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SELECTED MATHEMATICAL REVIEWS

related to the paper in the previous section by
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Borel, Armand; Serre, Jean-Pierre

Le théorème de Riemann-Roch.

Bulletin de la Société Mathématique de France **86** (1958), 97–136.

The advent of sheaf theory has, amongst many things, brought with it a great development of the classical theorem of Riemann-Roch. This paper is devoted to Grothendieck's version of the theorem. Grothendieck has generalized the theorem to the point where not only is it more generally applicable than the F. Hirzebruch's version [*Neue topologische Methoden in der algebraischen Geometrie*, Springer-Verlag, 1956; MR0082174], but it depends on a simpler and more natural proof.

The paper originated from the notes of a seminar devoted to the work of Grothendieck which the authors conducted in Princeton during the fall of 1957, and the opening phrase asserts their essentially editorial role.

The result of this unusual three-way collaboration is a remarkably clear, short, and highly motivated presentation of the Grothendieck theorem. Of necessity, such an exposition is primarily directed at the expert, and the paper is quite hard going for those of us who are not intimately acquainted with their basic references, in particular, with J.-P. Serre's paper "Faisceaux algébriques cohérentes" [Ann. of Math. (2) **61** (1955), 197–278; MR0068874], which unquestionably lays the foundation for Grothendieck's work. In their single-mindedness, the authors have also omitted an introduction, and start off at once with preparatory material towards theorem I.

This first goal of theirs is the following: Let $f\colon X \to Y$ be a proper map of quasi-projective varieties, let $\mathcal{F}$ be a coherent sheaf on X, and let the sheaves $R^q f(\mathcal{F})$ on Y be defined by $R^q f(\mathcal{F})_U = H^q(f^{-1}(U); \mathcal{F})$ (U open in X). Then these sheaves are also coherent.

This theorem has vital consequences for their study of the group $K(X)$ which they introduce next. If X is an algebraic variety (always over an arbitrary algebraically closed field) the group $K(X)$ is defined as follows. Let $F(X)$ denote the free abelian group generated by coherent sheaves over X. Also, if $E\colon 0 \to \mathcal{F}_1 \to \mathcal{F} \to \mathcal{F}_2 \to 0$ is a short exact sequence of such sheaves, let $Q(E)$ be the "word" $\mathcal{F} - (\mathcal{F}_1 + \mathcal{F}_2)$ in $F(X)$. Now define $K(X)$ as the quotient of $F(X)$ modulo the subgroup generated by $Q(E)$ as E ranges over the short exact sequences. (We call this construction the K-construction; it can clearly be applied to any category in which short exact sequences are defined.) For example, if p is a point, then $K(p) \approx \mathbf{Z}$ (= the ring of integers), the isomorphism being determined by attaching to a sheaf (which is merely a module over the ground-field in this case) its dimension. This homomorphism is denoted by $ch\colon K(p) \overset{\approx}{\to} \mathbf{Z}$.

As will be seen, the Riemann-Roch theorem is a comparison statement about $K(X)$ and the Chow ring $A(X)$ which is valid only on non-singular varieties. Accordingly, we will let $\mathfrak{A}$ denote the category of quasi-projective non-singular varieties and their proper maps. On this category $K(X)$ and $A(X)$ partake of both a covariant and a contravariant nature, and it is precisely to complete $K(X)$ to a covariant functor that theorem I is essential.

Grothendieck denotes this covariant homomorphism, induced by a map $f\colon X \to Y$ in $\mathfrak{A}$, by $f_!$, and defines it in this way: If $\mathcal{F}$ is a sheaf (coherent, algebraic, will be understood hereafter) then $f_!(\mathcal{F}) \in K(Y)$ shall be the class of the word $\sum_q (-1)^q R^q f(\mathcal{F})$ in $K(Y)$. Because the sum is finite on objects in $\mathfrak{A}$ this operation is well defined, and its linear extension to $F(X)$ is seen to vanish on words of the form $Q(E)$, thus inducing a homomorphism $f_!\colon K(X) \to K(Y)$.

The naturality condition $(f \circ g)_! = f_! \circ g_!$ is valid, and follows from the spectral sequence which relates $R^q(f \circ g)$ to $R^s f$ and $R^t g$. Thus the obvious "Euler characteristic" nature of $f_!$ is essential not only for the vanishing of $f_!$ on $Q(E)$, but also for the naturality! Note also that if $f\colon X \to p$ is the map onto a point, then $ch\, f_!(\mathcal{F})$ may be identified with $\sum(-1)^q \dim H^q(X;\mathcal{F}) = \mathfrak{X}(X;\mathcal{F})$; and it is an expression of this sort which was evaluated by Hirzebruch in his topological version of the Riemann-Roch theorem by a certain cohomology class. In short, $f_!$ is a very "good" notion.

In the Grothendieck theory, the role of cohomology is taken over by the Chow ring $A(X)$, of cycles under linear equivalence, the product being defined by intersection. On our category, $A(X)$ also has a covariant side to it, namely $f_*\colon A(X) \to A(Y)$, defined by the direct image of a cycle. However, f_* is only an additive homomorphism. The contravariant extension of $A(X)$, i.e., $f \to f^*$ where f^* is induced by the inverse image of a cycle, is of course a ring homomorphism; and these two operations are linked by the permanence law: $f_*((x) \cdot f^*(y)) = f_*(x) \cdot y, x \in A(X), y \in A(Y)$.

The contravariant properties of $K(X)$ are best brought out with the aid of the following theorem II: Let $K_1(X)$ be the group obtained by applying the K-construction to the category of algebraic vector bundles over $X, X \in \mathfrak{A}$. Also, let $\varepsilon\colon K_1(X) \to K(X)$ be the homomorphism defined by the operation which assigns to a bundle the sheaf of germs of its sections. Then ε is a bijection.

