

ALGEBRAIC HYPERSURFACES

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ABSTRACT. We give an introduction to the study of algebraic hypersurfaces, focusing on the problem of when two hypersurfaces are isomorphic or close to being isomorphic. Working with hypersurfaces and emphasizing examples makes it possible to discuss these questions without any previous knowledge of algebraic geometry. At the end we formulate the main recent results and state the most important open questions.

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Algebraic geometry started as the study of plane curves $C \subset \mathbb{R}^2$ defined by a polynomial equation and later extended to surfaces and higher dimensional sets defined by systems of polynomial equations. Besides using \mathbb{R}^n , it is frequently more advantageous to work with C^n or with the corresponding projective spaces $\mathbb{R}\mathbb{P}^n$ and $\mathbb{C}\mathbb{P}^n$.

Later it was realized that the theory also works if we replace \mathbb{R} or \mathbb{C} by other fields, for example the field of rational numbers \mathbb{Q} or even finite fields \mathbb{F}_q . When we try to emphasize that the choice of the field is pretty arbitrary, we use \mathbb{A}^n to denote *affine* n -space and \mathbb{P}^n to denote *projective* n -space.

Received by the editors May 29, 2018.

2010 *Mathematics Subject Classification*. Primary 14-02, 14J70; Secondary 14M20, 14E08, 14N25.

Partial financial support was provided by the NSF under grant number DMS-1362960 and by the Nemmers Prize of Northwestern University.

Conceptually, the simplest algebraic sets are *hypersurfaces*; these are defined by one equation. That is, an *affine algebraic hypersurface* of dimension n is the zero set of a polynomial $h(x_1, \dots, x_{n+1})$:

$$X = X(h) := (h(x_1, \dots, x_{n+1}) = 0) \subset \mathbb{A}^{n+1}.$$

We say that X is *irreducible* if h is irreducible, and we call $\deg h$ the *degree* of X .

Working over \mathbb{C} , one can (almost) harmlessly identify the algebraic hypersurface X with the corresponding subset of \mathbb{C}^{n+1} , but already over \mathbb{R} we have to be more careful. For example, $x_1^2 + x_2^2 + 1$ has no real zeros. Thus when we say “algebraic hypersurface over \mathbb{R} ”, we think of not just the real zero set but also the complex zero set. (One could also think about zero sets in even larger fields, for example $\mathbb{C}(t_1, \dots, t_m)$, but it turns out that knowing the zero set over \mathbb{C} , or over any algebraically closed field, determines the zero sets over any larger field.)

Thus in practice we usually think of an affine hypersurface $X = X(h)$ as a subset of \mathbb{C}^{n+1} , and we keep in mind that it may have been defined by an equation whose coefficients are real or rational. We use $X(\mathbb{C})$ to denote the set of complex solutions of the equation $h(\mathbf{x}) = 0$. If h has real or rational coefficients, then it also makes sense to ask about the real solutions $X(\mathbb{R})$ or rational solutions $X(\mathbb{Q})$.

We say that two hypersurfaces are *linearly isomorphic* if they differ only by a linear change of coordinates. In principle one can always decide whether two hypersurfaces are linearly isomorphic or not, though in practice the computations may be infeasibly long.

The main question we address in these notes is whether there are other, nonlinear maps and isomorphisms between algebraic hypersurfaces. The problem naturally divides into two topics.

Question 1. When are two algebraic hypersurfaces X_1 and X_2 isomorphic?

Question 2. When are two algebraic hypersurfaces X_1 and X_2 “almost” isomorphic?

Note that we have not yet defined what an “isomorphism” is and gave yet no hint what an “almost” isomorphism should be. Before we give these in Definition 13, we start with an example in Section 1.

Prerequisites. This article is aimed at readers with no prior knowledge of algebraic geometry. We focus on concrete examples that can be worked out by hand yet give a good illustration of the general principles. Thus we also avoid any uses of commutative algebra and of several complex variables, two topics that usually undergird algebraic geometry. The usual (topological) cohomology groups make a fleeting appearance in Section 6 and differential forms are used in Section 7.

However, the discussions of Section 10 need some concepts of algebraic geometry, mainly ampleness and the canonical class.

Throughout, we work with polynomials with complex coefficients and their complex solutions. For most claims this is not very important, but in some examples it is quite interesting to know what happens when we focus on rational solutions; this is the theory of Diophantine equations. We point out these, and connections with other topics, in *Side Remarks*. These can be skipped.

Algebraic geometers almost always work with systems of equations, leading to the notions of algebraic sets and varieties. For most of the results that we discuss,

these form the natural setting; see [Sha74, Mum76, Rei88, CLO92, Dol12, EH16] for introductions.

However, many definitions and results are much easier to state for hypersurfaces, and, once the foundational material is well understood, the basic difficulties are usually very similar.

1. STEREOGRAPHIC PROJECTION

The oldest example of an “almost” isomorphism between hypersurfaces is the stereographic projection. It may well have been invented—originally under the name *planisphaerium*—to create star charts that represent the celestial sphere in a plane. It has been used in mapmaking since the sixteenth century.

A theorem that Ptolemy attributes to Hipparchus (~190–120 BC) says that stereographic projection maps circles to circles. Halley (best known for his comet) proved in [Hal1695] that stereographic projection also preserves angles. Now we refer to these properties by saying that stereographic projection is conformal.

Instead of these metric properties, we are more interested in the actual formulas that define it.

3. Stereographic projection. The formulas for stereographic projection are nicest if we project the unit sphere

$$(3.1) \quad \mathbb{S}^n := (x_1^2 + \cdots + x_{n+1}^2 = 1) \subset \mathbb{R}^{n+1}$$

from the south pole $\mathbf{p}_0 := (0, \dots, 0, -1)$ to the plane $x_{n+1} = 0$ where we use y_1, \dots, y_n as coordinates instead of x_1, \dots, x_n ,

Geometrically, pick any point $\mathbf{p} \in \mathbb{S}^n$ (other than the south pole), and let $\pi(\mathbf{p})$ denote the intersection point of the line $\langle \mathbf{p}_0, \mathbf{p} \rangle$ with the hyperplane $(x_{n+1} = 0)$.

Algebraically, we get that

$$(3.2) \quad \pi(x_1, \dots, x_{n+1}) = \left(\frac{x_1}{1 + x_{n+1}}, \dots, \frac{x_n}{1 + x_{n+1}} \right)$$

with inverse

$$(3.3) \quad \pi^{-1}(y_1, \dots, y_n) = \left(\frac{2y_1}{1 + \Sigma}, \dots, \frac{2y_n}{1 + \Sigma}, \frac{1 - \Sigma}{1 + \Sigma} \right),$$

where $\Sigma = y_1^2 + \cdots + y_n^2$.

Here π is not defined at the points where $x_{n+1} = -1$, and π^{-1} is not defined at the points where $y_1^2 + \cdots + y_n^2 = -1$. In the real case we get a one-to-one map $\mathbb{S}^n \setminus (\text{south pole}) \cong \mathbb{R}^n$ and both π and π^{-1} are given by rational functions.

However, we know that we should also look at the complex case. Note that

$$\mathbb{S}^n(\mathbb{C}) := (x_1^2 + \cdots + x_{n+1}^2 = 1) \subset \mathbb{C}^{n+1}$$

is not a sphere, it is not even compact. Both indeterminacy sets

$$\mathbb{S}^n(\mathbb{C}) \cap (x_{n+1} = -1) \quad \text{and} \quad (y_1^2 + \cdots + y_n^2 = -1)$$

are $(n - 1)$ -dimensional hypersurfaces. We get a one-to-one map between the sets

$$\mathbb{S}^n(\mathbb{C}) \setminus (x_{n+1} = -1) \quad \text{and} \quad \mathbb{C}^n \setminus (y_1^2 + \cdots + y_n^2 = -1),$$

which is given by rational functions in both directions.

This will be our definition of a *birational map*, which is the usual name for the almost isomorphism of Question 2.

A number theoretic variant of stereographic projection is the following.

4. Side Remark: A diophantine equation. Consider the diophantine equation

$$(4.1) \quad a_1x_1^2 + \cdots + a_nx_n^2 + x_{n+1}^2 = 1 \quad \text{where } a_i \in \mathbb{Q}.$$

It determines the quadric hypersurface

$$(4.2) \quad Q^n := (a_1x_1^2 + \cdots + a_nx_n^2 + x_{n+1}^2 = 1).$$

Again projecting from the south pole $\mathbf{p}_0 := (0, \dots, 0, -1)$, we get the formulas

$$(4.3) \quad \pi(x_1, \dots, x_{n+1}) = \left(\frac{x_1}{1+x_{n+1}}, \dots, \frac{x_n}{1+x_{n+1}} \right)$$

with inverse

$$(4.4) \quad \pi^{-1}(y_1, \dots, y_n) = \left(\frac{2y_1}{1+\Sigma}, \dots, \frac{2y_n}{1+\Sigma}, \frac{1-\Sigma}{1+\Sigma} \right),$$

where now $\Sigma = a_1y_1^2 + \cdots + a_ny_n^2$.

Thus, we conclude that all rational solutions of equation (4.1) satisfying $x_{n+1} \neq -1$ can be written as in (4.4) for rational numbers y_1, \dots, y_n .

Note that in (4.1) we are working with the special case when $a_{n+1} = 1$. We could allow instead a_{n+1} to be arbitrary, but then we would need to project from the point $(0, \dots, 0, -\sqrt{a_{n+1}})$. If a_{n+1} is not a square, the resulting formulas give irrational solutions of the equation. See Theorem 7 for the right generalization.

