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SELECTED MATHEMATICAL REVIEWS

related to the paper in the previous section by
JESUS A. DE LOERA ET AL.

MR0187147 (32 #4601) 52.30; 05.00

Tverberg, H.

A generalization of Radon's theorem.

Journal of the London Mathematical Society **41** (1966), 123–128.

A set of points in R^n is r-divisible if it can be divided into r subsets in such a way that the convex hulls of the r subsets have a non-empty intersection. The author proves that any set of $r(n+1)-n$ points in R^n is r-divisible. The case $r=2$ is essentially Helly's theorem, and the case $n=2$ has previously been proved by the reviewer [Proc. Cambridge Philos. Soc. **55** (1959), 289–293; MR0109315]. The author's ingenious proof is entirely combinatorial; though elementary and fairly short, it is decidedly difficult. His main step is to show that if $Q, P_1, \cdots, P_s$ are algebraically independent, and the set $P_1, \cdots, P_s$ is r-divisible, then so is $Q, P_2, \cdots, P_s$; to do this, he pushes P_1 along a straight line towards Q, and shows that if T is the set of values of t for which $(1-t)P_1 + tQ$, $P_2, \cdots, P_s$ is r-divisible, then T, which is clearly closed, is also open.

B. J. Birch

From MathSciNet, May 2019

MR1171558 (93h:52008) 52A37; 52A35

Bárány, I.; Larman, D. G.

A colored version of Tverberg's theorem.

Journal of the London Mathematical Society. Second Series **45** (1992), *no.* 2, 314–320.

Let $P_n \subset \mathbf{R}^d$ be n distinct points and let $r \geq 1$ be such that $n \geq (d+1)\cdot r$. Let P_n be divided into $d+1$ subsets $C_1, C_2, \cdots, C_{d+1}$, each C_i having cardinality at least r. The "colored Tverberg problem" is to find the least value $N(r,d)$ of n such that if $n \geq N(r,d)$ and P_n is any set as above, then there exist points $p_{ij} \in C_i$, $i = 1, 2, \cdots, d+1$, $j = 1, 2, \cdots, r$, such that the sets $A_j := \{p_{ij}\}_{i=1}^{d+1}$, $j = 1, \cdots, r$, are mutually disjoint but $\bigcap_{j=1}^{r} \operatorname{conv}(A_j) \neq \emptyset$. For $r=2$, this is the "colored Radon problem". The authors prove that (i) $N(r,1) = 2r$; (ii) $N(r,2) = 3r$; (iii) $N(2,d) = 2(d+1)$. They also conjecture that $N(r,d) = r(d+1)$. The proof of (i) is a simple induction on r. As pointed out by the authors, (iii) is a well-known statement proved by many people independently (in the paper an unpublished elegant proof of L. Lovász, 1989, is reproduced). The proof of (ii) given in the paper is not simple. As indicated, (ii) might also be credited to J. Jaromczyk and G. Świątek ["The optimal constant for the colored version of Tverberg's theorem", Geom. Dedicata, to appear] who proved (ii) only for $r=3$ but whose proof can be extended easily to the general case $r \geq 3$. (An interesting problem would be to compare the two proofs.)

Béla Uhrin

From MathSciNet, May 2019

MR1193538 (94a:52007) 52A20

Kahn, Jeff; Kalai, Gil

A counterexample to Borsuk's conjecture. (English)

Bulletin of the American Mathematical Society (New Series) **29** (1993), *no.* 1, 60–62.

Sixty years ago K. Borsuk [Fund. Math. **20** (1933), 177–190; Zbl **6**, 424] conjectured that every bounded subset A of Euclidean n-space $\mathbf{R}^n$ containing at least two points can be partitioned into at most $n+1$ sets, each of diameter less than that of A. Without loss of generality we may restrict our attention to subsets of $\mathbf{R}^n$ of diameter 1. Letting $f(n)$ denote the smallest integer such that every set in $\mathbf{R}^n$ of diameter 1 admits a partition into at most $f(n)$ subsets, each of diameter less than 1, Borsuk's conjecture becomes the assertion that $f(n) = n+1$. The vertices of the regular n-simplex in $\mathbf{R}^n$ show that $f(n) \geq n+1$. Borsuk's conjecture has been verified for dimensions 2 and 3, and the best result to date concerning the first unresolved case, $n = 4$, is that a subset of $\mathbf{R}^4$ of diameter 1 admits a partition into at most nine sets, each of diameter less than 1. The conjecture has been confirmed in all dimensions for both centrally symmetric convex bodies and convex bodies with smooth boundary, and the conjecture seems to have been generally believed. Despite the labor of many mathematicians over the past sixty years to obtain polynomial growth bounds on $f(n)$, the results obtained to date place at best exponential bounds on the growth of $f(n)$ as n approaches ∞. See, for example, papers by M. Lassak [Bull. Acad. Polon. Sci. Sér. Sci. Math. **30** (1982), no. 9-10, 449–451 (1983); MR0703571] and O. Schramm [Mathematika **35** (1988), no. 2, 180–189; MR0986627].

