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Renewal theory is a classical branch of probability which generalizes the notion of Poisson processes. Given a sequence of nonnegative, i.i.d. random variables $(T_n)_{n\geq 0}$ such that $0 < \mathbb{E}[T_n] < \infty$, one defines the renewal process $(X_t)_{t\geq 0}$ by

$$X_t = \sup\Big\{N \geq 0 \ : \ \sum_{n=1}^{N} T_n \leq t\Big\}.$$

A standard interpretation for the above process is the following: if one considers a sequence of events occurring one after the other at random times $(T_n)_{n\geq 1}$ (e.g., T_n can be the lifetime of the nth light bulb), then X_t is the number of events that occurred up to time t, which in our example is the number of light bulbs that stopped working up to time t. Since the random variables $(T_n)_{n\geq 1}$ are independent, the Strong Law of Large Numbers (SLLN) implies that $\sum_{n=1}^{N} T_n \approx N\mathbb{E}[T_1]$ and therefore $X_t \approx t/\mathbb{E}[T_1]$. This is made precise by the following result:

Theorem 1 (Elementary Renewal Theorem (ERT))**.**

$$\lim_{t\to\infty} \frac{1}{t} X_t = \frac{1}{\mathbb{E}[T_1]} \quad a.s. \qquad\qquad and \qquad\qquad \lim_{t\to\infty} \frac{1}{t}\mathbb{E}[X_t] = \frac{1}{\mathbb{E}[T_1]}\,.$$

One can also regard the process $(T_n)_{n\geq 1}$ as generating a random semimetric on the natural numbers. Indeed, consider $\mathbb{N}$ with the standard edge set $\{\{n-1, n\} \ : \ n \geq 1\}$. For every $n \geq 1$, assign the edge $\{n-1, n\}$ a weight S_n, which can be thought of as the amount of time needed to cross it or as the distance between $n-1$ and n. Then X_t is the maximal location reachable up to time (distance) t when starting from 0.

First Passage Percolation (*FPP*) was introduced by Hammersley and Welsh [16] as a generalization of renewal theory and percolation to arbitrary graphs. Here, for the sake of simplicity, we will only consider the d-dimensional Euclidean lattice $\mathbb{Z}^d$ for $d \geq 1$. We denote by $\|\cdot\|_p$ the ℓ_p norm and by $\mathcal{E}_d$ the set of edges in $\mathbb{Z}^d$, i.e., pairs of vertices $x, y \in \mathbb{Z}^d$ such that $\|x-y\|_1 = 1$.

FPP is defined as follows: Let $(T_e)_{e\in\mathcal{E}_d}$ be a family of i.i.d. nonnegative random variables with distribution function F. We consider T_e as the time needed to cross the edge e (or as the length of the edge) and for a finite (or infinite) path γ, a sequence of edges $(e(n))_{n\geq 1}$ such that $e(n)\cap e(n+1) \neq \emptyset$ for every $n \geq 1$, we define the passage time of γ

$$T(\gamma) = \sum_{e\in\gamma} T_e\,.$$

The distance (passage time) between any pair of vertices $x, y \in \mathbb{Z}^d$ is

$$T(x,y) = \inf\{T(\gamma) \ : \ \gamma \text{ is a finite path from } x \text{ to } y\}$$

and the analogue of the renewal process $(X_t)_{t\geq 0}$ is defined to be

$$B(t) = \{x \in \mathbb{Z}^d \ : \ T(0,x) \leq t\} \qquad \forall t \geq 0.$$

If $F(0) = \mathbb{P}(T_e \leq 0) = 0$, then $T(\cdot,\cdot)$ almost surely defines a metric on $\mathbb{Z}^d$, and $B(t)$ is the random ball of radius t around the origin in it. For simplicity we will assume throughout the remainder of the review that $F(0) = 0$.