2. PROJECTIVE HYPERSURFACES

An affine hypersurface of dimension ≥ 1 is never compact over \mathbb{C} , thus we frequently work with its closure in $\mathbb{C}\mathbb{P}^{n+1}$.

Definition 5 (Projective space). We will think of the points of $\mathbb{C}\mathbb{P}^{n+1}$ as the lines through the origin in \mathbb{C}^{n+2} . Thus we use *homogeneous coordinates* $(X_0 : \cdots : X_{n+1})$, where at least one of the X_i must be nonzero. Furthermore,

$$(5.1) \quad (\lambda X_0 : \cdots : \lambda X_{n+1}) = (X_0 : \cdots : X_{n+1}) \quad \text{for all } \lambda \in \mathbb{C}^*.$$

A polynomial $h(X_0, \dots, X_{n+1})$ cannot be evaluated at a point of $\mathbb{C}\mathbb{P}^{n+1}$ since usually $h(\lambda X_0 : \cdots : \lambda X_{n+1}) \neq h(X_0 : \cdots : X_{n+1})$. However, if H is homogeneous of degree d , then

$$(5.2) \quad H(\lambda X_0 : \cdots : \lambda X_{n+1}) = \lambda^d H(X_0 : \cdots : X_{n+1}),$$

thus the zero set of a homogeneous polynomial is well defined. Thus, we get *projective hypersurfaces*

$$(5.3) \quad X = X(H) := (H(X_0 : \cdots : X_{n+1}) = 0) \subset \mathbb{P}^{n+1}.$$

One can go between the affine and the projective versions by the formulas

$$(5.4) \quad \begin{aligned} h(x_1, \dots, x_{n+1}) &= H(1, x_1, \dots, x_{n+1}) \quad \text{and} \\ H(X_0, \dots, X_{n+1}) &= X_0^{\deg h} \cdot h\left(\frac{X_1}{X_0}, \dots, \frac{X_{n+1}}{X_0}\right). \end{aligned}$$

Of course any of the X_i could play the special role of X_0 above, so we usually think of $\mathbb{C}\mathbb{P}^{n+1}$ as being covered by $n + 2$ charts, each isomorphic to \mathbb{C}^{n+1} .

The *cone* over X is the affine hypersurface

$$(5.5) \quad C_X = C_{X(H)} := (H(X_0, \dots, X_{n+1}) = 0) \subset \mathbb{A}^{n+2}.$$

This gives another way to go between projective and affine questions.

Note. It would be quite convenient to keep the notational distinction between affine coordinates x_i and projective coordinates X_i , but people usually use lower case x_i to denote both affine and projective coordinates.

Modern algebraic geometry usually considers the projective variant the basic object and the affine versions as the local charts on the projective hypersurface. Note, however, that in algebraic geometry the local charts are very big; they are always dense in the corresponding projective hypersurface.

Definition 6. A point $\mathbf{p} \in X(G)$ on a projective hypersurface is called *smooth* if $\frac{\partial G}{\partial x_i}(\mathbf{p}) \neq 0$ for some i . If we are over \mathbb{C} or \mathbb{R} , the implicit function theorem tells us that X is an n -dimensional submanifold of $\mathbb{C}\mathbb{P}^{n+1}$ or of $\mathbb{R}\mathbb{P}^{n+1}$ at its smooth points. Here of course over \mathbb{C} we count complex dimension, which is twice the real dimension.

The tangent plane $T_{\mathbf{p}}X$ (viewed as a hyperplane in \mathbb{P}^{n+1}) at a smooth point $\mathbf{p} := (p_0 : \cdots : p_{n+1})$ is given by the equation

$$(6.1) \quad \sum_i \frac{\partial G}{\partial x_i}(\mathbf{p}) \cdot x_i = 0.$$

(Our first inclination would be write $\sum_i \frac{\partial G}{\partial x_i}(\mathbf{p}) \cdot (x_i - p_i) = 0$ instead. This does not seem to make sense as an equation since it is not homogeneous. Luckily,

$$(6.2) \quad \sum_i \frac{\partial G}{\partial x_i}(\mathbf{p}) \cdot p_i = \deg G \cdot G(\mathbf{p}) = 0$$

since G is homogeneous, thus, after all, we do get (6.1) this way.)

The other points are *singular*. Thus the set of all singular points is defined by the equations

$$(6.3) \quad \frac{\partial G}{\partial x_0} = \cdots = \frac{\partial G}{\partial x_{n+1}} = 0.$$

It is thus a closed set which is easily seen to be nowhere dense if G is irreducible.

We have the following rather straightforward generalization of the formulas (3.2)–(3.3) for the stereographic projection.

Theorem 7. Consider the irreducible quadric hypersurface

$$Q^n := (G(x_0, \dots, x_{n+1}) = 0) \subset \mathbb{P}^{n+1}.$$

Let $\mathbf{p} \in Q^n$ be a smooth point with tangent plane $T_{\mathbf{p}}Q$. Pick any hyperplane $H \cong \mathbb{P}^n$ that does not contain \mathbf{p} . Then the following hold.

(1) Projection of \mathbb{P}^{n+1} from \mathbf{p} to H gives a one-to-one map

$$\pi : Q^n \setminus T_{\mathbf{p}}Q \longrightarrow H \setminus T_{\mathbf{p}}Q \cong \mathbb{A}^n.$$

(2) The coordinate functions of π are quotients of linear polynomials.

(3) The coordinate functions of π^{-1} are quotients of quadratic polynomials.

Side Remark. Furthermore, if the coefficients of G , of the equation of H , and the coordinates of \mathbf{p} are all in a field k , then so are the coefficients of π and π^{-1} .

3. RATIONAL AND BIRATIONAL MAPS

Definition 8 (Rational maps, affine case). The basic functions in algebraic geometry are polynomials $p(x_1, \dots, x_n)$ and their quotients

$$\phi(x_1, \dots, x_n) = \frac{p(x_1, \dots, x_n)}{q(x_1, \dots, x_n)}$$

are called *rational functions*. Since a polynomial ring has unique factorization, we may assume that p, q are relatively prime. In this case ϕ is not defined along the hypersurface ($q = 0$).

A *rational map* from \mathbb{A}^n to \mathbb{A}^m is a map given by rational coordinate functions

$$\Phi(x_1, \dots, x_n) = (\phi_1(x_1, \dots, x_n), \dots, \phi_m(x_1, \dots, x_n)),$$

where $\phi_i = p_i/q_i$. As before, Φ need not be everywhere defined.

Definition 9 (Rational maps, projective case). As we already noted in (5.2), a polynomial or a rational function $p(x_0, \dots, x_n)$ usually cannot be evaluated at a point of \mathbb{P}^n . However, if ϕ is a quotient of two homogeneous polynomials of the same degree, then

$$(9.1) \quad \phi(\lambda x_0 : \dots : \lambda x_n) = \phi(x_0 : \dots : x_n)$$

and ϕ defines a *rational function* on \mathbb{P}^n .

A *rational map* from \mathbb{P}^n to \mathbb{P}^m is a map given by rational functions

$$(9.2) \quad \Phi(x_0 : \dots : x_n) = (\phi_0(x_0 : \dots : x_n) : \dots : \phi_m(x_0 : \dots : x_n)).$$

We denote rational maps by a dashed arrow $--\rightarrow$.

The nonuniqueness of projective coordinates leads to two basic questions: When are two rational maps the “same”, and where is a rational map defined?

Given two rational maps $\Phi = (\phi_0, \dots, \phi_m)$ and $\Phi' = (\phi'_0, \dots, \phi'_m)$, they agree at a point \mathbf{x} iff

$$(9.3) \quad (\phi_0(\mathbf{x}) : \dots : \phi_m(\mathbf{x})) = \lambda \cdot (\phi'_0(\mathbf{x}) : \dots : \phi'_m(\mathbf{x}))$$

for some $\lambda \neq 0$. Here λ itself can depend on \mathbf{x} , thus it is a rational function $\lambda(\mathbf{x})$. Thus, two rational maps Φ and Φ' are considered the same if

$$(9.4) \quad \phi_0/\phi'_0 = \dots = \phi_m/\phi'_m.$$

Next write each ϕ_i is a quotient of two polynomials $\phi_i = p_i/q_i$. This suggests that ϕ_i is not be defined along the hypersurfaces ($q_i = 0$). Furthermore, every point in \mathbb{P}^m has at least one nonzero coordinate, so Φ is also not defined along the intersection of the hypersurfaces $(p_0 = 0) \cap \dots \cap (p_n = 0)$. Do we get the same answer if we start with Φ' ? Confusingly, the answer is no, but we can turn the nonuniqueness of projective coordinates to our advantage. Multiplying through with the least common denominator of the ϕ_i in (9.2) and canceling common factors, we can represent Φ as

$$(9.5) \quad \Phi(\mathbf{x}) = (P_0(\mathbf{x}) : \dots : P_m(\mathbf{x})),$$

where the P_i are polynomials that do not have a common factor. Such a representation of a rational map $\mathbb{P}^n --\rightarrow \mathbb{P}^m$ is unique (up to multiplication by a constant), and Φ is defined outside the common zero set of the P_i . From this we can read off two important properties.

