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Algebraic geometry II, by David Mumford and Tadao Oda, Texts and Readings in Mathematics, Vol. 73, Hindustan Book Agency, New Delhi, India, 2015, x+504 pp., ISBN 978-93-80250-80-9

When David Mumford published *Algebraic Geometry I* on complex projective varieties back in 1976 [25], he intended a second volume on schemes and a third volume on moduli problems. His notes on scheme theory were about two thirds complete at the time, but they lay dormant for several decades until his student Tadao Oda prepared them for publication, making improvements and adding several new sections. Further exercises and examples were provided by Ching-Li Chai, Vikraman Belaji, and D. S. Nagaraj, and an additional section on Seshadri constants was added by Belaji. All the new material is clearly marked. Thus *Algebraic Geometry II* can be viewed as Mumford's introduction to scheme theory, rounded out by contributions from Oda and his colleagues.

Since schemes are less familiar than complex varieties, we describe how varieties grew into schemes and what was gained in the transition. After giving some history and motivations leading up to Grothendieck's scheme theory, we explain how the category of schemes enlarges the category of varieties. We then show some of the new features made possible with scheme theory and close with a description of the new book.

1. Catalysts for the birth of scheme theory

Here we describe developments in algebraic geometry that led up to the creation of scheme theory; see surveys of Dieudonné [7,8] and Kleiman [17] for more details.

The introduction of imaginary numbers and points at infinity ushered in a golden age of projective geometry in the early 1800s, and it became commonplace for geometers to work in $\mathbb{P}^2_{\mathbb{C}}$ and $\mathbb{P}^3_{\mathbb{C}}$. Riemann's work on complex curves via abelian integrals and function fields in the 1850s had such a huge impact that today we use the term *Riemann surface* for a one-dimensional complex manifold. Dedekind and Weber [4] introduced divisors on curves and used discrete valuations on function fields to give algebraic analogues of many of Riemann's results. Brill and M. Noether further developed linear systems for curves, and these ideas were taken up around 1890 for surfaces with great success by the Italian school led by Castelnuovo, Enriques, and Severi. Working over $\mathbb{C}$, they freely used continuity and topology in their arguments, but sometimes did not carefully treat degenerate cases.

As abstract algebra grew in the early 1900s, it became natural to consider varieties over an arbitrary field k. The continuity arguments of the Italians failed over general fields, and along with it their notions of generic points and intersection multiplicity, but van der Waerden devised new versions of each [39, 40] by working in extension fields $k \subset K$. This period also saw Krull and E. Noether set a foundation for future work by pioneering the theory of ideals and modules over commutative rings. Number theory provided an impetus to study varieties over finite fields as F. K. Schmidt, building on E. Artin's 1921 thesis, defined a zeta function for curves over $\mathbb{F}_q$ that was analogous to the Riemann zeta function associated to curves over

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$\mathbb{C}$. Extending work of Hasse and Deuring from the 1930s, A. Weil built a theory of correspondences to prove the Riemann hypothesis for curves over finite fields [41, 43]. In the process he established foundations for abstract varieties over fields of arbitrary characteristic [42]. Zariski became a champion of algebraic methods in geometry, proving Bertini theorems in finite characteristic [48, 50], characterizing nonsingular points on varieties in terms of the ring of germs of regular functions [51], constructing the normalization of a variety [46] and resolutions of singularities [47, 49], and giving us the Zariski topology.

By 1950, Weil observed that the Zariski topology could be defined for his abstract varieties, which were constructed by gluing together affine varieties analogous to the construction of a manifold by gluing together Euclidean neighborhoods. Serre [33] transferred the sheaf theory of Leray to this setting, simplifying definitions and making gluing constructions easier, and Chevalley [3] studied the morphisms between them. The definition of abstract variety over a field was mostly worked out, but then following a suggestion of Cartier, Grothendieck embarked in 1957 on an enormous program to set new foundations for algebraic geometry that would absorb all previous developments and start from the category of *all* commutative rings. Grothendieck's schemes provide a geometric foundation that encompasses both arithmetic and projective geometry, fulfilling a goal going back to Kronecker [18]. With these new foundations, the field has grown immensely and flourished. The nearby fields of commutative algebra, number theory, homological algebra, and category theory have also benefited greatly.