We extend the definition of T and B to $\mathbb{R}^d$ by letting $[y]$ be the unique point in $\mathbb{Z}^d$ such that $y \in [y] + [-1/2, 1/2)^d$, setting $T(y,z) := T([y],[z])$ for all $y, z \in \mathbb{R}^d$ and defining

$$\widetilde{B}(t) = \{x \in \mathbb{R}^d \ : \ T(0,x) \leq t\} = \bigcup_{x \in B(t)} (x + [-1/2, 1/2)^d)\,.$$

Some of the fundamental questions we would like to understand about the model are the following: What can we say about $T(x,y)$ as $\|x - y\|_2 \to \infty$? How does $\widetilde{B}(t)$ looks like as $t \to \infty$? How do geodesics in the random metric T look like? How does the answer to the above questions depend on the distribution F? What can be said about the model on different graphs?

In the one-dimensional setting (i.e., $d = 1$) for every $N \geq 1$, there exists a unique path from 0 to N and therefore $T(0,N) = \sum_{n=1}^{N} T_{(n-1,n)}$. Consequently, by Theorem 1,

$$\lim_{t\to\infty} \frac{1}{t}(\widetilde{B}_t \cap [0,\infty)) = \lim_{t\to\infty} \frac{1}{t}\Big[-\frac{1}{2}, X_t + \frac{1}{2}\Big] = \Big[0, \frac{1}{\mathbb{E}[T_{0,1}]}\Big], \qquad \mathbb{P}\text{-a.s.}\,,$$

which answers all of the above questions for the graph $\mathbb{Z}$.

When $d \geq 2$, there is more than one path between any pair of points and therefore the situation is much more involved. Indeed, denote by $e_1 = (1,0,0,\ldots,0)$ the first coordinate vector in $\mathbb{Z}^d$. In the one-dimensional case $T(0, Ne_1) = \sum_{n=1}^{N-1} T_{(n-1,n)}$, that is the passage time is additive, and one can use the SLLN. However, when $d \geq 2$, one can only show that $T(0,Ne_1) \leq \sum_{n=1}^{N-1} T_{(ne_1,(n+1)e_1)}$, which is not sufficient for applying the SLLN.

Subadditivity and the shape theorem. Hammersley and Welsh [16] extracted from the study of FPP an important family of stochastic processes, called subadditive processes (see definition in Theorem 2), and they conjectured that the limit $\lim_{N\to\infty} \frac{X_N}{N}$ exists whenever $(X_n)_{n\geq 0}$ is subadditive. Their conjecture was proved by Kingman in his seminal work on the subadditive ergodic theorem [20, 21]. As of today the subadditive ergodic theorem has quite a few versions with different assumptions. We bring here a simplified version of a result by Liggett [23]:

Theorem 2 (Subadditive ergodic theorem)**.** *Let $(X_{m,n})_{0\leq m<n}$ be a family of nonnegative random variables that satisfies the following conditions:*

(1) $X_{0,n} \leq X_{0,m} + X_{m,n}$ *for all* $0 \leq m < n$.
(2) *The distribution of* $(X_{m,m+k})_{k\geq 1}$ *and* $(X_{m+1,m+1+k})_{k\geq 1}$ *is the same for all* $m \geq 0$.
(3) *For every* $k \geq 1$, *the sequence* $(X_{nk,(n+1)k})_{n\geq 1}$ *is stationary and ergodic.*
(4) $\mathbb{E}[X_{0,1}] < \infty$.

Then the limit $\lim_{n\to\infty} \frac{X_{0,n}}{n}$ *exists in* $\mathbb{P}$*-a.s. and in* L^1 *and is equal to* $\inf_{n\geq 1} \frac{1}{n}\mathbb{E}[X_{0,n}]$.