Claim 9.6. $\Phi : \mathbb{P}^n \dashrightarrow \mathbb{P}^m$ is everywhere defined iff the P_i in (9.5) have no common zero. □

Claim 9.7. Write $P_i(\mathbf{x}) = c_i \prod_j r_j(\mathbf{x})^{m(i,j)}$, where the c_i are constants, the r_j are irreducible polynomials, and $m(i, j) \geq 0$. Then Φ is defined outside the set

$$Z := \bigcup_{j_1 \neq j_2} (r_{j_1}(\mathbf{x}) = r_{j_2}(\mathbf{x}) = 0). \quad \square$$

Note that Z is made up of many pieces, but each of them is defined by the vanishing of two relatively prime polynomials. Thus we expect that Z has codimension ≥ 2 in \mathbb{P}^n . This is not hard to see; we state a general version as Theorem 12.

Definition 10 (Rational maps of hypersurfaces). A *rational function* on a hypersurface $X \subset \mathbb{P}^{n+1}$ is the restriction of a rational function ϕ on \mathbb{P}^{n+1} to X , provided the restriction makes sense. That is, when ϕ is defined on a dense open subset of X .

As before, a *rational map* Φ from a hypersurface $X \subset \mathbb{P}^{n+1}$ to \mathbb{P}^m is given by rational functions $(\phi_0 : \dots : \phi_m)$. If $\Phi(X)$ is contained in a hypersurface $Y \subset \mathbb{P}^m$, then Φ defines a rational map $\Phi : X \dashrightarrow Y$. (If we work over \mathbb{C} , then by $\Phi(X) \subset Y$ we mean that $\Phi(\mathbf{p}) \in Y$ whenever Φ is defined at $\mathbf{p} \in X$. In general we need the algebraic version given in Remark 15.)

We use three important basic properties of rational functions.

Claim 10.1. If X is irreducible and ϕ is defined at a single point of X , then it is defined on a dense open subset of X , cf. [Sha74, Sec.I.3.2].

Claim 10.2. If $X \subset \mathbb{A}^{n+1}$ is affine and $\phi|_X$ is everywhere defined, then there is a polynomial p on \mathbb{A}^{n+1} such that $p|_X = \phi|_X$. For example, $\frac{x^3-y}{x+y}$ has poles but its restriction to the x -axis is everywhere defined. We can take $p := x^2$.

This is usually proved as a consequence of Hilbert’s *Nullstellensatz*, see [Sha74, Sec.I.3.2].

Purity Principle 10.3. Let $\phi : X \dashrightarrow \mathbb{P}^1$ be a rational function on a hypersurface X . Then the set of points where ϕ is not defined has codimension 1 everywhere.

This is more subtle. In books it is usually treated as a combination of Krull’s principal ideal theorem and Serre’s S_2 property for hypersurfaces; however, this form goes back to Macaulay [Mac1916].

Side Remark. Readers may recognize Purity Principle 10.3 as an algebraic counterpart of Hartogs’s extension theorem in several complex variables.

Definition 11 (Morphisms). Φ is *defined* at a point $\mathbf{p} \in X$ if it has some representation

$$(11.1) \quad \Phi(\mathbf{x}) = (\phi_0(\mathbf{x}) : \dots : \phi_m(\mathbf{x}))$$

as in (9.2), where every coordinate function is defined at \mathbf{p} and at least one of them is nonzero. (The traditional terminology is “ Φ is regular” at \mathbf{p} , but be warned that “regular” has other, conflicting uses, even in algebraic geometry.)

A *morphism* is a rational map that is everywhere defined. We denote morphisms by a solid arrow \rightarrow .

Warning 11.2. A major difficulty is that, unlike for \mathbb{P}^n where we could write down the optimal representation in (9.5), there need not be a single form (11.1) that shows regularity at every point.

For example, consider the smooth hypersurface $Q := (x_1x_2 = x_3x_4) \subset \mathbb{P}^3$ and the rational map

$$\Phi : Q \dashrightarrow \mathbb{P}^1 \quad \text{given by} \quad (x_1:x_2:x_3:x_4) \mapsto (x_1:x_3).$$

It is defined outside $(x_1 = x_3 = 0)$. Next note that $x_1/x_4 = x_3/x_2$, thus Φ can also be written as

$$\Phi : (x_1:x_2:x_3:x_4) \mapsto (x_4:x_2),$$

hence Φ is defined outside $(x_4 = x_2 = 0)$. Thus Φ is everywhere defined on Q —it is a morphism.

However, no matter how we write $\Phi = (p_1:p_2)$ for some polynomials p_i , the system $p_1 = p_2 = x_1x_2 - x_3x_4 = 0$ always has a solution in \mathbb{P}^3 . Thus, unlike Claim 9.6, we cannot conclude that Φ is everywhere defined using just one representation of it.

A major theorem of Lefschetz implies that the situation is better for smooth hypersurfaces of dimension ≥ 3 ; see paragraph 34 for a discussion.

By contrast, the following generalization of (9.6) holds; see [Sha74, Sec.II.3.1].

Theorem 12. *Let X be a smooth hypersurface, and let $\Phi : X \dashrightarrow \mathbb{P}^m$ be a rational map. Then there is a closed subset $Z \subset X$ of codimension ≥ 2 such that Φ is defined on $X \setminus Z$. \square*

Now we come to the definition of “isomorphism” and “almost isomorphism.”

Definition 13. Let $\Phi : X \dashrightarrow Y$ be a rational map between hypersurfaces.

- (1) Φ is *birational* if there is a rational map $\Phi^{-1} : Y \dashrightarrow X$ that is the inverse of Φ . That is, $\Phi^{-1} \circ \Phi$ and $\Phi \circ \Phi^{-1}$ are both the identity, wherever they are defined. If this holds, then we say that X and Y are *birationally equivalent* or *birational*.
- (2) An n -dimensional hypersurface X is called *rational* if it is birational to \mathbb{P}^n .
- (3) Φ is an *isomorphism* if both Φ and Φ^{-1} are morphisms, that is, everywhere defined.

14. Why birational equivalence? The notion of birational equivalence usually seems unnatural to people coming from topology—where open dense subsets remember only the dimension of the manifold—but it should be more natural to people who work in analysis, measure theory, or probability, where ignoring measure 0 subsets is common in many questions.

Algebraic geometers feel that this concept is forced upon us, since many interesting maps, for example the stereographic projection, are not defined everywhere. Thus with many questions it is profitable to study first its birational version, and, once that is understood, try to see whether the question makes sense or is solvable up to isomorphism.

Remark 15. Note that an affine hypersurface is birational to its projective closure, thus its birationality can be checked using affine equations. Two hypersurfaces

$$\begin{aligned} X &:= (h(x_1, \dots, x_{n+1}) = 0) \subset \mathbb{A}_{\mathbf{x}}^{n+1} \quad \text{and} \\ Y &:= (g(y_1, \dots, y_{n+1}) = 0) \subset \mathbb{A}_{\mathbf{y}}^{n+1} \end{aligned}$$

are birational if there are rational maps

$$\begin{aligned} \Phi &= (\phi_1(x_1, \dots, x_{n+1}), \dots, \phi_{n+1}(x_1, \dots, x_{n+1})) \quad \text{and} \\ \Psi &= (\psi_1(y_1, \dots, y_{n+1}), \dots, \psi_{n+1}(y_1, \dots, y_{n+1})) \end{aligned}$$

with the following properties.

(15.1) Φ maps X to Y . In terms of equations this means that $g(\phi_1, \dots, \phi_{n+1})$ vanishes on X . That is, if we write $g(\phi_1, \dots, \phi_{n+1})$ as the quotient of two relatively prime polynomials, then the numerator is divisible by h .

(15.2) Ψ maps Y to X .

(15.3) $\Phi : X \dashrightarrow Y$ and $\Psi : Y \dashrightarrow X$ are inverses of each other. Note that

$$\Psi \circ \Phi = (\psi_1(\phi_1, \dots, \phi_{n+1}), \dots, \psi_{n+1}(\phi_1, \dots, \phi_{n+1})),$$

and this is the identity on X iff $x_i - \psi_i(\phi_1, \dots, \phi_{n+1})$ vanishes on X for every i . Similarly, $y_j - \phi_j(\psi_1, \dots, \psi_{n+1})$ vanishes on Y for every j .

4. THE MAIN QUESTIONS

Now we are ready to formulate the precise versions of Questions 1 and 2. In both cases the ideal complete answer would consist of two steps.

- Describe a set of *elementary* isomorphisms/birational maps between hypersurfaces.
- Prove that every isomorphism/birational map between hypersurfaces is elementary, or at least a composite of elementary maps.

So what are these elementary isomorphisms/birational maps?

Isomorphisms of projective hypersurfaces. For projective hypersurfaces $X \subset \mathbb{P}^{n+1}$ the elementary isomorphisms are those that are induced by an automorphism of \mathbb{P}^{n+1} . It is not hard to see that $\text{Aut}(\mathbb{P}^{n+1}) \cong \text{PGL}_{n+2}$ (cf. [Sha74, III.1.Exrc.17]), so these are exactly the linear isomorphisms. This leads to the projective version of Question 1.

Question 16. Let X_1, X_2 be projective algebraic hypersurfaces in \mathbb{P}^{n+1} , and let $\Phi : X_1 \rightarrow X_2$ be an isomorphism. Is Φ linear? That is, is there an automorphism $\Psi \in \text{Aut}(\mathbb{P}^{n+1}) \cong \text{PGL}_{n+2}$ such that $\Phi = \Psi|_{X_1}$?

We discuss almost complete answers to this in Section 6.