2. From varieties to schemes

To compare the content of *Algebraic Geometry I* and *Algebraic Geometry II*, we briefly define complex varieties and schemes, explaining how the category of schemes enlarges the category of varieties.

2.1. **The category of complex algebraic varieties.** Classic algebraic geometry has mainly focused on the zero locus of polynomial equations over $\mathbb{C}$. For $f_\alpha \in \mathbb{C}[x_1, \dots, x_n]$, the common zeros of f_α agree with those of the ideal $I = (f_\alpha)$ generated by the f_α, so we consider the zeros $Z(I) \subset \mathbb{C}^n$ of ideals $I \subset \mathbb{C}[x_1, \dots, x_n]$: the $Z(I)$ form the closed sets for the *Zariski topology* on $\mathbb{C}^n$. Setting $J = \sqrt{I} = \{f \in S : f^k \in I \text{ for some } k \geq 1\}$, it is easy to check that $Z(J) = Z(I)$ and that $\sqrt{J} = J$, meaning that J is a *radical* ideal. Hilbert's *Nullstellensatz* [15, I, 1.3A] says that $J \mapsto Z(J)$ gives a bijection between the set of radical ideals J and Zariski closed subsets of $\mathbb{C}^n$. Under this bijection, prime ideals correspond to *irreducible* closed subsets, those that are not unions of smaller closed sets: these are the *affine varieties*. These notions extend to zero sets of homogeneous ideals for projective space $\mathbb{CP}^n$, leading to *projective varieties* and their nonempty open subsets, *quasi-projective varieties*.

Given a variety Y and Zariski open subset $U \subset Y$, a function $f : U \to \mathbb{C}$ is *regular* if it is locally a rational function in the coordinates of an ambient $\mathbb{C}^n$. The set of all regular functions on U form a ring denoted $\mathcal{O}_Y(U)$. We obtain a category $\mathcal{V}$ of complex varieties by taking the morphisms $\psi : X \to Y$ to be continuous maps such that for each regular $f : U \to \mathbb{C}$, the composite $f \circ \psi : \psi^{-1}(U) \to \mathbb{C}$ is also regular. If the affine variety $Y \subset \mathbb{C}^n$ corresponds to the prime ideal $P \subset \mathbb{C}[x_1, \dots, x_n]$, then the natural map $\mathbb{C}^n[x_1, \dots, x_n]/P \to \mathcal{O}_Y(Y)$ is an isomorphism [15, I, 3.2 (a)] so that the global regular functions are restrictions of polynomial functions. Morphisms in

$\mathcal{V}$ are locally determined by homomorphisms of rings of regular functions, for if X is any complex variety and $Y \subset \mathbb{C}^n$ is affine, then pulling back regular functions gives a bijection [15, I, 3.5]

$$\mathrm{Hom}_{\mathcal{V}}(X,Y) \cong \mathrm{Hom}_{\mathbb{C}\text{-algebras}}(\mathcal{O}_Y(Y),\mathcal{O}_X(X)). \tag{1}$$

2.2. **The category of schemes.** The theory of varieties works just as well upon replacing $\mathbb{C}$ with any algebraically closed field k [15, I], but this is not enough for applications where one is interested in solutions to equations with coordinates in finite fields, the integers, or number fields. This suggests working over a larger category of commutative rings R, and Grothendieck decided to use all such rings having an identity. Points of an affine variety correspond to maximal ideals in the affine coordinate ring, but when one goes to the category of all commutative rings there are problems: the preimage of the maximal ideal under the natural inclusion $\mathbb{Z} \to \mathbb{Q}$ is $(0) \subset \mathbb{Z}$, an ideal which is prime, but not maximal. Scheme theory is based on a model in which any ring R becomes the ring of functions on some space, but what space? The answer follows.