To a topologist at least, this theorem is reminiscent of the Poincarè duality theorem. In any case, by identifying $K(X)$ with $K_1(X)$ one may induce the obvious (inverse image of a bundle) contravariant extension of $K_1(X)$ to $K(X)$. This homomorphism is denoted by $f^!$. Further, the ring structure of $K_1(X)$ induced by the tensor product of bundles is now also impressed on $K(X)$, and as the authors show, the permanence relation is again valid: $f_!(x \cdot f^!(y)) = f_!(x) \cdot y$ for a map f in $\mathfrak{A}$. This new interpretation of $K(X)$ (i.e., as $K_1(X)$) brings with it also a ring homomorphism $ch\colon K(X) \to A(X) \otimes \mathbf{Q}$ which is natural on the contravariant side (namely, $ch(f^{!*}x) = f^* ch(x)$) and agrees with our definition of ch on $K(p)$. This function is derived from the Chern character of bundles and can be characterized by: (1) If L is a line bundle over $X \in \mathfrak{A}$, then $ch(L) = e^c = 1 + c + c^2/2! + \cdots$, etc., where $c = c_1(L)$ is the class in $A(X)$ of the zeros of a generic rational section of L; (2) ch is a ring homomorphism; (3) the naturality condition already recorded. (See the next review [MR0116023].)

In general, the identification of $K_1(X)$ with $K(X)$ extends the notion of characteristic classes from vector bundles to coherent sheaves. We will, in particular, have need of the Todd-class, which on vector bundles is uniquely characterized by these conditions: (1) If L is a line bundle over an object X in $\mathfrak{A}$, then $T(L) = c/(1 - e^c)$, where $c = c_1(L)$ as defined earlier; (2) T is multiplicative: $T(E+F) = T(E) \cdot T(F)$; (3) $Tf^! = f^*T$ for maps in $\mathfrak{A}$.

This Todd class enters the answer to the following, in our context very natural, question: How does $ch\colon K(X) \to A(X) \otimes \mathbf{Q}$ behave under the covariant homomorphisms $f_!$ and f_*? The answer to this question is precisely the Riemann-Roch formula of Grothendieck: (Riemann-Roch theorem). Let f be a map $X \to Y$, in $\mathfrak{A}$. Then

$$ch\{f_!(x)\} \cdot T(Y) = f_*\{ch(x) \cdot T(X)\},$$

where $x \in K(X)$, and $T(X), T(Y)$ denote the values of the Todd class on the tangent bundles of X and Y respectively.

The Hirzebruch formula is an immediate corollary; just let f be the projection onto a point, and let x be represented by a locally free sheaf $\mathcal{F}$. Then the left-hand side reduces to $\mathfrak{X}(X; \mathcal{F})$ as remarked earlier, while the right-hand side gives the coefficient of $ch(\mathcal{F}) \cdot T(X)$ in the dimension of X, which Hirzebruch denotes by $\kappa_n(ch(\mathcal{F}) \cdot T(X))$.

The great advantage of Grothendieck's formulation is its dynamic nature. This enables one to prove the general theorem by considering special situations. Notably one concludes by the graph-construction that it is sufficient to prove the Riemann-Roch theorem in the following two cases: (a) $f\colon Y \times P \to Y$ is the projection onto Y, P being a projective space; (b) $f\colon Y \to X$ is a closed imbedding. These are then treated by quite different methods. To prove (a), the authors first prove a Künneth type theorem to the effect that $K(X) \otimes K(P) \to K(X \times P)$ is surjective. This fact, together with the Riemann-Roch formula for the projection of P onto a point—which is checked explicitly—proves (a). To establish (b), the authors first treat a special case of Riemann-Roch theorem for Y a divisor on X. This special case is quite simple and at the same time illuminating in that it essentially forces the Todd class upon one, once one seeks a formula for the extent to which ch and $f_!$ fail to commute. Here is the gist of the argument. Assume that $i\colon Y \subset X$ is a regular divisor of X, and let L be the line bundle it determines. Thus $c_1(L) = Y$ and $L|Y$ is the normal bundle of Y in X. We propose to compute both $ch\{i_!(y)\}$ and $i_*\{ch(y)\}$ and see by how much they differ when $y \in K(Y)$ is the class of the structure sheaf O_Y, or, interpreted in $K_1(Y)$, when Y is the class of the trivial bundle 1. Let $S(L^{-1})$ be the sheaf of germs of sections of L^{-1}. Then multiplication with a regular section in L which vanishes on Y gives rise to the exact sequence of sheaves

$$0 \to S(L^{-1}) \to O_X \to \hat{O}_Y \to 0,$$

where $\hat{O}_Y$ is the structure sheaf of Y trivially extended to X. Now one first verifies that $i_!(O_Y)$ is represented by $\hat{O}_Y$. It therefore follows from our exact sequence that $i_!(1) = 1 - L^{-1}$ (using the $K_1(X)$ version of $K(X)$), whence $ch(i_!(1)) = 1 - e^{-Y}$. On the other hand, $ch(1) = 1$, whence $i_*\{ch(1)\} = Y$. So then

$$ch(i_! 1) = i_*\{ch(1)\} \cdot T(L)^{-1} = i_*(T(i^! L)^{-1}),$$

the last step following from the permanence relation. This expression is equivalent to the Riemann-Roch formula with $y = 1$. Indeed if we multiply both sides by $T(X)$, use the permanence again on the right-hand side, and recall that $i^*T(X) = T(Y) \cdot T(i^! L)$ (because $i^! L$ is the normal bundle to Y and T is multiplicative), the above goes into $ch(i_!(1)T(X)) = i_*\{T(Y)\}$, which is just the special Riemann-Roch formula with X and Y reversed. Thus if a formula of the type we are seeking is at all possible, then the correction term will have to satisfy the axioms which were prescribed for T.

To complete the case of an imbedding $Y \subset X$, the authors blow up X along Y to obtain a new object X' in $\mathfrak{A}$, together with a projection $f\colon X' \to X$. The inverse image of Y under f' is then a regular divisor Y' of X', and by a series of ingenious arguments, the Riemann-Roch theorem for f_* is now reduced to the special Riemann-Roch theorem for the injection $Y' \to X'$. The reduction is in a sense the most difficult and certainly the most detailed step in the paper.

This then is a rough plan of the proof, and the methods of Serre [loc. cit.] essentially suffice to carry out the program. There are occasions, however, where the more abstract homological algebra of a previous paper by Grothendieck [Tôhoku Math. J. (2) **9** (1957), 119–221; MR0102537] is useful.

Although the paper pursues its goal relentlessly, it is nevertheless so rich in ideas and auxiliary results which are clearly more generally applicable, that I will not even try to do justice to them.