Isomorphisms of affine hypersurfaces. The projective case suggests that for affine hypersurfaces the elementary isomorphisms should be those that are induced by an automorphism of \mathbb{A}^{n+1} . A major difficulty is that $\text{Aut}(\mathbb{A}^{n+1})$ is infinite dimensional. For example, given any polynomials $g_i(x_{i+1}, \dots, x_{n+1})$, the map

$$\Phi : (x_1, \dots, x_{n+1}) \mapsto (x_1 + g_1, x_2 + g_2, \dots, x_{n+1} + g_{n+1})$$

is an automorphism of \mathbb{A}^{n+1} . These automorphisms and GL_n together generate the *tame subgroup* of $\text{Aut}(\mathbb{A}^{n+1})$. If $n = 1$, then we get the whole $\text{Aut}(\mathbb{A}^2)$ by [Abh77] but the analogous result does not hold in higher dimensions. The counterexample

$$(x, y, z) \mapsto (x - 2(xz + y^2)y - (xz + y^2)^2z, y + (xz + y^2)z, z)$$

was proposed by Nagata [Nag72] and proved in [SU04]; see [Kur16] for a survey and [AL92] for a complex analytic version.

Already, the simplest case of the affine isomorphism problem is a quite nontrivial result of [AM75] and [Suz74].

Theorem 17 (Abhyankar, Moh, and Suzuki). *Let $C \subset \mathbb{A}^2$ be a curve, and let $\Phi : C \rightarrow \mathbb{A}^1$ be an isomorphism. Then Φ extends to an isomorphism*

$$\Psi : (C \subset \mathbb{A}^2) \cong ((\text{coordinate axis}) \subset \mathbb{A}^2). \quad \square$$

The higher dimensional generalization of this is the Abhyankar–Sathaye conjecture (about which very little is known) and a series of counterexamples is proposed in [CD94]; see also [vdE00, Chap.5].

Conjecture 18. *Let $X \subset \mathbb{A}^{n+1}$ be a hypersurface, and let $\Phi : X \rightarrow \mathbb{A}^n$ be an isomorphism. Then Φ extends to an isomorphism*

$$\Psi : (X \subset \mathbb{A}^{n+1}) \cong ((\text{coordinate hyperplane}) \subset \mathbb{A}^{n+1}).$$

Interestingly, the union of the n coordinate hyperplanes ($x_1 \cdots x_n = 0$) has a unique embedding into \mathbb{C}^n by [Jel97]. For some recent results connecting the affine isomorphism problem with the methods of Section 9, see [CDP17].

Birational equivalence of hypersurfaces. It seems natural to formulate Question 2 in two variants. Note that an affine hypersurface is birational to its projective closure, so for the birationality questions there is no need to distinguish the affine and the projective cases.

Question 19. Which algebraic hypersurfaces are rational (as in Definition 13(2))?

Question 20. When are two algebraic hypersurfaces birational to each other?

Example 21 (Rational hypersurfaces). It has been long understood that even high degree hypersurfaces can be rational if they are very singular. Here are some examples of this.

(21.1) Let $X \subset \mathbb{P}^{n+1}$ be given by an equation

$$H_{d-1}(x_0, \dots, x_n)x_{n+1} + H_d(x_0, \dots, x_n) = 0,$$

where H_j is homogeneous of degree j . Projection from the point $(0 : \cdots : 0 : 1)$ to the hyperplane $(x_{n+1} = 0)$ is birational, the inverse is given by

$$(x_0 : \cdots : x_n) \mapsto \left(x_0 : \cdots : x_n : \frac{H_d(x_0, \dots, x_n)}{H_{d-1}(x_0, \dots, x_n)} \right).$$

If $d = 2$, then this is the stereographic projection. If $d \geq 3$, then $(0 : \cdots : 0 : 1)$ is a singular point of X .

(21.2) Let $X \subset \mathbb{P}^{2n+1}$ be a hypersurface of degree $2d + 1$ given by an equation of the form

$$\sum_{I, J, k} a_{I, J, k} \cdot M_I \cdot N_J \cdot x_k,$$

where M_I is a degree d monomial in the variables x_0, \dots, x_n and N_J is a degree d monomial in the variables x_{n+1}, \dots, x_{2n+1} and x_k is arbitrary. (In particular, X contains the linear subspaces $L_1 := (x_0 = \cdots = x_n = 0)$ and $L_2 := (x_{n+1} = \cdots = x_{2n+1} = 0)$.) Then X is rational. We will work out the $2d + 1 = 3$ case in detail in Proposition 24.

If $2d + 1 = 3$, then a general such X is smooth, but for $2d + 1 \geq 5$ it is always singular along the L_i .

(21.3) Take degree d homogeneous polynomials $p_0(\mathbf{x}), \dots, p_{n+1}(\mathbf{x})$. They define a rational map $\Phi : \mathbb{P}^n \dashrightarrow \mathbb{P}^{n+1}$. It is not hard to show that if the p_i are general, then Φ is a morphism whose image $Y = Y(p_0, \dots, p_{n+1})$ is a rational hypersurface of degree d^n . However, Y has very complicated self-intersections.

These and many other examples suggest that Questions 19 and 20 are reasonable only if the hypersurfaces are smooth or mildly singular. We will focus on the smooth cases.

Question 22. Which smooth algebraic hypersurfaces are rational?

Question 23. When are two smooth algebraic hypersurfaces birational to each other?

So far is not even clear that not all hypersurfaces are rational. We see later that this is not at all the case, but first we give some rational examples.

5. RATIONALITY OF CUBIC HYPERSURFACES

We proved in Theorem 7 that a smooth quadric hypersurface is rational. In this section we consider the rationality of cubic hypersurfaces. This is a topic with many interesting results and still unsolved questions. The simplest rationality construction is the following special case of (21.2).

Proposition 24. *Let $S \subset \mathbb{P}^3$ be a cubic surface that contains two disjoint lines. Then S is rational.*

Geometric Proof. Let the lines be L_1, L_2 . Pick points $\mathbf{p}_i \in L_i$. The line connecting them, which we call $\langle \mathbf{p}_1, \mathbf{p}_2 \rangle$, meets S in three points. We already know two of them, namely $\mathbf{p}_1, \mathbf{p}_2$. Let $\phi(\mathbf{p}_1, \mathbf{p}_2)$ be the third intersection point. This gives a rational map

$$(24.1) \quad \phi : L_1 \times L_2 \dashrightarrow S.$$

To get its inverse, pick $\mathbf{q} \in \mathbb{P}^3$ and let $\pi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ denote the projection from \mathbf{q} . Then $\pi(L_1), \pi(L_2)$ are two lines in \mathbb{P}^2 , hence they meet at a unique point \mathbf{q}' . Thus we get $\mathbf{p}_i \in L_i$ such that $\pi(\mathbf{p}_i) = \mathbf{q}'$. This gives

$$(24.2) \quad \psi : \mathbb{P}^3 \dashrightarrow L_1 \times L_2 \quad \text{such that} \quad \psi|_S = \phi^{-1}. \quad \square$$

Algebraic Proof. We can choose coordinates such that $L_1 = (x_0 = x_1 = 0)$ and $L_2 = (x_2 = x_3 = 0)$. Thus the equation of S can be written (nonuniquely) as

$$(24.3) \quad \sum_{i=0,1} \sum_{j=2,3} \ell_{ij}(\mathbf{x})x_i x_j,$$

where the $\ell_{ij}(\mathbf{x})$ are linear in the variables x_0, \dots, x_3 .

If $p_1 = (a_0:a_1:0:0)$ and $p_2 = (0:0:a_2:a_3)$, then the line connecting them is

$$(24.4) \quad \mathbb{P}^1 \ni (s:t) \mapsto (sa_0:sa_1:ta_2:ta_3) \in \mathbb{P}^3.$$

Substituting into (24.3), we get an equation

$$(24.5) \quad \sum_{i=0,1} \sum_{j=2,3} \ell_{ij}(sa_0:sa_1:ta_2:ta_3)sa_i ta_j = st \sum_{i=0,1} \sum_{j=2,3} \ell_{ij}(sa_0:sa_1:ta_2:ta_3)a_i a_j.$$

After dividing by st , the remaining equation is linear in s, t , thus it has a unique solution (up to a multiplicative constant)

$$(24.6) \quad (s:t) = \left(- \sum_{i=0,1} \sum_{j=2,3} \ell_{ij}(0:0:a_2:a_3)a_i a_j : \sum_{i=0,1} \sum_{j=2,3} \ell_{ij}(a_0:a_1:0:0)a_i a_j \right). \quad \square$$

This can be used to prove the following result of [Cle1866].

Corollary 25 (Clebsch). *Every smooth cubic surface over \mathbb{C} is rational.*

Hint of Proof. By Proposition 24 it is enough to find two disjoint lines. In fact, every smooth cubic surface over \mathbb{C} contains 27 lines. This was first proved by Cayley and Salmon in 1849 and published in [Sal1865]; see the next example or any of [Rei88, Chap.7], [Rei97, Chap.1], or [Sha74, Sec.IV.2.5] for proofs.