Let R be a commutative ring with identity 1. The *spectrum* of R is the topological space $\mathrm{Spec}\, R = \{p \subsetneq R : p \text{ is a prime ideal}\}$ with closed sets $V(I) = \{p \subset R : I \subset p\}$, $I \subset R$ an ideal. For $p \subset R$ prime, let $[p] \in \mathrm{Spec}\, R$ be the corresponding point. For each $[p] \in \mathrm{Spec}\, R$, *localization* [2, Ch. 3] inverts elements of $R - p$ to produce a ring R_p which is *local* in the sense that it has a unique maximal ideal pR_p. On $\mathrm{Spec}\, R$ there is a sheaf of rings $\mathcal{O}_{\mathrm{Spec}\, R}$ called the *structure sheaf*, whose sections over an open set $U \subset \mathrm{Spec}\, R$ are the functions f on U whose value at $[p] \in U$ lies in R_p and which are locally given by fractions from R, with the obvious restriction maps for inclusions $V \subset U$. One computes [29, p.7] that the stalk of $\mathcal{O}_{\mathrm{Spec}\, R}$ at $[p]$ is isomorphic to the local ring R_p. An *affine scheme* is a pair $(\mathrm{Spec}\, R, \mathcal{O}_{\mathrm{Spec}\, R})$ as above. A *scheme* is a pair $(X, \mathcal{O}_X)$ with topological space X, a sheaf of rings $\mathcal{O}_X$, and an open cover U_α such that $(U_\alpha, \mathcal{O}_X|_{U_\alpha})$ is an affine scheme. In particular, every point $x \in X$ comes with a local ring $(\mathcal{O}_{x,X}, \mathfrak{m}_x)$ and residue field $k(x) = \mathcal{O}_{x,X}/\mathfrak{m}_x$.

A morphism $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ in the category $\mathcal{S}$ of schemes consists of a continuous map $f : X \to Y$ of topological spaces and a homomorphism $f^* : \mathcal{O}_Y \to f_*\mathcal{O}_X$ of sheaves with the additional property that for $y = f(x)$, the induced homomorphism

$$f_x^* : \mathcal{O}_{y,Y} = \varinjlim_{y \in U} \mathcal{O}_Y(U) \to \varinjlim_{x \in V} \mathcal{O}_X(V) = \mathcal{O}_{x,X}$$

on local rings is a *local homomorphism*, meaning that $f_x^*(\mathfrak{m}_y) \subset \mathfrak{m}_x$. This is automatic for maps of germs of regular functions arising from a morphism of varieties because the functions map into $\mathbb{C}$, but here it is an axiom. A morphism of schemes $f : X \to Y$ with $Y = \mathrm{Spec}\, R$ an affine scheme induces a ring homomorphism $f^* : R \cong \mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$. Analogous to bijection (1), there is a strong connection to rings of functions: for Y affine there is a bijection [29, 1.3.7] induced by f^*, namely

$$\mathrm{Hom}_{\mathcal{S}}(X, \mathrm{Spec}\, R) \cong \mathrm{Hom}_{\mathrm{Rings}}(R, \mathcal{O}_X(X)). \tag{2}$$

2.2.1. *An example.* It is the structure sheaf and morphisms that make the category of schemes work. The structure sheaf is unusual because the targets of its functions vary from point to point, so we illustrate with an example. On the variety $\mathbb{C}$ the rational function $x^3/(x^2-1)$ defines a regular function on the Zariski open set $U = \mathbb{C} - \{-1, 1\} \subset \mathbb{C}$ whose value at $3 \in \mathbb{C}$ is $27/8$. On the scheme $\mathbb{A}^1 = \operatorname{Spec} \mathbb{C}[x]$, the fraction $x^3/(x^2-1) \in \mathbb{C}(x)$ also defines an element in $\mathcal{O}_{\mathbb{A}^1}(\tilde{U})$ over the corresponding open subset $\tilde{U} = \mathbb{A}^1 - V((x^2-1))$, but by considering it as an element of the local ring $\mathcal{O}_{y,\mathbb{A}^1}$ for each $y \in \tilde{U}$. At the point $y \in \mathbb{C}^1$ corresponding to the maximal ideal $p = (x-3)$ its value is $x^3/(x^2-1) \in \mathbb{C}[x]_{(x-3)} \cong \mathcal{O}_{y,\mathbb{A}^1}$. The image of $x^3/(x^2-1)$ in the residue field $k(y) \cong \mathbb{C}$ is $27/8$ as above.