The paper under review constructs an unexpected counterexample to Borsuk's conjecture. The construction shows that Borsuk's conjecture is false for $n = 1825$ and for every $n > 2144$. Even more surprising is the verification that $f(n)$ grows exponentially as n approaches ∞; in fact, the authors show that $f(n) \geq (1.1)^{\sqrt{n}}$, making earlier exponential bounds on $f(n)$ seem much more reasonable than formerly.

Interesting questions remain. From the paper under review: "It would be of considerable interest to have a better understanding of the asymptotic behavior of $\log f(n)$" and "of interest would be counterexamples in small dimensions". Concerning the latter statement, what is $f(4)$? What is the smallest value of n for which $f(n) > n+1$? The counterexamples constructed in this paper are examples of finite sets that violate the Borsuk bound of $n+1$. How many elements has the smallest counterexample to the conjecture?

Philip L. Bowers

From MathSciNet, May 2019

MR2156212 (2006d:55007) 55M20; 05A99, 05B30, 52B70

Prescott, Timothy; Su, Francis Edward

A constructive proof of Ky Fan's generalization of Tucker's lemma. (English)

Journal of Combinatorial Theory. Series A **111** (2005), *no.* 2, 257–265.

Tucker's combinatorial lemma, named after this reviewer's father A. W. Tucker, plays the same role for the Borsuk-Ulam theorem that Sperner's lemma plays for the

Brouwer fixed point theorem. It was presented at the first Canadian Mathematical Congress in 1945 but has recent applications to Kneser-type coloring theorems and fair-division problems. Tucker's original lemma, presented as a problem of dealing cards onto a table, states that for any labeling by $\pm 1, \pm 2$ of the vertices of a two-dimensional rectangular grid, if antipodal points on the boundary get opposite labels (i.e. summing to 0), then at least one pair of adjacent vertices also gets opposite labels. K. Fan's generalization [Ann. of Math. (2) **56** (1952), 431–437; MR0051506] is to label by $\pm 1, \ldots, \pm m$ the vertices of a barycentric subdivision of the octahedral subdivision of the sphere S^n so that antipodal vertices get opposite labels and adjacent vertices do not. The conclusion is that there are an odd number of n-simplices whose labels are an alternating sequence, strictly increasing in absolute value. In particular, no such labeling exists for $n = m$.

This paper presents a new constructive proof of Ky Fan's lemma applied to any antipodally symmetric triangulation of S^n that contains a flag of hemispheres. The proof defines an adjacency relationship for the labeled simplices such that the associated graph consists of disjoint paths beginning at a vertex and ending at an n-simplex having the desired type of labeling. Tucker's original proof, as well as Ky Fan's, depends on a nonconstructive parity argument. The clever constructive proof presented by the authors is closer in spirit to the constructive proofs of Sperner's lemma that inspired the 1970s work by M. J. Todd and others on the computation of fixed points.

Thomas W. Tucker
From MathSciNet, May 2019

MR2946447 14P05; 51Bxx, 52C10, 52C30

Solymosi, József; Tao, Terence

An incidence theorem in higher dimensions. (English)

Discrete & Computational Geometry. An International Journal of Mathematics and Computer Science **48** (2012), *no.* 2, 255–280.