Example 26. Consider the degree d hypersurface

$$X := (x_0^d + \dots + x_{2n+1}^d = 0) \subset \mathbb{P}^{2n+1}.$$

Divide the indices into $n + 1$ disjoint ordered pairs $(x_{\tau(i)}, x_{\sigma(i)}) : i = 0, \dots, n$. Fix d th roots $\epsilon_0, \dots, \epsilon_n$ of -1 . Then

$$L = L(\tau, \sigma, \epsilon) := (x_{\tau(i)} = \epsilon_i x_{\sigma(i)} : i = 0, \dots, n)$$

is a linear space of dimension n contained in X . Thus X contains at least

$$\frac{d^{n+1}(2n + 2)!}{2^{n+1}(n + 1)!}$$

linear subspaces of dimension n . Not all but many pairs of these linear subspaces are disjoint, for example $L(\tau, \sigma, \epsilon)$ is disjoint from $L(\tau', \sigma', \epsilon')$ if $\tau = \tau', \sigma = \sigma'$ but $\epsilon_i \neq \epsilon'_i$ for every i . For $n = 2, d = 1$ we get all 27 lines on the particular cubic surface.

The following is proved in [Seg43]; see [KSC04, Chap.2] for a modern treatment.

Side Theorem 27 (Segre). *The cubic surface*

$$S = S(a_0, a_1, a_2, a_3) := (a_0x_0^3 + a_1x_1^3 + a_2x_2^3 + a_3x_3^3 = 0) \subset \mathbb{P}^3$$

defined over a field k is not rational over k if for every permutation σ of the indices, the quotient $(a_{\sigma(0)}a_{\sigma(1)})/(a_{\sigma(2)}a_{\sigma(3)})$ is not a cube in k . □

28. Cubic 3-folds. In contrast to Corollary 25, smooth cubic 3-folds $X_3 \subset \mathbb{P}^4$ are all nonrational [CG72]. This is proved by analyzing the Hodge structure of the cohomology groups. This method has been very successful for many other three-dimensional cases, but extensions to higher dimensions are still lacking.

29. Cubic 4-folds. A cubic 4-fold is defined by a degree 3 homogeneous polynomial in six variables. These form a vector space of dimension $\binom{8}{3} = 56$. Since we do not care about multiplicative constants, we can think of the set of all cubic 4-folds as points in a \mathbb{P}^{55} . The smooth ones correspond to an open subset of it. (This is a general feature of algebraic geometry. Isomorphism classes frequently naturally correspond to points of another algebraic variety, called the *moduli space*.)

A long-standing conjecture that grew out of the works of [Mor40, Fan43] says that a general smooth cubic 4-fold is not rational. [Has99] describes countably many hypersurfaces $H_i \subset \mathbb{P}^{55}$ and conjectures that a cubic 4-fold X is not rational if the corresponding point $[X] \in \mathbb{P}^{55}$ is outside these hypersurfaces. See the collection [BHKV16] for recent surveys.

The strongest results, due to Russo and Staglianò [RS17], prove rationality for cubic 4-folds corresponding to three of the hypersurfaces H_i ($i = 14, 26, 38$ in Hassett's notation). Just to show that there are some rather subtle rationality constructions, here is one of the beautiful examples they discovered for H_{38} .

Pick ten general points $P_j \in \mathbb{P}^2$ and six general polynomials $p_i(x, y, z)$ of degree 10 that vanish with multiplicity 3 at all the points P_j . Let F be the surface obtained

as the image of the map

$$\Psi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^5 : (x:y:z) \mapsto (p_0(x:y:z) : \cdots : p_5(x:y:z)).$$

[RS17] shows that there are five linearly independent quintic polynomials q_0, \dots, q_4 on \mathbb{P}^5 that vanish along F with multiplicity 2. These give a rational map $\Phi : \mathbb{P}^5 \dashrightarrow \mathbb{P}^4$ whose general fiber is a conic that intersects F at five points. A conic and a cubic hypersurface typically have six points in common, and F accounts for five of these for the conics of the form $\Phi^{-1}(p)$. Thus, sending a point $p \in \mathbb{P}^4$ to the unique sixth intersection point in $X \cap \Phi^{-1}(p)$ gives the inverse of $\Phi|_X : X \dashrightarrow \mathbb{P}^4$.

It turns out that this construction works for a 54-dimensional subfamily H_{38} of the 55-dimensional family of all cubics.

The method of the proof of these claims in [RS17] shows both the power and the limitations of algebraic geometry. Consider for instance the claim about the five linearly independent quintic polynomials. Standard methods imply that there are at least five such linearly independent quintics. Then other semicontinuity arguments show that if there are exactly five in at least one concrete example, then there are exactly five in almost all cases. Moreover, it is enough to find one such example over a finite field. Russo and Staglianò programmed a computer to try out random cases until it hit one that worked. Since in algebraic geometry the term “almost all” usually means a dense open subset, there is every reason to believe that this leads to a solution.

The disadvantage is that while we proved rationality for smooth cubics corresponding to an open, dense subset in the 54-dimensional hypersurface $H_{38} \subset \mathbb{P}^{55}$, we have no idea which open, dense subset it is.

Fortunately, in this case the recent results of [NS17, KT17] imply that for any family of smooth hypersurfaces parametrized by a variety V , once rationality holds for hypersurfaces corresponding to points in an open, dense subset of V , then rationality holds for all hypersurfaces in the family, but this requires entirely different methods. In particular, rationality holds for all smooth cubics corresponding to H_{38} .

6. ISOMORPHISM OF HYPERSURFACES

The following theorem answers Question 16.

Theorem 30. *Let $X_1, X_2 \subset \mathbb{P}^{n+1}$ be irreducible hypersurfaces, and let $\Phi : X_1 \cong X_2$ be an isomorphism. Then Φ is obtained by a linear change of coordinates in \mathbb{P}^{n+1} , except possibly in the following three cases:*

- (1) $\dim X_1 = \dim X_2 = 1$ and $\{\deg X_1, \deg X_2\} = \{1, 2\}$,
- (2) $\dim X_1 = \dim X_2 = 1$ and $\deg X_1 = \deg X_2 = 3$, or
- (3) $\dim X_1 = \dim X_2 = 2$ and $\deg X_1 = \deg X_2 = 4$.

First we describe in detail the exceptional cases.

The first exceptional case is given by the stereographic projection of a plane conic to a line. This is the only dimension where the stereographic projection is an isomorphism.

The second exceptional case is $d = 3$ and $n = 1$, that is, degree 3 curves in \mathbb{P}^2 . This is the theory of elliptic curves and elliptic integrals. We give a description using the theory of the Weierstrass \wp -function; see [Sie69, Chap.1], [Sil86, Chap.6], [SS03, Chap.9], or many other books on complex analysis for details.

31. Side Example: Weierstrass \wp -function. Let $\Lambda \subset \mathbb{C}$ be a lattice. After multiplying by a suitable $c \in \mathbb{C}^*$, we can achieve that its generators are 1 and τ where $\text{Im}(\tau) > 0$. The Weierstrass $\wp(z)$ and its derivative $\wp'(z)$ satisfy an equation

$$\wp'(z)^2 = 4\wp^2(x) - g_4\wp(z) - g_6,$$

where the precise formulas for the g_i are not important for us. Thus $\sigma : z \mapsto (\wp(z):\wp'(z):1)$ gives an isomorphism

$$\sigma : \mathbb{C}/\Lambda \cong C_\tau := (y^2w = 4x^3 - g_4xw^2 - g_6w^3) \subset \mathbb{P}^2_{xyw}.$$

The origin is mapped to the point at infinity which is an inflection point of the curve.

For any $z_0 \in \mathbb{C}$, we can also use $\sigma_{z_0} : z \mapsto (\wp(z - z_0):\wp'(z - z_0):1)$ to get another isomorphism

$$\sigma_{z_0} : \mathbb{C}/\Lambda \cong C_\tau = (y^2w = 4x^3 - g_4xw^2 - g_6w^3) \subset \mathbb{P}^2_{xyw}.$$

Here the point $z_0 \in \mathbb{C}$ is mapped to the inflection point at infinity.

A linear automorphism of \mathbb{P}^2_{xyw} preserves inflection points, so the automorphism $\sigma_{z_0} \circ \sigma^{-1} : C_\tau \cong C_\tau$ cannot be linear if $\sigma(z_0)$ is not an inflection point.

With a little more work, we get that $\sigma_{z_0}\sigma^{-1} : C_\tau \cong C_\tau$ is linear iff $z_0 \in \frac{1}{3}\Lambda$.

Example 32. A more geometric way of obtaining a nonlinear automorphism of a smooth plane cubic $C \subset \mathbb{P}^2$ is the following. Pick a point $p_0 \in C$ and for $p \in C$ let $\tau_{p_0}(p)$ denote the third intersection point of the line $\langle p, p_0 \rangle$ with C . Thus τ_{p_0} is an involution of C , and it is linear iff p_0 is an inflection point of C .

In the description of Side Example 31, we can get τ_{p_0} as $z \mapsto 2z_0 - z$, where $\sigma(z_0) = p_0$.

The third exceptional case is $d = 4$ and $n = 2$, that is, degree 4 surfaces in \mathbb{P}^3 . Here there are many known examples of nonlinear isomorphisms, but these are not easy to find and we do not have a complete description. A rather large set of examples is the following.