2.2.2. *Scheme points as images of morphisms.* Let $x \in X$ be a point on a scheme. If $x = [p] \in \operatorname{Spec} R \subset X$ on an open affine subscheme, then $k(x) \cong R_p/pR_p$ and the ring homomorphism $R \to R_p/pR_p$ induces a morphism of schemes $\operatorname{Spec} k(x) \hookrightarrow \operatorname{Spec} R \subset X$ whose image is x. Note that since $k(x)$ is a field, it contains only one prime ideal so that $\operatorname{Spec} k(x)$ consists of one point.

2.3. **Complex varieties as schemes.** In view of bijections (1) and (2), one is naturally drawn to compare an affine variety $Y \subset \mathbb{C}^n$ to the affine scheme $\operatorname{Spec} \mathcal{O}_Y(Y)$. To do this right, let $\mathcal{S}(\mathbb{C})$ be the category whose objects are pairs (X, π_X) with X a scheme and $\pi_X : X \to \operatorname{Spec} \mathbb{C}$ a morphism; the morphisms in $\mathcal{S}(\mathbb{C})$ are scheme morphisms $f : X \to Y$ satisfying $\pi_Y \circ f = \pi_X$. If Y is a complex variety, the constant functions form a subring $\mathbb{C} \subset \mathcal{O}_Y(Y)$ and $\tilde{Y} = \operatorname{Spec} \mathcal{O}_Y(Y) \to \operatorname{Spec} \mathbb{C}$ is naturally in $\mathcal{S}(\mathbb{C})$: thus the assignment $Y \mapsto \tilde{Y}$ is a functor from affine complex varieties to $\mathcal{S}(\mathbb{C})$ which extends to a fully faithful functor $\mathcal{V} \to \mathcal{S}(\mathbb{C})$ [15, II, 2.6] (see also [29, §2.1]) so that $\mathcal{V}$ embeds into $\mathcal{S}(\mathbb{C})$.

3. An advertisement for scheme theory

The category of schemes is much richer than the category of varieties. Schemes are topologically more flexible in that they can have multiple irreducible components, though this is mostly a matter of terminology because one can in any event consider unions of varieties. Of much greater mathematical significance is the local structure: schemes can carry extra local infinitesimal data (scheme structure) in the form of zero divisors in their local rings. Because schemes use all commutative rings, there are applications to number theory, and through the canonical morphism to $\operatorname{Spec} \mathbb{Z}$ one can compare families of varieties as the characteristic of the ground field changes. Often schemes can be constructed which parametrize objects of interest in algebraic geometry such as families of schemes and vector bundles. We illustrate these facets.

3.1. **Topology and generic points.** Unlike varieties, schemes can have multiple irreducible components. A scheme X is irreducible if and only if it is the closure of a point $\eta \in X$, called its *generic point* [29, 1.3.2], so the map $p \mapsto \overline{\{p\}}$ gives a bijective correspondence between points of X and irreducible closed subschemes of X. Consequently, nonempty open subsets U, V of an irreducible scheme X must intersect (both contain the generic point), so the Zariski topology is almost never Hausdorff. Many local properties (such as smoothness) hold at η if and only if they hold on a dense open subset $U \subset X$, making a connection between the generic point $\eta \in X$ and general points of X.

3.1.1. *Ambient spaces.* Classic algebraic geometry deals with subvarieties in $\mathbb{A}^n_k$ or $\mathbb{P}^n_k$ over a field k. Similarly modern algebraic geometry often deals with subschemes of the ambient spaces $\mathbb{A}^n_k = \operatorname{Spec} k[x_1, \ldots, x_n]$ and $\mathbb{P}^n_k$, defined by gluing together open affine subsets $U_i \cong \mathbb{A}^n_k$. These constructions work over any ring R to form

$\mathbb{A}^n_R$ and $\mathbb{P}^n_R$. These spaces are irreducible if R is an integral domain, since then $[(0)] \in \mathbb{A}^n_R$ is a generic point.