It seems to me appropriate to close this review with a word of thanks to the authors for presenting us with such an informal and tightly knit account of so many interesting ideas. To my mind, this is the best method of mathematical communication. Also, an account of this sort was especially needed in view of Grothendieck's chilling announcement which puts the topics discussed here into Chapter 12—if we start counting with 1—of his already bulging foundation [Inst. Hautes Études Sci. Publ. Math. No. 4 (1960)].

R. Bott

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MR0139181 (25 #2617) 57.30; 14.52

Atiyah, M. F.; Hirzebruch, F.

Vector bundles and homogeneous spaces.

Proc. Sympos. Pure Math., Vol. III, 7–38, *American Mathematical Society, Providence, R.I.*, 1961.

This paper summarizes some of the authors' researches on the K-theory, with special emphasis on the K-groups of a homogeneous space.

The authors start by defining the functors K^{-n} $(n \geq 0)$ on the category of pairs of finite CW complexes. This is done by setting $K^{-n}(X, Y)$ equal to the homotopy classes of basepoint-preserving maps of the nth suspension of X/Y into $\mathbf{Z} \times B_U$, where B_U is the universal base-space of the infinite unitary group. (When Y is vacuous, X/Y is defined as the disjoint union of X with a point * which plays the role of basepoint.) They then interpret the periodicity format $\Omega^2 B_U \simeq B_U$ as a canonical isomorphism $K^{-n}(X, Y) \overset{\simeq}{\to} K^{-(n+2)}(X, Y)$, and thereby extend the definition of K^n to all integers. Now they observe with the aid of the Puppe sequence that the resulting functors $\{K^n\}$ satisfy all the axioms of a cohomology theory save the dimension axiom. They also define a graded ring structure for the functor $K^{(n)}$ and are finally led to the "abbreviated" functor $(X, Y) \to K^*(X, Y) = K^0(X, Y) + K^{-1}(X, Y)$ from pairs (X, Y) to $\mathbf{Z}_2$-graded rings.

The study of this functor is now based on the following three of its properties: (1) If p is a point, then $K^n(p) = \mathbf{Z}$ (n even), $K^n(p) = 0$ (n odd). (This is a restatement of the corresponding formula for $\pi_n(\mathbf{Z} \times B_U)$.) (2) The usual

Chern-character extends to give a natural transformation of cohomology theories $K^*(X,Y) \rightsquigarrow H^*(X,Y;\mathbf{Q})$ which is a ring homomorphism. (Here the salient fact is that the adjoint of the periodicity map, i.e., the map $j\colon S^2 \times B_U \to B_U$ takes the universal character $\underline{ch} \in H^{**}(B_U;\mathbf{Q})$ into $X \otimes \underline{ch}$ where X generates $H^2(S^2)$.) (3) There is a spectral sequence with E_2-term $H^*(X,Y)$ which converges to a graded group associated to $K^*(X,Y)$. Further, the differential operators in this sequence raise dimension by an odd number. (This theorem may be interpreted as the proper generalization of the Eilenberg-Steenrod uniqueness theorem; a spectral sequence of the type $H^*(X,Y;K^*(p)) \Rightarrow K^*(X,Y)$ exists whenever K^* satisfies all the axioms of Eilenberg-Steenrod, save possibly the dimension axiom.)

As an example of the power of this approach we cite the following immediate corollary of (3): If $H^*(X,\mathbf{Z})$ is free of torsion, then $K^*(X)$ is (unnaturally) isomorphic to $H^*(X)$, $\mathbf{Z}_2$-graded by the even and odd dimensional parts.

The more delicate results announced in this paper depend on the "differentiable Riemann-Roch theorem" of the author [Bull. Amer. Math. Soc. **65** (1959), 276–281; MR0110106]. With the aid of both these tools the authors are able to make considerable progress in their program to prove the K-analogues of the theorems about the ordinary cohomology of homogeneous spaces and the classifying spaces of compact Lie groups.

R. Bott

From MathSciNet, October 2019

MR0338129 (49 #2895) 18F25

Quillen, Daniel

Higher algebraic K-theory. I.

Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972), pp. 85–147. Lecture Notes in Math., Vol. 341, *Springer, Berlin*, 1973.

This article is the foundation paper of higher K-theory. In addition its techniques allow one to apply the methods of homotopy theory to pure algebra whereas formerly this had been possible only with homology and cohomology theory. In this respect the long range impact of this paper may be as great as the fundamental papers of homological algebra of the fifties.

First, the author assigns to every small category $\mathbf{C}$ a CW complex, $B\mathbf{C}$, called its classifying space. The space $B\mathbf{C}$ is just the geometrical realization of the nerve $N\mathbf{C}$ of the category $\mathbf{C}$, where n-simplexes of $N\mathbf{C}$ are given by diagrams $C_0 \xrightarrow{f_1} C_1 \to \cdots \xrightarrow{f_n} C_n$ of $\mathbf{C}$. This construction is quite old, but first appears in print in a paper by G. Segal [Inst. Hautes Études Sci. Publ. Math. No. **34** (1968), 105–112; MR0232393]. If C is an object of $\mathbf{C}$, it corresponds to a vertex of $B\mathbf{C}$, hence one defines the homotopy groups $\pi_n(\mathbf{C},C)$ to be $\pi_n(B\mathbf{C},C)$ and the homology groups $H_n(\mathbf{C}) = H_n(B\mathbf{C})$. For example, if G is a group (i.e., a category with one object, all of whose morphisms are invertible) the BG is a space of type $K(G,1)$, so that $H_n(G)$ is the usual Eilenberg-MacLane integral homology of G.

A functor $f\colon \mathbf{C} \to \mathbf{D}$ between small categories induces a map $Bf\colon B\mathbf{C} \to B\mathbf{D}$, and a natural transformation τ between two functors $f_0, f_1\colon \mathbf{C} \to \mathbf{D}$ gives rise to a homotopy between Bf_0 and Bf_1. Using this, one sees that a pair of adjoint functors, $F\colon \mathbf{C} \to \mathbf{D}$ and $G\colon \mathbf{D} \to \mathbf{C}$, are such that BF and BG are inverse homotopy equivalences. In particular, a category $\mathbf{C}$ having an initial or terminal object is such that $B\mathbf{C}$ is contractible.