One can apply the subadditive ergodic theorem to the process $X_{m,n} = T(me_1, ne_1)$ which satisfies conditions (1)–(3) and prove

Theorem 3 (Theorem 2.19 in [18]). *Assume*[1] *that* $\mathbb{E}[T(0,e_1)] \in (0,\infty)$. *Then there exists a constant* $\mu(e_1) \in (0,\infty)$ *such that*

$$\lim_{N\to\infty} \frac{T(0,Ne_1)}{N} = \mu(e_1), \qquad \textit{a.s. and in } L^1 .$$

In fact, using that in $\mathbb{Z}^d$ there are $2d$ disjoint paths $(\gamma_j)_{j=1}^{2d}$ from 0 to e_1, and therefore $X_{0,1} = T(0,e_1) \le \min\{T(\gamma_1),\dots,T(\gamma_{2d})\}$, one can characterize the cases when $\mathbb{E}[T(0,e_1)] < \infty$.

Lemma 4. $\mathbb{E}[T(0,e_1)] < \infty$ *if and only if* $\mathbb{E}[\min\{Y_1,\dots,Y_{2d}\}] < \infty$, *where* $(Y_j)_{j=1}^{2d}$ *are i.i.d. random variables distributed according to* F.

One can repeat the argument above for an arbitrary rational direction $x \in \mathbb{Q}^d$ instead of e_1, thus proving the existence of a function $\mu : \mathbb{Q}^d \to (0,\infty)$ such that

$$\lim_{n\to\infty} \frac{T(0,nx)}{n} = \mu(x), \qquad \text{a.s. and in } L^1 .$$

The existence of the limit for all $x \in \mathbb{Q}^d$ together with the subadditivity of T, implies that μ is a norm on $\mathbb{Q}^d$ with a unique continuous extension to a norm on $\mathbb{R}^d$. Furthermore, μ is invariant under symmetries of $\mathbb{Z}^d$ that preserve the origin. Combining all of the above together with a uniform bound on the growth rate in different directions (see [5, Theorem 2.17] for detail) one can prove the following shape theorem:

Theorem 5 (Shape theorem for FPP, Cox and Durrett [9]). *Let* F *be any distribution such that* $\mathbb{E}[T_{0,e_1}] < \infty$, *and define the deterministic set* $\mathcal{B} = \{x \in \mathbb{R}^d \ : \ \mu(x) \le 1\}$. *Then, for every* $\varepsilon > 0$,

$$\mathbb{P}\Big((1-\varepsilon)t\mathcal{B} \subset \widetilde{B}(t) \subset (1+\varepsilon)t\mathcal{B} \textit{ for all sufficiently large } t\Big) = 1 .$$

Furthermore, $\mathcal{B}$ *is convex, compact, has a nonempty interior, and has the symmetries of* $\mathbb{Z}^d$ *that fix the origin.*

Despite the success of the subadditive ergodic theorem, it does not give us an explicit expression for the function μ or for the shape $\mathcal{B}$. Determining the value of μ and, in particular $\mu(e_1)$ as a function of F, is one of the fundamental and challenging problems in FPP. Nevertheless, partial results regarding the possible limiting shapes and the relation between shapes of different distribution do exist. For example, we consider the following.

(1) The limiting shape and the function μ are known to be continuous with respect to weak convergence; see [10, 15, 18]. That is, if $F_n \Rightarrow F$ and if μ_n, μ are respective limits, then $\lim_{n\to\infty} \mu_n(x) = \mu(x)$ for every $x \in \mathbb{R}^d$.
(2) If $F(1^-) = 0$ and $F(1)$ is sufficiently large, then the shape theorem is known to have flat edges in its boundary yet not to be polygonal; see [2, 12, 24]. Note that it is conjectured that the shape is strictly convex whenever F is continuous.

[1] Note that neither the positivity nor the finiteness of $\mathbb{E}[T(0,e_1)]$ is needed; see [18] for the former and [7] for the later.

FIGURE 1. On the left, $B(200)$ for exponential 1 distribution. On the right, $B(500)$ for the distribution taking the values 1 and 20 with probability 1/2.