3.1.2. *Degenerate conics I.* Consider the family $C_t = \operatorname{Spec}\mathbb{C}[x,y]/(xy-t) \subset \mathbb{A}^2_{\mathbb{C}}$ for $t \in \mathbb{C}$. Then C_t is an irreducible conic for $t \neq 0$, but the limiting curve as $t \to 0$ is $C_0 = \operatorname{Spec}\mathbb{C}[x,y]/(xy)$, which has two irreducible components in the lines $\operatorname{Spec}\mathbb{C}[x,y]/(x)$ and $\operatorname{Spec}\mathbb{C}[x,y]/(y)$.

3.2. **Infinitesimal structure and nilpotents.** Unlike varieties, local rings on schemes can have zero divisors. This innovation allows schemes to encode multiplicity and is a key in making the category of schemes work.

3.2.1. *Degenerate conics II.* Consider the family $C_t = \operatorname{Spec}\mathbb{C}[x,y]/(ty-x^2) \subset \mathbb{A}^2_{\mathbb{C}}$ for $t \in \mathbb{C}$. Then C_t is an irreducible conic for $t \neq 0$, but the limit as $t \to 0$ is a double line $C_0 = \operatorname{Spec}\mathbb{C}[x,y]/(x^2)$, whose local rings have the nilpotent element x.

3.2.2. *Dual numbers.* The algebra of dual numbers over $\mathbb{C}$ is $\mathbb{C}[\epsilon]/(\epsilon^2)$. Mumford describes the scheme $\operatorname{Spec}\mathbb{C}[\epsilon]/(\epsilon^2)$, which consists of a point with nilpotent element ϵ, as a "disembodied tangent vector" [29, §5.1]. For a scheme $X \subset \mathbb{P}^n_{\mathbb{C}}$, giving a morphism $\operatorname{Spec}\mathbb{C}[\epsilon]/(\epsilon^2) \to X$ is equivalent to giving a closed point $x \in X$ and a nonzero Zariski tangent vector $v \in T_{x,X} = \operatorname{Hom}_{\mathbb{C}}(\mathfrak{m}_x/\mathfrak{m}_x^2, \mathbb{C})$.

3.3. K**-valued points and number theory.** If X is a scheme and R is a ring, the set of *R-valued points of* X is the set $X(R) = \operatorname{Hom}_{\mathcal{S}}(\operatorname{Spec} R, X)$. When $R = K$ is a field, giving an element of $X(K)$ is equivalent to giving a point $x \in X$ (the image of $\operatorname{Spec} K$) and a field extension $k(x) \subset K$. If X is a complex variety with corresponding scheme $\tilde{X}$, then $\tilde{X}(\mathbb{C}) = X$ recovers the points of the variety.

3.3.1. *Diophantine equations.* For polynomials $f_\alpha \in \mathbb{Z}[x_1,\dots,x_n]$ and a ring R, the set $\{(a_1,\dots,a_n) \in R^n : f_\alpha(a_1,\dots,a_n) = 0 \text{ for all } \alpha\}$ of solutions to the corresponding Diophantine equations is given by $X(R)$, where X is the scheme

$$X = \operatorname{Spec}\mathbb{Z}[x_1,\dots,x_n]/(f_1,\dots,f_m).$$

Indeed, each $\varphi \in X(R)$ is given by a homomorphism

$$\varphi^* : \mathbb{Z}[x_1,\dots,x_n]/(f_1,\dots,f_n) \to R$$

via bijection (2), which is given by images a_i of x_i satisfying $f_i(a_1,\dots,a_n) = 0$. Usually, one seeks solutions in $R = \mathbb{Z}$ or $R = \mathbb{Q}$.

3.3.2. *Fermat's last theorem.* According to Wiles and Taylor [37, 45], the only $\mathbb{Q}$-valued points of $X = \operatorname{Spec}\mathbb{Z}[x,y,z]/(x^n+y^n-z^n)$ with $n > 2$ satisfy $x = 0$ or $y = 0$.