The author has developed techniques, which he calls Theorems A and B, for computing $\pi_n(\mathbf{C}, C)$. To state these, we require the notions of the fibres f/D of a functor $f\colon \mathbf{C} \to \mathbf{D}$, where D is an object of $\mathbf{D}$. The category f/D has objects (C, u) where $C \in$ object $\mathbf{C}$ and $u\colon fC \to D$ is an arrow of $\mathbf{D}$. A morphism $(C, u) \to (C', u')$ is an arrow $C \overset{v}{\to} C'$ such that $u' \cdot f(v) = u$. One defines analogously categories $D\backslash f$ whose objects are pairs (C, u), $D \overset{u}{\to} fC$.

Theorem A: If, for each D, $B(f/D)$ is contractible, then $Bf\colon B\mathbf{C} \to B\mathbf{D}$ is a homotopy equivalence.

Theorem B: If for each arrow $D \to D'$ in $\mathbf{D}$ the map $B(f/D) \to B(f/D')$ is a homotopy equivalence, then for any $D_0 \in \operatorname{obj} \mathbf{D}$, the diagram $B(f/D_0) \to B(\mathbf{C}) \overset{Bf}{\to} B(\mathbf{D})$ is a fibre space. That is, the homotopy theoretic fibre of the map Bf is $B(f/D_0)$, where the functor $f/D_0 \to \mathbf{C}$ is given by $(C, u) \mapsto C$. Consequently one has the exact homotopy sequence $\cdots \to \pi_{n+1}(\mathbf{D}, D_0) \to \pi_n(f/D_0, (C_0, 1)) \to \pi_n(\mathbf{C}, C_0) \to \pi_n(\mathbf{D}, D_0) \to \cdots$, where $C_0 \in \operatorname{obj} \mathbf{C}$ is such that $fC_0 = D_0$.

The K-theory exact sequences arise eventually as applications of Theorem B. There are alternative versions of Theorems A and B involving the categories $D\backslash f$. In addition, if the functor $f\colon \mathbf{C} \to \mathbf{D}$ is pre-fibred or precofibred, then one can replace the fibre categories by simpler categories $f^{-1}(D)$ in the statements of theorems A and B.

The author formalizes the notion of a category $\mathbf{P}$ with exact sequences. One may define this to be a full additive subcategory $\mathbf{P}$ of an abelian category $\mathbf{A}$ such that $\mathbf{P}$ is closed under extensions in $\mathbf{A}$, i.e., if $0 \to P' \to A \to P'' \to 0$ is exact in $\mathbf{A}$ with P', P'' in $\mathbf{P}$, then $A \in \operatorname{obj} \mathbf{P}$ also. The author calls $\mathbf{P}$ an exact category and calls a diagram in $\mathbf{P}$ a short exact sequence if it is such in $\mathbf{A}$. The notion of exact category is shown to be intrinsic, admitting a characterization not involving the ambient category $\mathbf{A}$. For a short exact sequence $0 \to P' \overset{i}{\to} P \overset{j}{\to} P'' \to 0$, in $\mathbf{P}$, one calls i an injective arrow and j a surjective arrow.

Next he defines a category $Q\mathbf{P}$ such that obj $Q\mathbf{P} = \operatorname{obj} \mathbf{P}$. The arrows from P to P' in $Q\mathbf{P}$ are isomorphism classes of diagrams $P \overset{j}{\leftarrow} P_1 \overset{i}{\to} P'$ in $\mathbf{P}$, where j is surjective and i is injective. The composition is defined by requiring surjective arrows [resp., injective arrows] to compose as they do in $\mathbf{P}$, while to compose the injective arrow $P \overset{i}{\to} P'$ with the surjective arrow $P' \overset{j}{\leftarrow} P''$, one forms the bicartesian square in $\mathbf{P}$,

$$\begin{array}{ccc} P & \xrightarrow{\ i\ } & P' \\ {\scriptstyle j'}\big\uparrow & & {\scriptstyle j}\big\uparrow \\ P_1 & \xrightarrow{\ i'\ } & P'', \end{array}$$

and defines the composite to be the isomorphism class of the diagram $P \overset{j'}{\leftarrow} P_1 \overset{i'}{\to} P''$. Next, the author establishes Theorem 1: $\pi_1(Q\mathbf{P}, 0) = K_0(\mathbf{P})$, where $K_0(\mathbf{P})$ is the Grothendieck group of $\mathbf{P}$ with relations given by short exact sequences of $\mathbf{P}$. With this as a guide, he defines $K_n(\mathbf{P}) = \pi_{n+1}(Q\mathbf{P}, 0)$, $n \geq 1$.

As an example, if A is a ring and $\mathbf{P}(A)$ is the category of finitely generated projective A-modules, then the author has shown that $\Omega(BQ\mathbf{P}(A) \simeq BGlA^+ \times K_0(A)$, where $BGlA^+$ is the space obtained from $BGl(A)$ by killing $\mathcal{E}(A)$ in such a way as to preserve the integral homology [the author, *New developments in topology* (Proc. Sympos. Algebraic Topology, Oxford, 1972), pp. 95–103, Cambridge Univ. Press, London, 1974; MR0335604]. This result has not appeared in print yet, although an outline of the proof by L. Breen may be found in "Un théorème de finitude en K-théorie (d'après D. Quillen)", Séminaire Bourbaki, 1973/74, Exp. No. 438, Springer, Berlin (to appear). Consequently, $K_n(\mathbf{P}(A)) = K_n(A)$, where $K_n(A) =: pi_n(BGlA^+ \times K_0A)$. This is known to agree with Bass's K_1 and Milnor's K_2 of rings.

There follows a plethora of results about the groups $K_n\mathbf{P}$, of which we quote only two.

Corollary 1 to Theorem 2: Let $\mathbf{M}'$ and $\mathbf{M}''$ be exact categories and let $0 \to F' \to F \to F'' \to 0$ be an exact sequence of exact functors from $\mathbf{M}' \to \mathbf{M}''$; then $F_* = F_*{}' + F_*{}''\colon K_i(\mathbf{M}') \to K_i(\mathbf{M}'')$.