Fluctuations and concentration. The shape theorem provides us with a first-order approximation for $T(0,\cdot)$, namely $T(0,x) = \mu(x) + o(\|x\|_1)$ as $\|x\|_1 \to \infty$ almost surely. Now, we wish to discuss the error term, i.e., $T(0,x) - \mu(x)$. The error is composed of a random part $T(0,x) - \mathbb{E}[T(0,x)]$ (the fluctuations) and a deterministic part $\mathbb{E}[T(0,x)] - \mu(x)$. Here we will concentrate on the former, referring the reader to [5] for further detail and a discussion on the nonrandom error.

The most basic control on the fluctuations of a random variable is given by its variance. It was predicted in the physics literature (see [17, 29]) that there exists a dimension dependent constant $\chi = \chi(d)$ such that $\mathrm{Var}(T(0,x)) \sim \|x\|_2^{2\chi}$ as $\|x\|_2 \to \infty$. Furthermore, this constant is conjectured to be universal in the sense that it should not depend on the distribution F (as long as F satisfies certain mild conditions) and to be equal to $1/2$ for $d=1$ and $1/3$ for $d=2$.

In fact, the case $d=1$ is relatively simple. Indeed, let us recall that for $d=1$, the term $T(0,Ne_1) = \sum_{n=1}^{N} T_{(n-1,n)}$ is a sum of i.i.d. random variables. The Central Limit Theorem (CLT) implies that the fluctuations are of order $N^{1/2}$ provided F has a finite second moment.

If the distribution F is bounded (i.e., $F(M) = \mathbb{P}(T_e \le M) = 1$ for some $M \in (0,\infty)$), then $T(0,ne_1) \le Mn$ and therefore $\mathrm{Var}(T(0,ne_1)) \le M^2n^2$. The first rigorous nontrivial bound on the variance was given by Kesten [18, Theorem 5.16] who showed that $\mathrm{Var}(T(0,ne_1)) \le \frac{Cn^2}{\log^\alpha n}$ for some constants $C,\alpha \in (0,\infty)$ depending only on d. Although the bound is only logarithmically better than the trivial bound (at least for bounded distributions), the proof is far from being trivial. Later on (see [19]) Kesten improved his previous result showing that $C_1 \le \mathrm{Var}((T(0,x)) \le C_2\|x\|_1$ for some $C_1, C_2 \in (0,\infty)$ and all $x \in \mathbb{Z}^d$, provided the second moment of F is finite and positive.

As it turns out, improving the variance bounds, both from below and from above, is quite a difficult task. As of today, the best known upper bound on the variance for $d \ge 2$ is $\mathrm{Var}(T(0,x)) \le \frac{C\|x\|_1}{\log\|x\|_1}$ for some $C \in (0,\infty)$. As in the original work of Kesten and despite the the improvement being only logarithmic, the proof is highly nontrivial. The first proof of sublinear variance for $T(0,x)$ is due to Benjamini, Kalai, and Schramm [6]. Their proof used the notion of influence in Boolean functions together with a hypercontractivity inequality by Talagrand. Their proof only applies to Bernoulli distribution, taking two values $0 < a < b < \infty$

with equal probability. The proof was generalized later on to Gamma distributions [14] and to all distributions with $\int x^2(\log x)_+ F(dx) < \infty$; see [11]. Regarding lower bounds, the state of the art is the following result (see also [8]):

Theorem 6 (Newman and Piza [26])**.** *Let $d = 2$, and let I be the infimum of the support of F. If F has a finite and positive second moment and $F(I)$ is sufficiently small* (*see* [26] *for a precise statement*), *then* $\mathrm{Var}(T(0,x)) \geq B \log \|x\|_1$ *for some constant $B > 0$.*

Geodesics and the wandering exponent. The last type of question related to FPP that we will discuss is about the geometry of geodesics with respect to the random metric T. A path γ from x to y is called *geodesic* if $T(\gamma) = T(x,y)$. The first natural question regarding geodesics is whether they exist. Following several results with different assumptions on the distribution F, the question was finally settled in the two-dimensional case.