3.3.3. *The Weil conjectures.* If $X \subset \mathbb{P}^n_{\mathbb{F}_q}$ is a subscheme, there is an associated zeta function $Z(X,t) = \exp(\sum_{m=1}^\infty (N_m/m)t^m) \in \mathbb{Q}[[t]]$, where N_m be the number of points in $X(\mathbb{F}_{q^m})$. For example, when $X = \mathbb{A}^1_{\mathbb{F}_q} = \operatorname{Spec}\mathbb{F}_q[x]$, we have $N_m = q^m$ points so that $Z(\mathbb{A}^1,t) = 1/(1-qt)$. Hasse proved the Riemann hypothesis about the zeros for $Z(X,t)$ when X is a nonsingular curve of genus $g = 1$. Weil later proved the general case $g \geq 1$ and in 1948 made influential conjectures about the zeta function when X is a smooth projective variety [44]. These were proved by Deligne in 1973 [5].

3.4. **Variation of characteristic.** Each scheme X has a unique morphism $X \to \operatorname{Spec}\mathbb{Z}$ given by $x \mapsto [(\operatorname{char} k(x))]$, so we can think of X as fibered by $X_p = X \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec}\mathbb{Z}/p\mathbb{Z}$ for $p \neq 0$ and $X_0 = X \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec}\mathbb{Q}$. For a complex variety $V \subset \mathbb{A}^n_{\mathbb{C}}$ or $V \subset \mathbb{P}^n_{\mathbb{C}}$, one can construct a finitely generated $\mathbb{Z}$-algebra $R \subset \mathbb{C}$ by adjoining the coefficients of the defining equations for V to $\mathbb{Z}$. The same equations define a scheme $X \subset \mathbb{A}^n_R$ or $X \subset \mathbb{P}^n_R$ such that $V \cong X \times_{\operatorname{Spec} R} \operatorname{Spec}\mathbb{C}$ and the map $X \to \operatorname{Spec} R \to \operatorname{Spec}\mathbb{Z}$ is of finite type. Properties of V over $\mathbb{C}$ may be influenced by the fibers X_p and conversely [29, §4.1].

3.4.1. *Mori's proof of Hartshorne's conjecture.* S. Mori used these ideas to prove that the only n-dimensional smooth complete variety X over an algebraically closed field k with ample tangent bundle T_X is $X = \mathbb{P}^n_k$, as conjectured by Hartshorne [14]. He proved the theorem by covering X with rational curves and using them to construct an isomorphism. The amazing feature of Mori's proof [20] is that it requires an argument in characteristic $p > 0$, even if char $k = 0$: he produces the rational curves over a finite field with the help of the Frobenius morphism and then deforms them to characteristic zero. At this point there is still no proof that avoids this idea.

3.5. **Representable functors and moduli spaces.** An aspect that sets algebraic geometry apart from other branches of mathematics is that many families of objects of interest (schemes, vector bundles, morphisms) are parametrized by schemes: the study of these *moduli spaces* has been a very active area of research going back to work of Riemann, Plücker, and Cayley in the mid-1800s. We illustrate with some examples, working over a fixed algebraically closed field k.

3.5.1. *Grassmann varieties.* On projective space $\mathbb{P}^n$ over k there is the *Euler sequence*

$$0 \to \Omega_{\mathbb{P}^n} \otimes \mathcal{O}(-1) \to \mathcal{O}^{n+1}_{\mathbb{P}^n} \xrightarrow{\pi} \mathcal{O}_{\mathbb{P}^n}(1) \to 0 \tag{3}$$

in which $\mathcal{O}(1)$ is dual to the tautological bundle $\mathcal{O}(-1)$, $\pi(a_0, \dots, a_n) = \sum a_i x_i$ and $\Omega_{\mathbb{P}^n}$ is the cotangent bundle. For $x \in \mathbb{P}^n$, the map $\pi_x : k^{n+1} \to k$ on fibers ranges over all quotients: taking $x_0, \dots, x_n$ as basis for k^{n+1}, we have $Z(\ker \pi_x) = \{x\}$.