Corollary 1 to Theorem 3: Let $\mathbf{P}$ be an exact subcategory of the exact category $\mathbf{M}$ and assume that $\mathbf{P}$ is closed under extension in $\mathbf{M}$; Assume further that (a) for every exact sequence $0 \to \mathbf{M}' \to \mathbf{M} \to \mathbf{M}'' \to 0$ in $\mathbf{M}$ with M, M'' in $\mathbf{P}$, then M' is in $\mathbf{P}$ too, (b) given $j\colon M \to P$ surjective, there exists a surjective arrow $j'\colon P' \to P$ and an arrow $f\colon P' \to M$ in $\mathbf{M}$ such that $jf = j'$; let $\mathbf{P}_n$ be the full subcategory of $\mathbf{M}$ consisting of M having $\mathbf{P}$ resolutions of length $\leq n$ (i.e., there exists an exact sequence $0 \to P_n \to \cdots \to P_0 \to M \to 0$ with $P_i \in \operatorname{obj}\mathbf{P}$), and put $\mathbf{P}_\infty = \bigcup \mathbf{P}_n$; then $K_i(\mathbf{P}) \overset{\cong}{\to} K_i(\mathbf{P}_1) \overset{\cong}{\to} \cdots \overset{\cong}{\to} K_i(\mathbf{P}_\infty)$.

As an application of this result, the author proves the Corollary 2 to Theorem 3: If A is a regular ring, then $K_i(A) = K_i(\operatorname{Mod} f(A))$, where $\operatorname{Mod} f(A)$ is the category of finitely generated A modules (a regular ring is a left Noetherian ring such that every finitely generated left module has finite projective dimension).

If $f\colon A \to B$ is a ring homomorphism such that the A-module B has a finite resolution by finitely generated projectives, then the author defines the transfer map $f_*\colon K_i(B) \to K_i(A)$. If in addition A and B are commutative and $f^* = (B \otimes_A (\cdot))^*\colon K_i(A) \to K_i(B)$, then he proves the projection formula $f_*(f^*x \cdot y) = x \cdot f_*(y)$ for $x \in K_0(A)$ and $y \in K_i(B)$.

There follow two results about the K-theory of Abelian categories. Theorem 4: (Dévissage) Suppose that $\mathbf{B}$ is a nonempty full subcategory of $\mathbf{A}$, closed under taking subobjects, quotient objects, and finite products in $\mathbf{A}$; suppose that every object M of $\mathbf{A}$ has a finite filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$ such that $M_j/M_{j-1} \in \operatorname{obj}\mathbf{B}$ for each j; then $K_i(\mathbf{B}) \overset{\cong}{\to} K_i(\mathbf{A})$.

Theorem 5: (Localization): Let $\mathbf{B}$ be a Serre subcategory of $\mathbf{A}$ (so that, in addition to the hypotheses of Theorem 4, $\mathbf{B}$ is closed under extensions in $\mathbf{A}$); then there is a long exact sequence

$$\cdots \to K_{n+1}(\mathbf{A}/\mathbf{B}) \to K_n(\mathbf{B}) \to K_n(\mathbf{A}) \to K_n(\mathbf{A}/\mathbf{B}) \to \cdots$$

A corollary of these results is the exact sequence $\cdots \to K_{n+1}(F) \to \coprod_{\mathbf{m}} K_n(A/\mathbf{m}) \to K_n(A) \to K_n(F) \to \cdots$, where A is a Dedekind ring with field of fractions F, and $\mathbf{m}$ runs over the maximal ideals of A.

The remainder of the article is devoted to applications to ring theory and to algebraic geometry. The main result of § 6 is Theorem 8: If A is a Noetherian ring, then there are canonical isomorphisms (i) $K_i{}'(A[t]) \cong K_i{}'(A)$, (ii) $K_i{}'(A[t, t^{-1}]) \cong K_i{}'(A) \oplus K'_{i-1}(A)$. Here

$$K_i{}'(A) =: K_i(\operatorname{Mod} f(A)).$$

Corollary (Fundamental theorem for regular rings): If A is regular, then $K_i(A[t]) = K_i(A)$ and $K_i(A[t,t^{-1}]) = K_i(A) \oplus K_{i-1}(A)$.

Section 7 is devoted to the foundation of K-theory of schemes. For a scheme X one sets $K_q(X) = K_q(\mathbf{P}(X))$, where $\mathbf{P}(X)$ is the category of vector bundles on X. If X is a Noetherian scheme, one sets $K_q{}'(X) = K_q(\mathbf{M}(X))$, where $\mathbf{M}(X)$ is the abelian category of coherent sheaves of $\mathcal{O}_X$-modules. If X is regular, then $K_q(X) = K_q{}'(X)$. In general K_q is a contravariant functor whereas $K_q{}'$ is a contravariant functor for flat morphisms of schemes. The author proves a projection formula (§ 7 Proposition 2.10): Suppose that $f\colon X \to Y$ is proper and of finite Tor dimension; assume that X and Y have ample line bundles; then for $x \in K_0(X)$ and $y \in K_q{}'(\gamma)$ one has $f_*(x \cdot f^*y) = f_*(x) \cdot y \in K_q{}'(y)$, where $f_*(x) \in K_0(\gamma)$.

If Z is a closed subscheme of X and $U = X - Z$, then (Proposition 3.2) there is a long exact sequence $\cdots \to K'_{q+1}(U) \to K_q{}'(Z) \to K_q{}'(X) \to K_q{}'(U) \to \cdots$, from which follows the Mayer-Vietoris sequence $\cdots \to K'_{q+1}(U \cap V) \to K_q{}'(U \cup V) \to K_q{}'(U) \oplus K_q{}'(V) \to K_q(U \cap V) \to \cdots$ for open sets U, V in X.

Next a homotopy property (Proposition 4.1): Let $f\colon P \to X$ be a flat map whose fibres are affine spaces; then $f^*\colon K_q{}'(X) \to K_q{}'(P)$ is an isomorphism.

The author considers the filtration of $\mathbf{M}(X)$ by Serre subcategories $\mathbf{M}_p(X)$ of coherent sheaves whose support is of codimension $\geq p$. If $x \in X$, denote $k(x)$ the residue class field at x. Let X_p be the set of points $x \in X$ such that the Krull dimension of $\mathcal{O}_{X,x}$ is p.