Theorem 7 (Wierman and Reh [28])**.** *For any distribution F, there almost surely exists a geodesic between any two points in $\mathbb{Z}^2$.*

In dimension $d \geq 3$, geodesics are known to exist for bounded distributions; however, for general distributions the existence is currently an open problem.

The geometry of geodesics in FPP was studied extensively, although many questions remain open. Let us mention two results regarding their size and diameter. A lower bound on the size of geodesics was proved by Kesten in [18].

Theorem 8 ([18] Proposition 5.8)**.** *There are constants $a, C \in (0,\infty)$ such that*

$$\mathbb{P}\left(\exists \text{ a self-avoiding path } \gamma \text{ containing } 0 \text{ s.t. } |\gamma| \geq n \text{ and } T(\gamma) < an\right) \leq e^{-Cn},$$

and therefore

$$\liminf_{\|x\|_1 \to \infty} \frac{T(0,x)}{\|x\|_1} \geq a, \qquad \text{almost surely}.$$

The diameter of the set of geodesics was studied by the authors of the book, Auffinger, Damron, and Hanson.

Theorem 9 ([4] Theorem 6.2)**.** *Let $\overline{\mathrm{GEO}}(0,x)$ denote the set of vertices in $\mathbb{Z}^d$ belonging to at least one geodesic*[2] *from 0 to x. Then, there exist constants $M, C \in (0,\infty)$ such that for every $x \in \mathbb{Z}^d$*

$$\mathbb{P}(\mathrm{diam}(\overline{\mathrm{GEO}}(0,x)) \geq M\|x\|_\infty) \leq e^{-C\|x\|_\infty}.$$

In particular, almost surely

$$\limsup_{\|x\|_\infty \to \infty} \mathrm{diam}(\overline{\mathrm{GEO}}(0,x))/\|x\|_\infty \leq M.$$

One of the most natural asymptotic problems related to geodesics is estimating the distance of the geodesics from the ℓ^1 norm. Since FPP metric can be thought of as a random perturbation of the ℓ^1 metric, one may wonder how close the two are. One way to quantify this problem is described in the question, How large is the distance between the geodesic from 0 to ne_1 and the line $L_{e_1} = \{te_1 : t \geq 0\}$? Denote by $D(n)$ the maximal distance of a point in $\overline{\mathrm{GEO}}(0, ne_1)$ from L_{e_1}. Large deviation results by Kesten [18] imply that if e_1 is an exposed point of the limiting

[2] Note that if F is continuous, then almost surely there is only one geodesic between any pair of points.

shape (i.e., some continuous linear functional on $\mathbb{R}^d$ attains its strict maximum over $\mathcal{B}e_1$), then

$$\lim_{\varepsilon\to 0}\liminf_{n\to\infty}\mathbb{P}(D(n) < \varepsilon n) = 1\,,$$

i.e., the geodesic from 0 to ne_1 stay at distance $o(n)$ from L_{e_1}.

In order to better discuss estimations for $D(n)$, we introduce the wandering exponent ξ. Informally, it is a positive number such that $D(n) \sim n^{\xi}$. Like the exponent related to variance χ, it is believed that ξ is universal in the sense that, for a fixed dimensions d, it is the same for all reasonable distributions. Furthermore, it is believed that for $d = 2$, one should have $\xi = \frac{2}{3}$ and that the scaling relation $\chi = 2\xi - 1$ holds for all distributions with uniform positive curvature of the limit shape (see below). There is no agreement on the "correct" way to define the exponent ξ, and various upper and lower bounds were proven for different definitions of it. Note that as of today there is no single distribution for which the relation $\chi = 2\xi - 1$ is proven rigorously.