In this interesting article the authors generalize (a weaker form of) the Szemerédi-Trotter theorem that bounds the number of point-line incidences in the plane to points versus real algebraic varieties. The main theorem proven is the following: For any given integers $k, d \geq 0$, $d \geq 2k$ and real numbers $\varepsilon > 0$, $c_0 \geq 1$ there is a constant $a = a(k, \varepsilon, c_0) > 0$ such that

$$|I| \leq a|P|^{2/3+\varepsilon}|L|^{2/3} + 3|P|/2 + 3|L|/2$$

holds, whenever $P \subset \mathbf{R}^d$ are finitely many points, L are finitely many real algebraic varieties in $\mathbf{R}^d$, each of dimension k and degree at most c_0 (to be made precise), I is a set of (not necessarily all) pairs (p, l) with $p \in P$, $l \in L$ and $p \in l$, and P, L and I satisfy certain pseudo-line type axioms which we explain next. Each $l \in L$ is in fact assumed to be the restriction to $\mathbf{R}^d$ of a complex algebraic variety l', and the dimension and degree refer to those of l'. As to the axioms: (i) if $l, l' \in L$, $l \neq l'$ then $|\{p \in P \mid (p, l), (p, l') \in I\}| \leq c_0$, (ii) if $p, p' \in P$, $p \neq p'$ then $|\{l \in L \mid (p, l), (p', l) \in I\}| \leq c_0$, (iii) if $(p, l) \in I$ then p is a smooth, in the real sense, point of l (there is a unique tangent space $T_p l$ of l at p, which is a k-dimensional real affine space containing p) and (iv) if $(p, l), (p, l') \in I$, $l \neq l'$ then $T_p l \cap T_p l' = \{p\}$ (so l and l' intersect transversally at p).

The proof of the main theorem is given in Section 5, the main tools being "the polynomial ham sandwich theorem and induction on both the dimension and the number of points". In addition to a comparison with existing results, Section 2 contains several applications of the main theorem, for example one on the sums versus products theme: If $A \subset \mathbf{R}^{k\times k}$ are n matrices such that each $M - M'$, $M, M' \in A$, $M \neq M'$ is non-singular and $V, W \subset \mathbf{R}^k$ are two n-element sets of column vectors, then $|V+W|+|AW| \gg_{k,\varepsilon} n^{5/4-\varepsilon}$ for $\varepsilon > 0$. Section 3 sketches the proof of a particular case of the main theorem, the "cheap" (i.e., with bound involving ε) complex Szemerédi-Trotter theorem (for points and lines in $\mathbf{C}^2$). Section 4 reviews required notions and facts from algebraic geometry, such as dimension or degree. Lemmas on relation between degree and complexity of a variety and on smooth points are proven (sometimes by referring to the literature). The appendix contains two theorems, with proofs, giving bounds on the numbers of connected components of various real semi-algebraic sets.

Martin Klazar

From MathSciNet, May 2019

MR3336834 52A35; 05A18, 55S91

Blagojević, Pavle V. M.; Matschke, Benjamin; Ziegler, Günter M.

Optimal bounds for the colored Tverberg problem. (English)

Journal of the European Mathematical Society (JEMS) **17** (2015), *no.* 4, 739–754.

The colored Tverberg problem is stated as follows: Determine the smallest number $t = t(d,r)$ such that for every collection $\mathcal{C} = C_0 \uplus \cdots \uplus C_d$ of points in $\mathbb{R}^d$ with $|C_i| \geq t$, there are r disjoint subcollections $F_1, \ldots, F_r$ of $\mathcal{C}$ satisfying

(A) $|C_i \cap F_j| \leq 1$ for every $i \in \{0, \ldots, d\}$ and $j \in \{1, \ldots, r\}$, and

(B) $\operatorname{conv}(F_1) \cap \cdots \cap \operatorname{conv}(F_r) \neq \emptyset$.

This problem originated from Tverberg's 1966 theorem saying that any family of $(d+1)(r-1)+1$ points in $\mathbb{R}^d$ can be partitioned into r sets whose convex hulls intersect. In 1990, I. Bárány, Z. Füredi and L. Lovász [Combinatorica **10** (1990), no. 2, 175–183; MR1082647] dealt with the case of three triangles in the plane. In 1992, Bárány and D. G. Larman [J. London Math. Soc. (2) **45** (1992), no. 2, 314–320; MR1171558] formulated the above general problem and solved the planar case. In addition, they also posed the Bárány-Larman conjecture: $t(d,r) = r$ for all $r \geq 2$ and $d \geq 1$. This conjecture became one of the main components of the colored Tverberg problem.