Theorem 5.4: There is a spectral sequence $E_1{}^{p,q}(X) = \coprod_{x \in X_p} K_{-p-q}(k(x)) \to K_{-n}{}'(X)$ that is convergent if X has finite Krull dimension. The spectral sequence is contravariant for flat morphisms. Following this the author states the conjectures of the reviewer [*Algebraic K-theory, I: Higher K-theories* (Proc. Conf. Seattle Res. Center, Battelle Memorial Inst., 1972), pp. 211–243, Lecture Notes in Math., Vol. 341, Springer, Berlin, 1973] If X is the spectrum of a regular local ring, then the sequence (given by differentials in the E_1-term of the spectral sequence) (5.9) $0 \to K_n{}'(X) \to \coprod_{x \in X_0} K_n(k(x)) \to \coprod_{x \in X_1} K_{n-1}(k(x)) \to \cdots \to \coprod_{x \in X_n} K_0(k(x)) \to 0$ is exact. The author proves Theorem 5.11: Let R be an algebra of finite type over a field and let S be a finite set of primes of R such that R_p is regular for each $p \in S$; let A be the regular semi-local ring obtained by localizing R with respect to S; then (5.9) is exact for $X = \operatorname{Spec} A$.

Next he identifies a differential in the spectral sequence Proposition 5.14: Let X be a regular scheme of finite type over a field; then the image of $d_1\colon \coprod_{x \in X_{p-1}} K_1(k(x)) \to \coprod_{x \in X_p} K_0(k(x)) = \coprod_{x \in X_p} \mathbf{Z}$ is the subgroup of codimension p cycles which are linearly equivalent to zero. Consequently $E_2{}^{p,-p}(X)$ is canonically isomorphic to the group $A^p(X)$ of cycles in codimension p modulo linear equivalence. The proof involves a formula of C. Chevalley [*Séminaire C. Chevalley, 2e année: 1958: Anneaux de Chow et applications*, Exp 2, Secrétariat mathématique, Paris, 1958; MR0110704] for the multiplicity of a given divisor in the divisor of a rational function. From the work that precedes, the author deduces Theorem 5.19: For a regular scheme X of finite type over a field, there is a canonical isomorphism $H^p(X, \mathbf{K}_p) = A^p(X)$. Here $\mathbf{K}_p$ is the shear on the Zariski topology of X associated to the presheaf $U \to K_p(U)$.

In § 8 the author generalizes the projective bundle theorem for K_0 [*Théorie des intersections et théorème de Riemann-Roch* (Séminaire de Géométrie Algébrique du Bois-Marie, 1966/1967, SGA6), Exp. VI, p. 365, Théorème 1.1, Lecture Notes in Math., Vol. 225, Springer, Berlin, 1971]. The main result is § 8 Theorem 2.1: Let E be a vector bundle of rank r over a scheme S and let $X = \mathrm{Proj}(SE)$, the associated projective scheme; if S is quasi-compact, then one has isomorphisms $K_q(S)^r \overset{\cong}{\to} K_q(x)$, $(a_i)_{0\le i<r} \mapsto \sum_{i=0}^{r-1} z^i \cdot f^*(a_i)$, where $z \in K_0(X)$ is the class of the canonical bundle $\mathcal{O}_X(-1)$ and $f\colon X \to S$ is the structure map. There follows a computation of the K-theory of the projective line generalizing a result of H. Bass for K_0 [*Algebraic K-theory,* Chapter XII, § 9, Benjamin, New York, 1968; MR0249491].

Finally, in § 9, there is an application to Severi-Brauer schemes and Azumaya algebras.

{For more complete bibliographic information about the collection in which this article appears, including the table of contents, see MR0325307; MR0325308; MR0325309.}

Stephen M. Gersten
From MathSciNet, October 2019

MR0675529 (84i:12007) 12A62; 12G05, 14C35, 14F12, 14F15, 14G25, 18F25
Merkur′ev, A. S.; Suslin, A. A.
K-cohomology of Severi-Brauer varieties and the norm residue homomorphism.
Izvestiya Akademii Nauk SSSR. Seriya Matematicheskaya **46** (1982), *no.* 5, 1011–1046, 1135–1136.

In this outstanding paper the authors prove that the residue norm homomorphism $R_{F,n}\colon K_2(F)/nK_2(F) \to H^2(F,\mu_n^{\otimes 2})$ is an isomorphism, where F is any field of characteristic coprime to n. In the case when F contains ζ, a primitive nth root of unity, $R_{F,n}$ is defined by the association $\{a,b\} \to A_\zeta(a,b)$, where $\{a,b\}$ is a symbol and $A_\zeta(a,b)$ is a cyclic algebra (a central simple F-algebra which is split by a cyclic extension of F of degree n) generated by x,y with the relations $x^n = a$, $y^n = b$, $xy = \zeta yx$. (Note that $H^2(F,\mu_n^{\otimes 2}) \simeq H^2(F,\mu_n) \simeq_n \mathrm{Br}(F)$.) The proof of the above-mentioned result is based on very clever calculations of K-cohomology of Brauer-Severi varieties. Gersten's spectral sequence, K-cohomology Chern classes, and "K-theoretic" Riemann-Roch due to Shekhtman and Gillet are among the variety of tools used by the authors.

It follows from the main result of the paper under review that any central simple F-algebra of exponent n is similar to a product of cyclic algebras of exponent n in the case when $\mu_n \subset F$. In particular, every such algebra has an abelian splitting field. The authors also prove the K_2-analogue of Hilbert's Theorem 90. Namely, let E/F be a cyclic extension of degree n coprime to $\mathrm{char}\, F$ and let σ be a generator of $\mathrm{Gal}\, E/F$. Then $\ker(K_2(E) \overset{N_{E/F}}{\to} K_2(F)) = K_2(E)^{1-\sigma}$. (Note that according to Hilbert's Theorem 90 the equality above holds with K_2 replaced by K_1.) It follows that in the case when $\mu_n \subset F$, ${}_nK_2(F) = \{\mu_n, F^*\}$. The last chapter of the paper presents refinements of Bloch's results on codimension-two cycles.

{English translation: Math. USSR-Izv. **21** (1983), no. 2, 307–340.}

Maksymilian Boratyński
From MathSciNet, October 2019

MR1376246 (97e:14030) 14F99; 14C25, 14F20, 19E15

Suslin, Andrei; Voevodsky, Vladimir

Singular homology of abstract algebraic varieties.

Inventiones Mathematicae **123** (1996), *no.* 1, 61–94.