Let us provide an intuition for the conjecture of the scaling relation $\chi = 2\xi - 1$ under the assumption of uniform positive curvature in e_1, that is $\mu(u+z) - \mu(u) \sim |z|^2$ for all $z = te_2$, with $|t|$ sufficiently small. The reader is referred to [3] for a more general and detailed discussion. Let $T'(0, ne_1)$ be an independent copy of $T(0, ne_1)$. Then

$$n^{2\chi} \sim \mathrm{Var}(T(0, ne_1)) = \mathbb{E}[(T(0, ne_1) - T'(0, ne_1))^2]\,.$$

Since the geodesic from 0 to ne_1 stays within distance n^{ξ} from L_{e_1} it follows that "with high probability" the geodesic from $n^{\xi}e_2$ to $n^{\xi}e_2 + ne_1$ and the geodesic from 0 to ne_1 are in disjoint boxes and are thus independent. Therefore,

$$\begin{aligned} n^{2\chi} &\sim \mathbb{E}[(T(0, ne_1) - T(n^{\xi}e_2, n^{\xi}e_2 + ne_1))^2] \sim (\mu(ne_1 - n^{\xi}e_2) - \mu(ne_1))^2 \\ &= n^2\big(\mu(e_1 - n^{\xi-1}e_2) - \mu(e_1)\big)^2 \sim n^{2+4(\xi-1)} = n^{4\xi-2}\,, \end{aligned}$$

where in the step before last we used the uniform curvature assumption.

Growth models and other motivations. We end this review with a discussion on some related models and further motivations for studying FPP. It was noted already in the original paper of Hammersley and Welsh [16] that FPP has numerous interpretations and applications. Since then, the list of applications has lengthened considerably. A very incomplete list includes fluid flow models [16], infection models [27], defects in solids with imperfections [22], and growth models [1, 13, 25]. As an example, we discuss Eden growth model.

The Eden model was introduced in 1961 by Murray Eden [13] as a two-dimensional growth process of cells. The model can be described as follows: Start with a single cell at the origin in $\mathbb{Z}^d$. Assume the nth step ends with a finite, connected set A_n in $\mathbb{Z}^d$ containing the origin. Then, in the next step one defines A_{n+1} as $A_n \cup \{a_{n+1}\}$, where a_{n+1} is chosen uniformly at random (and independently from all previous choices) from the outer (vertex) boundary of A_n, i.e., from $\partial A_n = \{y \in \mathbb{Z}^d \setminus A_n \ :\ \exists x \in A_n \text{ such that } \|x - y\|_1 = 1\}$.

The Eden model can be constructed from (a variant of) FPP in $\mathbb{Z}^d$ with exponential with mean 1 distribution sampled at random times. Indeed, let $(X_v)_{v\in\mathbb{Z}^d}$ be a family of i.i.d. exponential 1 random variables, and define vertex FPP by declaring $T(\gamma)$, for a path γ, to be the sum of the weights of the vertices along γ. Denoting

$\sigma_0 = 0$ and for $n \geq 1$

$$\sigma_n = \inf\{t \geq 0 \ : \ |B(t)| = n+1\},$$

where $B(t)$ is the ball associated with FPP, one can verify that $\{A_n \ : \ n \geq 0\}$ and $\{B(\sigma_n) \ : \ n \geq 0\}$ have the same distribution.

The last observation allows us to conclude that $(A_n)_{n\geq 1}$ has a limit shape, namely, $A_n = B(\sigma_n) \approx \sigma_n \mathcal{B}$, where $\mathcal{B}$ is the limiting shape of vertex FPP with exponential 1 distribution.

About the book. The wonderful book by Auffinger, Damron, and Hanson provides an up-to-date and thorough discussion of one of the most classical models in probability theory. The book is readable and user friendly, yet provides the reader with detailed proofs and intuition for classical and modern results. It is therefore an excellent source for an advanced course on the subject. In addition, the book contains dozens of open questions, and it is thus a natural tool for researchers who are interested in the subject.

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