The paper under review deals with the topological versions of the colored Tverberg problem and the Bárány-Larman conjecture. The important contributions of the paper under review are stated as follows:

Theorem A. Let $r \geq 2$ be prime, $d \geq 1$, and $N = (r-1)(d+1)$. Let Δ_N be an N-dimensional simplex with a partition of its vertex set into $m+1$ parts (color classes)

$$\mathcal{C} = C_0 \uplus \cdots \uplus C_m$$

with $|C_i| \leq r-1$ for all i. Then for every continuous map $f\, \Delta_N \longrightarrow \mathbb{R}^d$, there is a colored r-partition given by disjoint faces $F_1, \ldots, F_r$ of Δ_N whose images under f intersect, that is,

(A) $|C_i \cap F_j| \leq 1$ for every $i \in \{0, \ldots, m\}$ and $j \in \{1, \ldots, r\}$, and

(B) $f(F_1) \cap \cdots \cap f(F_r) \neq \emptyset$.

This theorem is a strengthening of the topological Tverberg theorem in the prime case.

Theorem B. If $r+1$ is prime, then $t(d,r) = tt(d,r) = r$, where $tt(d,r)$ is the topological version of $t(d,r)$.

One of the key ingredients in the proof is relative equivariant obstruction theory.

Zhi Lü

From MathSciNet, May 2019

MR3726616 52A35; 05C15, 68U05

Mirzakhani, Maryam; Vondrák, Jan

Sperner's colorings and optimal partitioning of the simplex. (English)

A journey through discrete mathematics, 615–631, *Springer, Cham*, 2017.

In this paper the authors study a number of variations of Sperner's lemma, a classical result in combinatorial topology. To state some of their results, let $\mathbf{e}_1, \ldots, \mathbf{e}_k$ denote the standard basis for $\mathbb{R}^k$. Given $q \in \mathbb{N}$, let $\Delta_{k,q}$ denote the $(k-1)$-dimensional simplex

$$\Delta_{k,q} := \mathrm{conv}(q\mathbf{e}_1, \ldots, q\mathbf{e}_k) = \Big\{\mathbf{x} \in \mathbb{R}^k : \sum_{i=1}^{k} x_i = q \text{ and } \mathbf{x} \geq \mathbf{0}\Big\}.$$

Also let $H_{k,q}$ denote the set of simplices $S(\mathbf{b}) := \mathrm{conv}(\mathbf{b}+\mathbf{e}_1, \ldots, \mathbf{b}+\mathbf{e}_k)$ for $\mathbf{b} \in \Delta_{k,q-1} \cap \mathbb{N}^k$.

These simplices of $H_{k,q}$ are all contained in $\Delta_{k,q}$ and can be extended to a triangulation $\mathbf{T}$ of $\Delta_{k,q}$. Sperner's lemma shows that any Sperner-admissible colouring of the vertices of $\mathbf{T}$ contains a $(k-1)$-dimensional simplex whose vertices receive k distinct colours. On the other hand, these multicoloured simplices need not appear in $H_{k,q}$. The authors prove a tight lower bound on the number of non-monochromatic simplices of $H_{k,q}$ that must appear under a Sperner-admissible colouring. They also construct a Sperner-admissible colouring in which these simplices receive at most four colours, provided $k \geq 4$ and $q \geq k^2$.

The second focus of the paper is a variant of the Knaster-Kuratowski-Mazurkiewicz lemma, a geometric version of Sperner's lemma. This result says that given closed sets $A_1, \ldots, A_k$ with $\Delta_{k,1} = \bigcup_{i\in[k]} A_i$ and $A_i \subset \Delta_{k,1} \cap \{\mathbf{x} : x_i > 0\}$ for all $i \in [k]$, we have $\bigcap_{i\in[k]} A_i \neq \emptyset$. The authors prove an optimal lower bound on the measure of $\bigcup_{i\neq j}(A_i \cap A_j)$, showing that it is minimised when $A_1, \ldots, A_k$ are Voronoi cells in $\Delta_{k,1}$ induced from the points $\{\mathbf{e}_1, \ldots, \mathbf{e}_k\}$.

The paper concludes with a large number of open problems.

Eoin Long

From MathSciNet, May 2019