In one of his most influential papers, A. Weil [Bull. Amer. Math. Soc. **55** (1949), 497–508; MR0029393] proved the "Riemann hypothesis for curves over functions fields", an analogue in positive characteristic algebraic geometry of the classical Riemann hypothesis. In contemplating the generalization of this theorem to higher-dimensional varieties (subsequently proved by P. Deligne [Inst. Hautes Études Sci. Publ. Math. No. 43 (1974), 273–307; MR0340258] following foundational work of A. Grothendieck), Weil recognized the importance of constructing a cohomology theory with good properties. One of these properties is functoriality with respect to morphisms of varieties. J.-P. Serre showed with simple examples that no such functorial theory exists for abstract algebraic varieties which reflects the usual (singular) integral cohomology of spaces. Nevertheless, Grothendieck together with M. Artin [*Théorie des topos et cohomologie étale des schémas. Tome 1*, Lecture Notes in Math., 269, Springer, Berlin, 1972; MR0354652; *Tome 2*, Lecture Notes in Math., 270, 1972; MR0354653; *Tome 3*, Lecture Notes in Math., 305, 1973; MR0354654] developed étale cohomology which succeeds in providing a suitable Weil cohomology theory provided one considers cohomology with finite coefficients (relatively prime to residue characteristics). This theory, presented in J. Milne's book [*Étale cohomology*, Princeton Univ. Press, Princeton, N.J., 1980; MR0559531], relies on a new formulation of topology and sophisticated developments in sheaf theory.

In the present paper, the authors offer a very different solution to the problem of providing an algebraic formulation of singular cohomology with finite coefficients. Indeed, their construction is the algebraic analogue of the topological construction of singular cohomology [see, e.g., E. H. Spanier, *Algebraic topology*, McGraw-Hill, New York, 1966; MR0210112], thereby being much more conceptual. Their algebraic singular cohomology with (constant) finite coefficients equals étale cohomology for varieties over an algebraically closed field. The proof of this remarkable fact involves new topologies, new techniques, and new computations reminiscent of the earlier work of Artin and Grothendieck.

To understand the authors' construction, we recall the classical theorem of A. Dold and R. Thom [Ann. of Math. (2) **67** (1958), 239–281; MR0097062]. This asserts that the singular homology of a CW complex X is naturally isomorphic to the homotopy groups of the simplicial abelian group $(\mathrm{Sing}_.(\coprod_{d\geq 0} S^dX))^+$, the group completion of the singular complex of the topological abelian monoid $\coprod_{d\geq 0} S^dX$. Now, if X is an algebraic variety, so are its symmetric products. Moreover, homotopy groups of the simplicial abelian group $(\mathrm{Sing}_.(\coprod_{d\geq 0} S^dX))^+$ can be computed as the homology of the associated chain complex, which we denote by $(\mathrm{Sing}_.(\coprod_{d\geq 0} S^dX))^\sim$. The construction of Suslin-Voevodsky, first proposed by Suslin in a talk in Luminy in 1987, is to replace the singular complex by its algebraic analogue. Algebraic singular simplices were exploited years ago by M. Karoubi and O. Villamayor [C. R. Acad. Sci. Paris Sér. A-B **269** (1969), A416–A419; MR0251717] and more recently used by S. Bloch in his formulation of higher Chow groups [Adv. in Math. **61** (1986), no. 3, 267–304; MR0852815].

The fundamental theorem of Suslin-Voevodsky is that if X is an algebraic scheme of finite type over an algebraically closed field k of characteristic $p \geq 0$ and if n

is an integer prime to p, then the étale cohomology of X with $\mathbf{Z}/n$ coefficients can be computed as $\mathrm{Ext}^*((\mathrm{Sing}^{\mathrm{alg}}_{\cdot}(\coprod_{d\geq 0} S^d X))^+, \mathbf{Z}/n)$. (The published statement assumes that the ground field k is of characteristic 0; as the authors soon realized, recent work of J. de Jong giving a weak version of resolution of singularities for varieties over fields of positive characteristic enables this extension to arbitrary characteristic.) Although the formulation of this theorem is relatively elementary, its proof involves sophisticated techniques of abstract algebraic geometry as well as insights from algebraic K-theory. Indeed, the authors encountered this theorem as a part of a sweeping approach to motivic cohomology and algebraic K-theory [see, e.g., V. Voevodsky, A. Suslin and E. Friedlander, *Cycles, transfers, and motivic homology theories*, Ann. of Math. Stud., to appear]. Underlying the authors' approach to (motivic) cohomology is the utilization of algebraic cycles. Maps from a normal variety S (e.g., a standard algebraic simplex Δ^k) to a symmetric product of X correspond to cycles on $S \times X$ finite and surjective over X.

The geometric heart of the proof is the authors' determination of the relative algebraic singular homology of a relative curve in terms of the relative Picard group, just as a key first ingredient for étale cohomology is the understanding of the étale cohomology of curves. This computation leads to a general form of the rigidity theorem of O. Gabber [in *Algebraic K-theory, commutative algebra, and algebraic geometry (Santa Margherita Ligure, 1989)*, 59–70, Contemp. Math., 126, Amer. Math. Soc., Providence, RI, 1992; MR1156502] and H. A. Gillet and R. W. Thomason [J. Pure Appl. Algebra **34** (1984), no. 2-3, 241–254; MR0772059] which played a key role in Suslin's proof of the Quillen-Lichtenbaum conjecture for an arbitrary algebraically closed field [A. A. Suslin, J. Pure Appl. Algebra **34** (1984), no. 2-3, 301–318; MR0772065]. Namely, the authors consider homotopy invariant presheaves with transfers, a basic structure which now plays a central role in their approach to motivic cohomology. The example of most interest for the present work is the "free sheaf generated by X", whose values on standard algebraic simplices determine the chain complex $(\mathrm{Sing}^{\mathrm{alg}}_{\cdot}(\coprod_{d\geq 0} S^d X))^{\sim}$. This example fits their general context of presheaf with transfers thanks to the theorem that any "qfh-sheaf" admits the structure of a presheaf with transfers.

An essential ingredient in the authors' approach to cohomology is a further generalization of the étale topology in which proper maps arising in resolutions of singularities occur as coverings. Voevodsky's "h-topology" and its quasi-finite version leading to qfh-sheaves [cf. Selecta Math. (N.S.) **2** (1996), no. 1, 111–153] play an important role. The authors' rigidity theorem asserts the equality of various Ext-groups from sheaves associated to a homotopy invariant presheaf F with transfers to $\mathbf{Z}/n$, where these Ext-groups are computed in the étale topology and various topologies associated to the h-topology. Much of the formal effort in establishing their comparison theorems consists in analyses and manipulations of resolutions of sheaves for these topologies.