Pulling back (3) along a morphism $f : X \to \mathbb{P}^n$ gives a surjection $\mathcal{O}^{n+1}_X \to f^*\mathcal{O}_{\mathbb{P}^n}(1)$ and conversely any line bundle quotient $\mathcal{O}^{n+1}_X \to \mathcal{L}$ uniquely defines such a morphism f. Thus the functor $\operatorname{Hom}_{\mathcal{S}}(-, \mathbb{P}^n)$ from schemes to sets is naturally equivalent to the functor $F(-)$ which assigns to a scheme X the set of all line bundle quotients $\mathcal{O}^{n+1}_X \to \mathcal{L}$ on X. We say that $\mathbb{P}^n$ *represents* the functor F.

More generally, the contravariant functor $F(-)$ taking X to the set of rank $r+1$ vector bundle quotients $\mathcal{O}^{n+1}_X \to \mathcal{E}$ is represented by the Grassmann variety $\mathbb{G}(n, r)$, whose k-valued points are in bijective correspondence with rank $r+1$ vector space quotients of k^{n+1} up to isomorphism. As a variety, $\mathbb{G}(n, r)$ is smooth of dimension $(n-r)(r+1)$. Taking x_i as a basis for k^{n+1}, the kernel generates the ideal of a linear subspace $L \subset \mathbb{P}^n$ of dimension r so that $\mathbb{G}(n, r)$ parametrizes all r-dimensional linear subspaces of $\mathbb{P}^n$.

3.5.2. *Hilbert schemes.* Each closed subscheme $X \subset \mathbb{P}^n$ has a unique largest defining ideal $I \subset S = k[x_0, \dots, x_n]$ and *Hilbert function* $h_X(m) = \dim_k (S/I)_m$ giving the vector space dimension of the graded pieces of S/I. There is a unique polynomial $P_X(t) \in \mathbb{Q}[t]$ such that $h_X(m) = P_X(m)$ for all $m \gg 0$, the *Hilbert polynomial* of X. Now define the contravariant functor $F_n^{P(t)}$ taking a scheme X to the set of all closed subschemes $Y \subset X \times \mathbb{P}^n$ such that for each $x \in X$, the fiber $Y_x = Y \times_X \operatorname{Spec} k(x) \subset \mathbb{P}^n_{k(x)}$ has Hilbert polynomial $P(t)$. Grothendieck proved that $F_n^{P(t)}$ is represented by a projective scheme $\operatorname{Hilb}_n^{P(t)}$ called the Hilbert scheme [10]. In particular, the k-valued points $\operatorname{Hilb}_n^{P(t)}(k)$ are in bijective correspondence with the set of closed subschemes $Y \subset \mathbb{P}^n$ having Hilbert polynomial $P(t)$. Hilbert schemes are fascinating objects. They are connected [13] but are typically reducible and have arbitrarily bad singularities [38].

3.5.3. *Space curves.* The study of curves $C \subset \mathbb{P}^3_{\mathbb{C}}$ goes back over 100 years. If C is nonsingular and connected, then $P_C(t) = dt + (1 - g)$, where g is the topological genus of the compact real surface C and d is the number of points in $C \cap H$ for a general plane $H \subset \mathbb{P}^3$: a straight line $\mathbb{P}^1 \cong C \subset \mathbb{P}^3$ has Hilbert polynomial $t + 1$. Halphen [12] and M. Noether [30] classified such curves up to degree 20 in the early 1880s. The Hilbert polynomial defines degree d and (arithmetic) genus g for any one-dimensional closed subscheme $C \subset \mathbb{P}^3$ by the formula above, but here the degree measures $C \cap H$ with multiplicity and the genus g can be negative: the union of two disjoint lines has degree $d = 2$ and genus $g = -1$. There are only a few nonsingular connected curves of degree $d \leq 4$: plane curves, the twisted cubic, the rational quartic, and the elliptic quartic. Nowadays we have classifications of curves without embedded points of degree $d \leq 4$ [31,32], most of which have negative genus and consist of multiplicity structures supported on lines. The corresponding Hilbert schemes have many irreducible components, some of which are everywhere singular [19].