Example 33. Start with $\mathbb{P}^3_{\mathbf{x}} \times \mathbb{P}^3_{\mathbf{y}}$ and four bilinear hypersurfaces given by equations $H_k := (\sum_{ij} a^k_{ij}x_iy_j = 0)$. Set $S := H_1 \cap \dots \cap H_4$. In order to compute its projection to $\mathbb{P}^3_{\mathbf{x}}$, we view the H_k as linear equations in the \mathbf{y} -variables. The coefficient matrix is

$$\mathbf{B} = (b_{kj}) \quad \text{where} \quad b_{kj} = \sum_i a^k_{ij}x_i.$$

A given point $\mathbf{p} \in \mathbb{P}^3_{\mathbf{x}}$ is in $\pi_{\mathbf{x}}(S)$ iff $\mathbf{B}(\mathbf{p})\mathbf{y}^t = 0$ has a nonzero solution \mathbf{y} and this in turn holds iff $\det \mathbf{B}(\mathbf{p}) = 0$. Thus

$$\pi_{\mathbf{x}}(S) = (\det \mathbf{B}(\mathbf{x}) = 0) \subset \mathbb{P}^3_{\mathbf{x}}.$$

Similarly, the image of the projection to $\mathbb{P}^3_{\mathbf{y}}$ is given by

$$\pi_{\mathbf{y}}(S) = (\det \mathbf{C}(\mathbf{y}) = 0) \quad \text{where} \quad \mathbf{C} = (c_{ki}) = \left(\sum_j a^k_{ij}y_j \right).$$

It is not hard to check that, for general a^k_{ij} , the projections give isomorphisms

$$S \cong (\det \mathbf{B}(\mathbf{x}) = 0) \quad \text{and} \quad S \cong (\det \mathbf{C}(\mathbf{y}) = 0).$$

In particular, the quartic surfaces $(\det \mathbf{B} = 0)$ and $(\det \mathbf{C} = 0)$ are isomorphic to each other. By Cramer’s rule the isomorphism between them can be given by the coordinate functions

$$\phi_j(\mathbf{x}) = (-1)^{j-1} \det \mathbf{B}_{4j}(\mathbf{x}),$$

where \mathbf{B}_{4j} is the submatrix obtained by removing the fourth row and j th column. So the $\phi_j(\mathbf{x})$ are cubic polynomials.

In principle it could happen that $(\det \mathbf{B}(\mathbf{x}) = 0)$ and $(\det \mathbf{C}(\mathbf{y}) = 0)$ are isomorphic by a linear isomorphism. This does not occur for general a_{ij}^k , though this needs proof. See [SS16, Ogu16] for this and other such examples.

One can play the same game with $\mathbb{P}_{\mathbf{x}}^{n+1} \times \mathbb{P}_{\mathbf{y}}^{n+1}$ and $n + 2$ bilinear hypersurfaces. It turns out that for $n \geq 3$ the projections are birational but they are not isomorphisms, and the image hypersurfaces are always singular. Nonetheless, this construction gives very interesting examples of mildly singular (in fact with terminal singularities) hypersurfaces that are birational to each other in an unexpected way.

34. Comments on the Proof of Theorem 30. The theorem is an easy consequence of some big theorems, and one should prove the big theorems. I give only the bare outlines and references.

Assume first that $\dim X \geq 3$. The main point is to show that rational maps $X \dashrightarrow \mathbb{P}^m$ have a unique optimal representation, analogous to (9.5). We start with a single rational function.

Claim 34.1. Let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of dimension $n \geq 3$. Let ϕ be a rational function on X . Then there are homogeneous polynomials p, q such that $\phi = (p/q)|_X$ and

- (a) the (closure of the) zero set of ϕ is exactly $X \cap (p = 0)$ and
- (b) the (closure of the) polar set of ϕ is exactly $X \cap (q = 0)$.

A key observation of Lefschetz is that although Claim 34.1 is algebraic, the hardest part of it is topological. The closure of the zero set of ϕ is a subset $Z \subset X(\mathbb{C})$ of real codimension 2. It has a homology class in $H_{2n-2}(X(\mathbb{C}), \mathbb{Z})$, we prefer to use instead its dual cohomology class $[Z] \in H^2(X(\mathbb{C}), \mathbb{Z})$. So the topological version of Claim 34.1 asks whether $[Z] \in H^2(X(\mathbb{C}), \mathbb{Z})$ is obtained as the restriction of an element of $H^2(\mathbb{C}\mathbb{P}^{n+1}, \mathbb{Z})$. The following is a consequence of the Lefschetz hyperplane theorem; see [Lef50], [Gro68] or [GH78, p.156] for proofs.

Theorem 34.2. *Let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface. Then the natural map*

$$(34.2.a) \quad H^i(\mathbb{C}\mathbb{P}^{n+1}, \mathbb{Z}) \rightarrow H^i(X(\mathbb{C}), \mathbb{Z})$$

is an isomorphism for $i < n$. □

We already hinted that Claim 34.1 needs the case $i = 2$ (hence the assumption $n \geq 3$), and going from the topological to the algebraic version also needs (34.2.a) for $i = 1$.

From Claim 34.1 it is easy to derive the following generalization of (9.5).

Claim 34.3. Let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of dimension $n \geq 3$, and let $\Phi : X \rightarrow \mathbb{P}^m$ be a nonconstant morphism. Then it can be written as

$$\Phi = (P_0(\mathbf{x}) : \cdots : P_m(\mathbf{x})), \quad \text{where}$$

- (a) the P_i are homogeneous polynomials of the same degree d and
- (b) they do not have a common zero on X . □

Finally, it is straightforward to conclude that if Φ is an injection, then the degree of its image is $d^{\dim X} \cdot \deg X$. Thus $X \hookrightarrow \mathbb{P}^{n+1}$ is the unique smallest degree embedding of X into any projective space, up to linear coordinate changes.

As we saw in Warning 11.2, Claim 34.1 does not hold if $\dim X = 2$, and, by Example 33, the case $\dim X = 2, \deg X = 4$ is exceptional. A different approach for all cases where $\dim X \geq 2, \deg X \neq \dim X + 2$, will be discussed in paragraph 43.

Finally, for $n = 1$, a theorem of Noether [Noe1882, Sec.5] describes all low degree embeddings of a plane curve into some projective space. A particular case of this says that an irreducible plane curve of degree ≥ 4 has a unique embedding into \mathbb{P}^2 . A complete proof is given by Hartshorne [Har86], see also [ACGH85, p.56].

7. NONRATIONALITY OF LARGE DEGREE HYPERSURFACES

We start with the following answer to Question 22.

Theorem 35. *Let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of degree d . If $d \geq n + 2$, then X is not rational. More strongly, the image of every rational map $\Phi : \mathbb{P}^n \dashrightarrow X$ has dimension $\leq n - 1$.*

The stronger property is usually stated as “ X is not unirational”. The theorem will be a direct consequence of Propositions 40 and 42, and the same method also answers Question 23 for large degree hypersurfaces.

Theorem 36. *Let $X_1, X_2 \subset \mathbb{P}^{n+1}$ be smooth hypersurfaces of degrees d_1, d_2 that are birational to each other. Assume that $d_1 \geq n + 3$. Then $d_1 = d_2$ and the X_i can be obtained from each other by a linear change of coordinates.*

In order to prove Theorem 35, we need to study algebraic differential forms.

Definition 37. Let X be a smooth hypersurface of dimension n . As we noted in Definition 6, it is also a real manifold of dimension $2n$. An *algebraic differential m -form* is a differential m -form that locally can be written as linear combination of terms

$$(37.1) \quad \phi(\mathbf{x}) \cdot d\psi_1(\mathbf{x}) \wedge \cdots \wedge d\psi_m(\mathbf{x}),$$

where ϕ and the ψ_i are rational functions. An algebraic differential form is *defined* at a point $p \in X$ if it can be written with summands as in (37.1) where ϕ and the ψ_i are all defined at p .

We will be especially interested in n -forms, which are the *algebraic volume forms*. (It is not hard to check that if σ is an algebraic volume form, then $\sigma \wedge \bar{\sigma}$ is a usual volume form, (up to a power of $\sqrt{-1}$), albeit possibly with zeros and poles.)

Example 38 (Algebraic volume forms on \mathbb{C}^n). On \mathbb{C}^n the standard volume form is $dz_1 \wedge \cdots \wedge dz_n$. If ψ is a rational function, then

$$d\psi = \sum_i \frac{\partial \psi}{\partial z_i} dz_i,$$

hence every algebraic volume form can be written as

$$(38.1) \quad \sigma := \phi(\mathbf{z}) \cdot dz_1 \wedge \cdots \wedge dz_n,$$

where ϕ is a rational function. This is undefined precisely where ϕ is undefined. Thus, as we noted in Definition 8, either σ is everywhere defined or it is undefined exactly along a hypersurface.

The projective case behaves quite differently.

