of the main developments in the subject since [27] first appeared. Additional viewpoints to some of the material in this volume can be found in the surveys [2, 32].

In sum, these are excellent books that are bound to become standard references for the theory of $\mathcal{H}(b)$ spaces.

References

[1] J. M. Anderson and J. Rovnyak, *On generalized Schwarz-Pick estimates*, Mathematika **53** (2006), no. 1, 161–168 (2007). MR2304058 ↑540

[2] J. A. Ball and V. Bolotnikov, *de Branges-Rovnyak Spaces: Basics and Theory* (2014), available at `https://arxiv.org/abs/1405.2980v1`. ↑541

[3] J. A. Ball and T. L. Kriete III, *Operator-valued Nevanlinna-Pick kernels and the functional models for contraction operators*, Integral Equations Operator Theory **10** (1987), no. 1, 17–61. MR868573 ↑539

[4] A. Baranov, E. Fricain, and J. Mashreghi, *Weighted norm inequalities for de Branges-Rovnyak spaces and their applications*, Amer. J. Math. **132** (2010), no. 1, 125–155. MR2597508 ↑540

[5] A. Beurling, *On two problems concerning linear transformations in Hilbert space*, Acta Math. **81** (1948), 17. MR0027954 ↑536

[6] A. Blandignères, E. Fricain, F. Gaunard, A. Hartmann, and W. T. Ross, *Direct and reverse Carleson measures for $\mathcal{H}(b)$ spaces*, Indiana Univ. Math. J. **64** (2015), no. 4, 1027–1057. MR3385785 ↑540

[7] V. Bolotnikov and A. Kheifets, *A higher order analogue of the Carathéodory-Julia theorem*, J. Funct. Anal. **237** (2006), no. 1, 350–371. MR2239269 ↑540

[8] A. Brown and P. R. Halmos, *Algebraic properties of Toeplitz operators*, J. Reine Angew. Math. **213** (1963/1964), 89–102. MR0160136 ↑537

[9] N. Chevrot, D. Guillot, and T. Ransford, *De Branges-Rovnyak spaces and Dirichlet spaces*, J. Funct. Anal. **259** (2010), no. 9, 2366–2383. MR2674117 ↑540

[10] C. Costara and T. Ransford, *Which de Branges-Rovnyak spaces are Dirichlet spaces (and vice versa)?*, J. Funct. Anal. **265** (2013), no. 12, 3204–3218. MR3110499 ↑540

[11] R. B. Crofoot, *Multipliers between invariant subspaces of the backward shift*, Pacific J. Math. **166** (1994), no. 2, 225–246. MR1313454 ↑540

[12] L. de Branges, *A proof of the Bieberbach conjecture*, Acta Math. **154** (1985), no. 1-2, 137–152. MR772434 ↑539

[13] L. de Branges, *Unitary linear systems whose transfer functions are Riemann mapping functions*, Operator theory and systems (Amsterdam, 1985), Oper. Theory Adv. Appl., vol. 19, Birkhäuser, Basel, 1986, pp. 105–124. MR903695 ↑539

[14] L. de Branges, *Underlying concepts in the proof of the Bieberbach conjecture*, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Berkeley, Calif., 1986), Amer. Math. Soc., Providence, RI, 1987, pp. 25–42. MR934213 ↑539

[15] L. de Branges and J. Rovnyak, *Square summable power series*, Holt, Rinehart and Winston, New York-Toronto, Ont.-London, 1966. MR0215065 ↑537

[16] K. de Leeuw and W. Rudin, *Extreme points and extremum problems in H_1*, Pacific J. Math. **8** (1958), 467–485. MR0098981 ↑539

[17] B. M. Davis and J. E. McCarthy, *Multipliers of de Branges spaces*, Michigan Math. J. **38** (1991), no. 2, 225–240. MR1098860 ↑540

[18] S. R. Garcia, J. Mashreghi, and W. T. Ross, *Introduction to model spaces and their operators*, Cambridge Studies in Advanced Mathematics, vol. 148, Cambridge University Press, Cambridge, 2016. MR3526203 ↑537

[19] A. Hartmann, D. Sarason, and K. Seip, *Surjective Toeplitz operators*, Acta Sci. Math. (Szeged) **70** (2004), no. 3-4, 609–621. MR2107530 ↑540

[20] B. A. Lotto, *Inner multipliers of de Branges's spaces*, Integral Equations Operator Theory **13** (1990), no. 2, 216–230. MR1038151 ↑540

[21] N. K. Nikol′skiĭ, *Treatise on the shift operator*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 273, Springer-Verlag, Berlin, 1986. Spectral function theory; With an appendix by S. V. Hruščev [S. V. Khrushchëv] and V. V. Peller; Translated from the Russian by Jaak Peetre. MR827223 ↑537, 540

[22] N. K. Nikolski, *Operators, functions, and systems: an easy reading. Vol. 2*, Mathematical Surveys and Monographs, vol. 93, American Mathematical Society, Providence, RI, 2002. Model operators and systems; Translated from the French by Andreas Hartmann and revised by the author. MR1892647 ↑540

[23] N. K. Nikolski, *Operators, functions, and systems: an easy reading. Vol. 1*, Mathematical Surveys and Monographs, vol. 92, American Mathematical Society, Providence, RI, 2002. Hardy, Hankel, and Toeplitz; Translated from the French by Andreas Hartmann. MR1864396 ↑540

[24] N. K. Nikol′skiĭ and V. I. Vasyunin, *Notes on two function models*, The Bieberbach conjecture (West Lafayette, Ind., 1985), Math. Surveys Monogr., vol. 21, Amer. Math. Soc., Providence, RI, 1986, pp. 113–141. MR875237 ↑539

[25] D. Sarason, *Exposed points in H^1. I*, The Gohberg anniversary collection, Vol. II (Calgary, AB, 1988), Oper. Theory Adv. Appl., vol. 41, Birkhäuser, Basel, 1989, pp. 485–496. MR1038352 ↑539

[26] D. Sarason, *Exposed points in H^1. II*, Topics in operator theory: Ernst D. Hellinger memorial volume, Oper. Theory Adv. Appl., vol. 48, Birkhäuser, Basel, 1990, pp. 333–347. MR1207406 ↑539

[27] D. Sarason, *Sub-Hardy Hilbert spaces in the unit disk*, University of Arkansas Lecture Notes in the Mathematical Sciences, vol. 10, John Wiley & Sons, Inc., New York, 1994. A Wiley-Interscience Publication. MR1289670 ↑541

[28] D. Sarason, *Local Dirichlet spaces as de Branges-Rovnyak spaces*, Proc. Amer. Math. Soc. **125** (1997), no. 7, 2133–2139. MR1396993 ↑540

[29] D. Sarason, *Algebraic properties of truncated Toeplitz operators*, Oper. Matrices **1** (2007), no. 4, 491–526. MR2363975 ↑540

[30] B. Sz.-Nagy and C. Foiaş, *Harmonic analysis of operators on Hilbert space*, Translated from the French and revised, North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., Inc., New York; Akadémiai Kiadó, Budapest, 1970. MR0275190 ↑537

[31] B. Sz.-Nagy, C. Foias, H. Bercovici, and L. Kérchy, *Harmonic analysis of operators on Hilbert space*, Revised and enlarged edition, Universitext, Springer, New York, 2010. MR2760647 ↑537

[32] D. Timotin, *A short introduction to de Branges–Rovnyak spaces*, Invariant subspaces of the shift operator, Contemp. Math., vol. 638, Amer. Math. Soc., Providence, RI, 2015, pp. 21–38. MR3309347 ↑541

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