3.5.4. *Moduli spaces of genus g curves.* Riemann stated that the set M_g of isomorphism classes of genus g complex curves were characterized by $3g - 3$ complex parameters varying continuously for $g \geq 2$. Severi [35] proved this rigorously in 1921 by using connectedness of the space $H^{n,w}$ of n-sheeted simple coverings of $\mathbb{P}^1$ with w branch points [16]. Mumford constructed M_g as a coarse moduli scheme using geometric invariant theory [28]. Over fields of characteristic $p > 0$, Deligne and Mumford [6] constructed a compactification $\overline{M}_g$ of M_g by adding nodal curves to show irreducibility of M_g in arbitrary characteristic. Later compactifications were constructed by Mumford via Chow varieties [26] and Gieseker via Hilbert schemes [9]. Nowadays many compactifications are considered.

4. The book under review

Grothendieck published his theory of schemes in the monumental work *Éléments de géométrie algébrique* (EGA) [11]. Early practitioners of scheme theory relied on EGA as the primary source, but the level of generality and hundreds of pages in length could be intimidating, so gentler introductions began to appear. Mumford's *Lectures on curves on an algebraic surface* [23] from 1966 might be the first book to contain a concise introduction to schemes. A few years later Mumford's Harvard lecture notes on schemes, which were originally mimeographed and bound with a red cover by the math department, were published by Springer under the title *Red Book of Varieties and Schemes* [27]. Shafarevich's *Basic Algebraic Geometry* [36] appeared in 1974 and Hartshorne's popular *Algebraic Geometry* [15] in 1977. Now many introductions to schemes are available.

Algebraic Geometry II by Mumford and Oda delivers standard results in scheme theory with additional topics. Chapters 1–3 cover the definition of a scheme X and its functor $\mathrm{Hom}_{\mathcal{S}}(-, X)$, basic properties (reduced, irreducible, separated, finite type, proper), and the Proj construction, including twists of sheaves, blowups, line bundles, and divisors. Chapter 4 discusses base extension from X/k to $\overline{X}/\overline{k}$ ($\overline{k}$ is the algebraic closure of k) and the action of $\mathrm{Gal}(\overline{k}/k)$, comparison of fibers over $\operatorname{Spec}\mathbb{Z}$ in varying characteristic, flatness, dimension of fibers of a morphism, and Hensel's lemma. Chapter 5 covers nonsingular points, smooth morphisms, normality, and Zariski's main theorem. A short Chapter 6 on group schemes is followed by a lengthy Chapter 7 on cohomology, spectral sequences, ampleness criteria, and intersection numbers. Chapter 8 features the Riemann–Roch theorem, Serre's GAGA principle, de Rham cohomology, characteristic p phenomena, and deformation theory. The book closes with two results which require schemes, Mori's existence theorem for rational curves and Belyi's three point theorem.

The explanations and illustrations in this book are excellent. Longer proofs are often broken down into steps for better readability. It is especially useful that alternative arguments and constructions are given in several places: for instance, tricks of the trade in computing Čech cohomology are illustrated using four different methods (§7.4–§7.7). The book does a great job of making connections between scheme theory and other areas of mathematics, such as number theory (Kronecker's big picture), complex manifolds (Serre's GAGA principle [34] and de Rham cohomology), algebraic topology (the algebraic fundamental group), and group actions. Throughout we see Mumford's enthusiasm for Grothendieck's work, as he often seems to pull the reader aside to show us some slick idea or result of Grothendieck. I especially enjoyed Chapters 4, 6, and 8.

This book might be too much as a first introduction to schemes. The pace is brisk, with sheaf theory a brief appendix and schemes defined already on page 11. As a researcher Mumford excelled at producing interesting examples [1, 21, 22, 24] and indeed the examples given are excellent, but there are probably too few of them for the beginner. For example, in the first chapter only the two sections discussing the functor defined by a scheme have *any* examples, and some chapters have none. The exercises are interesting and good, but again there might not be enough for the active reader. Only the Čech cohomology theory is given without Grothendieck's derived functor cohomology, but it is observed that these theories agree for separated schemes.

In view of the comments above, I would recommend *Algebraic Geometry II* as a second introduction to scheme theory, or to supplement another book with more exercises and examples. I would have benefited from reading this book as a graduate student, but it could not replace the books [2, 15] I learned from.

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