Lemma 39. *There are no everywhere defined algebraic volume forms on \mathbb{P}^n .*

Proof. We use projective coordinates $X_0 : \dots : X_n$ and show the claim using just two of the affine charts. Let these be

$$\begin{aligned} (x_1, \dots, x_n) &= (X_1/X_0, \dots, X_n/X_0), \\ (y_0, \dots, y_{n-1}) &= (X_0/X_n, \dots, X_{n-1}/X_n). \end{aligned}$$

Thus, we can transition between these charts by the formulas

$$x_1 = \frac{y_1}{y_0}, \dots, x_{n-1} = \frac{y_{n-1}}{y_0}, x_n = \frac{1}{y_0}.$$

This gives that

$$dx_1 \wedge \dots \wedge dx_n = d\frac{y_1}{y_0} \wedge \dots \wedge d\frac{y_{n-1}}{y_0} \wedge d\frac{1}{y_0} = \frac{(-1)^n}{y_0^{n+1}} \cdot dy_0 \wedge \dots \wedge dy_{n-1}.$$

By Example 38, any algebraic volume form on the first chart $\mathbb{C}_{\mathbf{x}}^n$ is of the form $f(\mathbf{x}) \cdot dx_1 \wedge \dots \wedge dx_n$. On the second chart $\mathbb{C}_{\mathbf{y}}^n$ it becomes

$$\frac{(-1)^n}{y_0^{n+1}} f\left(\frac{y_1}{y_0}, \dots, \frac{y_{n-1}}{y_0}, \frac{1}{y_0}\right) \cdot dy_0 \wedge \dots \wedge dy_{n-1}.$$

Thus, it has a pole of order $\geq n + 1$ along $(y_0 = 0)$, so it is not defined there. □

Proposition 40. *Let X be a smooth projective hypersurface that has an everywhere defined, nonzero, algebraic volume form σ_X . Then X is not rational. In fact, there is not even a rational map $\Phi : \mathbb{P}^n \dashrightarrow X$ with dense image.*

Proof. By Claim 9.6 (or the more general Theorem 12), there is a closed subset $Z \subset \mathbb{P}^n$ of codimension ≥ 2 such that Φ is defined on $\mathbb{P}^n \setminus Z$. Thus $\Phi^* \sigma_X$ is an algebraic volume form on $\mathbb{P}^n \setminus Z$. By the purity of poles noted in Example 38, $\Phi^* \sigma$ must extend to an everywhere defined algebraic volume form on \mathbb{P}^n . This is impossible by Lemma 39. □

Side Remark. The second part of Proposition 40 is one result that is quite different in positive characteristic. If we are in characteristic p , then $(x^p)' = px^{p-1}$ is identically 0. Thus the pullback of a nonzero algebraic volume form can be identically zero.

Thus, should we write down algebraic volume forms on hypersurfaces. We start with the affine case.

41. Algebraic volume forms on affine hypersurfaces. Let $X \subset \mathbb{C}^{n+1}$ be a hypersurface. As in [Sha74, Sec.III.6.4], it is quite easy to write down a nowhere zero volume form on X^{sm} , the smooth part of X .

Claim 41.1. For any hypersurface $X = (h = 0) \subset \mathbb{C}^{n+1}$, the volume forms

$$(41.2) \quad \sigma_i := \frac{(-1)^i}{\partial h / \partial z_i} dz_1 \wedge \dots \wedge \widehat{dz_i} \wedge \dots \wedge dz_{n+1}$$

patch together to a volume form σ_X that is defined and nowhere zero on X^{sm} .

Proof. By definition, $h|_X = 0$, thus $dh = \sum_j \frac{\partial h}{\partial z_j} dz_j$ is also zero on X . Wedging it with all the dz_i except dz_{i_1} and dz_{i_2} gives that $\sigma_{i_1} = \sigma_{i_2}$. Thus, the σ_i patch together to a rational volume form on X .

It is clear that σ_i is defined and nowhere zero on the open set where $\partial h / \partial z_i \neq 0$. Thus at least one of the σ_i is defined at a smooth point of X , so σ_X is defined and nowhere zero on X^{sm} . □

As a counterpart/consequence of Claim 10.2 we obtain the following.

Claim 41.3. Let $X \subset \mathbb{A}^{n+1}$ be a smooth hypersurface and ρ an everywhere defined algebraic volume form on X . Then it can be written as $\rho = p(\mathbf{x})\sigma_X$, where p is a polynomial on \mathbb{A}^{n+1} . □

We already noted the following analogue of Claim 10.1 for $X = \mathbb{C}^n$, the general case is proved similarly using Claim 10.1.

Purity Principle 41.4. Let ρ be an algebraic volume form on a smooth hypersurface X . Then either ρ is everywhere defined or it is not defined along a codimension 1 subset of X .

Proposition 42. *Let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of degree d . If $d \geq n + 2$, then it has a nonzero, everywhere defined, algebraic volume form.*

More precisely, the vector space of everywhere defined, algebraic volume forms on X is naturally isomorphic to the vector space of homogeneous polynomials of degree $d - n - 2$ in $n + 2$ variables.

Proof. We can combine the computations of paragraph 41 and Lemma 39 to determine the volume forms on smooth hypersurfaces, but the following way may be quicker.

As we discussed in Definition 5, $\mathbb{C}\mathbb{P}^{n+1}$ is the quotient of $\mathbb{C}^{n+2} \setminus \{0\}$ by the \mathbb{C}^* -action

$$(42.1) \quad m_\lambda : (X_0, \dots, X_{n+1}) \mapsto (\lambda X_0, \dots, \lambda X_{n+1}),$$

which is induced by the vector field $v_X := \sum X_i \frac{\partial}{\partial X_i}$.

We can thus think of a volume form on $\mathbb{C}\mathbb{P}^{n+1}$ as a \mathbb{C}^* -equivariant volume form on $\mathbb{C}^{n+2} \setminus \{0\}$, contracted by the vector field v_X . By Claim 41.3 every volume form on $\mathbb{C}^{n+2} \setminus \{0\}$ extends to a volume form on \mathbb{C}^{n+2} .

Next, let $X = X(H) \subset \mathbb{C}\mathbb{P}^{n+1}$ be a smooth hypersurface of degree d , and let $C_X := (H = 0) \subset \mathbb{C}^{n+2}$ be the cone (5.5) over it. Denote the cone minus its vertex by C_X° . We can thus view a volume form on X as a \mathbb{C}^* -equivariant volume form on C_X° contracted by v_X . We wrote down a nowhere zero volume form on C_X° in (41.2). Choose the chart where it is given by

$$(42.2) \quad \Sigma_X := \left\{ \frac{\pm 1}{\partial H / \partial X_{n+1}} dX_0 \wedge \dots \wedge dX_n \right\}.$$

In particular,

$$(42.3) \quad m_\lambda^* \Sigma_X = \lambda^{n+1-(d-1)} \Sigma_X.$$

Thus if G is any homogeneous polynomial, then

$$(42.4) \quad m_\lambda^* (G \Sigma_X) = \lambda^{\deg G + n + 2 - d} \cdot G \Sigma_X.$$

Hence $G \Sigma_X$ is \mathbb{C}^* -invariant iff $\deg G = d - n - 2$.

Let us work out explicitly what we get in terms of the forms σ_i given in (41.2). Contracting $G\Sigma_X$ by v_X gives

$$(42.5) \quad \sum X_i \frac{\partial}{\partial X_i} (G\Sigma_X) = \sum_{i=0}^n X_i G \cdot \frac{dX_0 \wedge \cdots \wedge \widehat{dX_i} \wedge \cdots \wedge dX_n}{\partial H / \partial X_{n+1}}.$$

We need to pull this back to X by the map $(x_1, \dots, x_{n+1}) \mapsto (1:x_1:\cdots:x_{n+1})$. The pullback of dX_0 is then 0, and the only term that survives is

$$(42.6) \quad G(1, x_1, \dots, x_{n+1}) \frac{dx_1 \wedge \cdots \wedge dx_n}{\partial H(1, x_1, \dots, x_{n+1}) / \partial x_{n+1}} = G(1, x_1, \dots, x_{n+1}) \cdot \sigma_{n+1},$$

where σ_{n+1} is as in (41.2). □

43. Comments on the Proofs of Theorems 30 and 36. We proved in Proposition 42 that the vector space of homogeneous polynomials of degree $\deg X - n - 2$ in $n + 2$ variables (an invariant of the embedding $X \hookrightarrow \mathbb{P}^{n+1}$) is naturally isomorphic to the vector space of everywhere defined algebraic volume forms on X (an intrinsic invariant of X).

We are completely done with both theorems if $d = n + 3$. Then the volume forms tell us the linear polynomials, and they define the embedding $X \hookrightarrow \mathbb{P}^{n+1}$.

We are almost done if $d > n + 3$; the missing ingredient is that $H^2(X(\mathbb{C}), \mathbb{Z})$ is torsion free for $\dim X \geq 2$. This is another special case of the Lefschetz hyperplane theorem.

If $d < n + 3$, then we can use the duals of holomorphic volume forms. These are wedge products of tangent vector fields. We can pull back a vector field by an isomorphism but not by an arbitrary map. This is why Theorem 30 works for $d < n + 3$ but Theorem 36 does not. □

8. NONRATIONALITY OF LOW DEGREE HYPERSURFACES

In the previous section we proved that a smooth hypersurface $X \subset \mathbb{P}^{n+1}$ of degree $\geq n + 2$ is not rational. Here we discuss what is known when the degree is $\leq n + 1$. Most of the results use different—and much more advanced—methods, so we give the statements only, with barely a hint of how they can be proved.

We proved in Theorem 7 that every irreducible quadric hypersurface is rational over \mathbb{C} and, by Corollary 25, every smooth cubic surface is rational.

We already noted that smooth cubic 3-folds are all nonrational, and the same holds for smooth quartic 3-folds [IM71]; we say more about this in Section 9.

Another method, using differential forms over fields of positive characteristic, was introduced in [Kol95]. We proved in (42.5) that if $\deg X < n + 2$, then there are no everywhere defined n -forms on X . One can show that there are also no everywhere defined algebraic r -forms on X for $r \leq n$.

It turns out that while the first of these claims continues to hold over fields of positive characteristic, there are some singular hypersurfaces that carry $(n - 1)$ -forms. After dealing with some technical issues involving these singularities, one can sometimes conclude that these hypersurfaces are nonrational. However, this happens in positive characteristic. Nonetheless, one can use some general theorems going back to Matsusaka and Mumford [MM64] to get similar conclusions over \mathbb{C} .

A serious drawback is that we can prove nonrationality only for hypersurfaces $X = (\sum_I a_I x^I = 0)$ whose coefficients satisfy countably many conditions of the

form $p_j(a_I) \neq 0$, where the p_j are (pretty much unknown) polynomials. We refer to such hypersurfaces as *very general*.

Theorem 44 ([Kol95]). *Very general hypersurfaces $X_d \subset \mathbb{P}^{n+1}$ of degree $d \geq \frac{2}{3}n+3$ are not rational. Moreover, X_d is not birational to any product $\mathbb{P}^1 \times Y$, where Y is a hypersurface (or variety) of dimension $n - 1$.*

I stress that we really do not know which hypersurfaces are covered by the theorem, though some concrete examples were written down in [Kol96, Sec.V.5].

Another argument, this time using degeneration to certain singular varieties and the topology of their resolution, was introduced by Voisin [Voi15]. It was used by many authors to prove nonrationality of varieties. For hypersurfaces the strongest result is proved by [Sch18].

Theorem 45 (Schreieder). *Very general hypersurfaces $X_d \subset \mathbb{P}^{n+1}$ of degree $d \geq 2 + \log_2 n$ are not rational for $n \geq 3$.*

9. RIGIDITY OF LOW DEGREE HYPERSURFACES

Much less is known about Question 20 for hypersurfaces $X_d \subset \mathbb{P}^{n+1}$ whose degree is $3 \leq d \leq n + 1$. We studied the degree 3 case in Section 5. The answer in the $d = n + 1$ case is one of the crowning achievements of the Noether–Fano rigidity theory.

I call it the Noether–Fano–Segre–Iskovskikh–Manin–Pukhlikov–Corti–Cheltsov–de Fernex–Ein–Mustață–Zhuang theorem, although the contributions of Fano, Iskovskikh, and Pukhlikov were the most substantial. See [Kol18] for a detailed survey.

Theorem 46. *Let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of degree $n + 1$. Assume that $n \geq 3$. Then X is not birational to any other smooth hypersurface.*

47. *A short history of Theorem 46.* The first similar result is Max Noether’s description of all birational maps $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ [Noe1870], whose method formed the basis of all further developments.

Theorem 46 was first stated by Fano for 3-folds [Fan1908, Fan1915]. His arguments contain many of the key ideas, but they also have gaps. I call this approach the *Noether–Fano method*. The first complete proof for 3-folds, along the lines indicated by Fano, is in Iskovskikh and Manin [IM71]. Iskovskikh and his school used this method to prove similar results for many other 3-folds; see [Isk79, Sar80, IP99, Isk01]. This approach was gradually extended to higher dimensions by Pukhlikov [Puk87, Puk98, Puk02] and Cheltsov [Che00]. These results were complete up to dimension 8, but needed some additional general position assumptions in higher dimensions. A detailed survey of this direction is in [Puk13].

The Noether–Fano method and the Minimal Model Program were brought together by Corti [Cor95]. (See [Kol87] for a by-now outdated but elementary introduction and [KM98] for a detailed treatment.) Corti’s technique has been very successful in many cases, especially for 3-folds; see [CR00] for a detailed study and [KSC04, Chap.5] for an introduction. However, usually one needs some special tricks to make the last steps work, and a good higher dimensional version proved elusive for a long time.

New methods involving multiplier ideals were introduced by de Fernex, Ein, and Mustață [dFEM03]; these led to a more streamlined proof that worked up to dimension 12. The proof of Theorem 46 was finally completed by de Fernex [dF16].

The recent paper of Zhuang [Zhu18] makes the final step of the Corti approach much easier in higher dimensions. The papers [SZ18, Zhu18, LZ18] contain more general results and applications.

10. CONNECTIONS WITH THE CLASSIFICATION OF VARIETIES

So far I have studiously avoided assuming prior knowledge of algebraic geometry. However, in order to explain the place of these results in algebraic geometry, it becomes necessary to use the basic theory, as in [Sha74, Vol.I]. In particular we need to know the notions of ampleness and canonical class K_X .

The classification theory of algebraic varieties—developed by Enriques for surfaces and extended by Iitaka and then Mori to higher dimensions—says that every variety can be built from three basic types:

- *General type*: K_X is ample,
- *Calabi–Yau*: K_X is trivial, and
- *Fano*: $-K_X$ is ample;

see [Kol14] for an introduction. Moreover, in the Fano case the truly basic ones are those that satisfy $H^2(X(\mathbb{C}), \mathbb{Z}) \cong \mathbb{Z}$. (It is better think of the last condition as X having *class number* equal to 1. That is, every divisor D on X is linearly equivalent to a (possibly rational) multiple of $-K_X$.)

The computations of Proposition 42 show that if $X \subset \mathbb{P}^{n+1}$ is a smooth hypersurface, then it is of general type iff $\deg X \geq n+3$, it is Calabi–Yau iff $\deg X = n+2$, and it is Fano iff $\deg X \leq n+1$.

As in Theorem 36, one can see that if two varieties X_1, X_2 on the basic type list are birationally equivalent, then they have the same type. Furthermore, in the general type case they are even isomorphic; this generalizes Theorem 36. In the Calabi–Yau case, X_1 and X_2 need not be isomorphic, but the possible birational maps between two Calabi–Yau varieties are reasonably well understood, especially in dimension 3; see, for example, [Kol89, Kol91, CK16].

It is very useful to think of Fano varieties as representatives of *rationally connected varieties*, but this would have taken us in another direction; see [Kol01] for an overview and [Kol96, AK03] for more detailed treatments.

Thus, from the general point of view, for low degree cases, the best variant of Question 20 is the following.

Definition 48. A Fano variety X with class number 1 is called *weakly superrigid* if every birational map $\Phi : X \dashrightarrow Y$ to another Fano variety Y with class number 1 is an isomorphism.

Question 49. Which Fano varieties are weakly superrigid?

The adjective “weakly” is not standard; I use it just to avoid further definitions. The correct definition of superrigid allows Y to have terminal singularities and to be a Mori fiber space; see [Puk95, Che05]. The proofs in the Noether–Fano theory are designed to prove superrigidity. However, there should be many varieties that are weakly superrigid but not superrigid; see Question 54.

11. OPEN PROBLEMS ABOUT HYPERSURFACES

The following questions are stated in the strongest forms that are consistent with known examples. I have no reason to believe that the answer to any of them

is positive, and there may well be rather simple counterexamples. As far as I know, there has been very little work on low degree hypersurfaces beyond cubics in dimension 4.

All the known rationality constructions work for even dimensional cubics only, so the following is still open.

Question 50. Is there any odd dimensional, smooth, rational cubic hypersurface?

We leave to the reader the easy task of generalizing Proposition 24 to the following higher dimensional version.

Proposition 51. *Let $X \subset \mathbb{P}^{2n+1}$ be a cubic hypersurface that contains two disjoint linear subspaces of dimension n . Then X is rational. \square*

This leaves the following open.

Question 52. Is the general even dimensional, smooth, cubic hypersurface of dimension ≥ 4 nonrational?

We saw that quadrics are rational and Proposition 51 gives examples of cubic hypersurfaces that are rational. However, no rationality construction is known for smooth hypersurfaces of higher degree. Example (21.2) gives higher degree rational hypersurfaces $X_{2d+1}^{2n} \subset \mathbb{P}^{2n+1}$, but they are always singular. However, the singularities are mild when $d \leq n$. To be precise, they are canonical if $d \leq n$ and terminal if $d \leq n - 1$. (See [KM98, Sec.2.3] or [Kol13] for introductions to such singularities.) Thus, the following problem is still open.

Question 53. Is there any smooth, rational hypersurface of degree ≥ 4 ?

Understanding birational maps between Fano hypersurfaces is even harder. Theorem 46 deals with smooth hypersurfaces $X \subset \mathbb{P}^{n+1}$ of degree $n + 1$.

Question 54. Is every birational map between smooth hypersurfaces of degree ≥ 5 an isomorphism?

Here ≥ 5 is necessary since there are some smooth quartics with nontrivial birational maps.

Example 55. Let $X \subset \mathbb{P}^{2n+1}$ be a quartic hypersurface that contains two disjoint linear subspaces L_1, L_2 of dimension n .

As in Proposition 24, for every $\mathbf{p} \in \mathbb{P}^{2n+1} \setminus (L_1 \cup L_2)$ there is a unique line $\ell_{\mathbf{p}}$ through \mathbf{p} that meets both L_1, L_2 . This line meets X in four points; two of these are on L_1, L_2 . If $\mathbf{p} \in X$, then this leaves a unique fourth intersection point, call it $\Phi(\mathbf{p})$. Clearly, Φ is an involution which is not defined at \mathbf{p} if either $\mathbf{p} \in L_1 \cup L_2$ or if $\ell_{\mathbf{p}} \subset X$.

To get a concrete example, for $n \geq 2$ consider the smooth, quartic hypersurface

$$X := (x_0^4 + \cdots + x_{2n+1}^4 = 0) \subset \mathbb{P}^{2n+1}.$$

Then $\text{Aut}(X)$ is finite (probably of order $4^{2n+1} \cdot (2n+2)!$) but combining Example 26 with the above observation gives many birational involutions on X . Most likely these involutions generate an infinite subgroup of $\text{Bir}(X)$.

ACKNOWLEDGMENTS

These notes are based on lectures given at Northwestern University in April and May of 2018. The hospitality of Northwestern University, and especially of M. Popa, gave an ideal time to write them up in expanded form. I received many helpful comments and references from V. Cheltsov, A. Corti, T. de Fernex, L. Ein, Y. Liu, and B. Totaro.

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