Eric M. Friedlander

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MR1427618 (98h:14055a) 14L15; 20C20, 20G10
Friedlander, Eric M.; Suslin, Andrei
Cohomology of finite group schemes over a field.
Inventiones Mathematicae **127** (1997), *no.* 2, 209–270.

MR1443546 (98h:14055b) 14L15; 20C20, 20G10
Suslin, Andrei; Friedlander, Eric M.; Bendel, Christopher P.
Infinitesimal 1-parameter subgroups and cohomology.
Journal of the American Mathematical Society **10** (1997), *no.* 3, 693–728.

MR1443547 (98h:14055c) 14L15; 20C20, 20G10
Suslin, Andrei; Friedlander, Eric M.; Bendel, Christopher P.
Support varieties for infinitesimal group schemes.
Journal of the American Mathematical Society **10** (1997), *no.* 3, 729–759.

Many major advances in the last two decades in modular representation theory have been made through the investigation and use of the cohomological spectrum. For finite groups, L. Evens [Trans. Amer. Math. Soc. **101** (1961), 224–239; MR0137742] proved that the cohomology ring is Noetherian. Subsequent results by D. Quillen followed [Ann. of Math. (2) **94** (1971), 549–572; ibid. (2) **94** (1971), 573–602; MR0298694] on the structure of the spectrum of this cohomology ring. In 1977, J. Alperin defined the important notion of the complexity of a module by looking at the rate of growth of the minimal projective resolution of the given module. Major progress in this area during this same time period can be attributed to J. F. Carlson [J. Algebra **85** (1983), no. 1, 104–143; MR0723070], who defined a cohomological variety for a given module whose dimension equals the complexity of the module. These varieties are often referred to as support varieties. A stratification theorem providing some understanding about the structure of these support varieties (generalizing the prior work of Quillen) was proved by G. S. Avrunin and L. L. Scott [Invent. Math. **66** (1982), no. 2, 277–286; MR0656624].

In a series of papers [see, e.g., Invent. Math. **86** (1986), no. 3, 553–562; MR0860682], Friedlander and B. J. Parshall extended the theory of support varieties for restricted Lie algebras. In this setting they proved that the cohomology ring is finitely generated, and this result together with work of J. C. Jantzen [Abh. Math. Sem. Univ. Hamburg **56** (1986), 191–219; MR0882415] provided a realization of these varieties as a subvariety of the given Lie algebra. For finite-dimensional graded connected cocommutative Hopf algebras, finite generation of the cohomology ring was proved by C. Wilkerson [Trans. Amer. Math. Soc. **264** (1981), no. 1, 137–150; MR0597872], and in later work by A. M. Bajer and H. Sadofsky [J. Pure Appl. Algebra **94** (1994), no. 2, 115–126; MR1282834]. These graded Hopf algebras arise naturally when one looks at finite-dimensional Hopf subalgebras of the Steenrod algebra. A theory of support varieties was developed within this context, similar to the one for finite groups, by Nakano and J. H. Palmieri ["Support varieties for the Steenrod algebra", Math. Z., to appear]. Aside from these specific examples, it was not known in general whether the cohomology ring of an arbitrary finite-dimensional cocommutative Hopf algebra is finitely generated and whether a theory of supports could be developed. For many years the general question of finite generation remained as an open, difficult and elusive problem. The three papers under review provide both important and exciting breakthroughs in this direction.

In the first paper, the authors prove the finite generation of cohomology for a finite group scheme (or equivalently a finite-dimensional cocommutative Hopf algebra). The proof given by the authors involves first embedding a finite group scheme G into some general linear group GL_n and then constructing universal extension classes in certain degrees. These extensions in essence provide the generators for the cohomology ring. An interesting twist on this theme is the use of the polynomial functors developed by V. Franjou, J. Lannes and L. Schwartz in order to construct such extensions. The theory of such functors is related to the study of modules for the classical Schur algebras. Other interesting results in this paper involve computation of the cohomology of the infinite general linear group over a finite field.

With finite generation of cohomology firmly established, Friedlander, Suslin and Bendel are now armed to investigate the spectrum of the cohomology ring for infinitesimal group schemes in the second paper. The authors work in the context of schemes rather than varieties. The first step is to construct a scheme $V_r(G)$ consisting of homomorphisms from one-dimensional additive subgroups to the given group scheme G. This scheme in the case when the height equals one is the well-known scheme of p-nilpotent elements in $\mathrm{Lie}(G)$. More generally, $V_r(G)$ can be identified with r-tuples of p-nilpotent, pairwise commuting matrices of GL_n given a closed embedding of G into GL_n. This object was first defined and studied by D. Gross. The authors proceed to show that there exists a natural homomorphism of rings $\Phi\colon H^{2*}(G,k) \to k[V_r(G)]$, where $k[V_r(G)]$ is the coordinate algebra of $V_r(G)$. Much of the paper is devoted to determining characteristic classes associated to one-parameter subgroups of GL_n. This information is valuable because it enables the authors to give a concrete interpretation of the composition of $V_r(G) \to \mathrm{Spec}(H^{2*}(G,k)) \to \mathrm{gl}_n^{(r)\times r}$, where the second composition is the map induced by a natural map obtained by looking at the universal classes defined in the first paper.

The third paper involves using the information obtained in the second paper to give a concrete realization of support varieties for infinitesimal group schemes. A key ingredient to the results is the use of embeddings of one-parameter subgroups into G to show that cohomology classes are detected modulo nilpotence on these subgroups. One of the main results which uses this detection principle is that the map Φ is indeed an inseparable isogeny. This allows the authors to give an explicit non-cohomological realization of the support variety of a rational G-module as a certain subscheme of $V_r(G)$. At the end of this paper several applications, including some generalizations of known results in the group algebra case, are given for infinitesimal group schemes. Explicit calculations of support varieties for simple and induced modules for Frobenius kernels of SL_2 are also included.

These three papers serve as an important contribution to the understanding of the cohomology and representation theory of finite group schemes. The authors manage to answer difficult questions in a very elegant manner. The advent of these new and beautiful ideas makes one optimistic that more interesting results in this direction will arise in the not too distant future.

Daniel K. Nakano

From MathSciNet, October 2019

MR1744945 (2001g:14031) 14F42; 19D45, 19E15, 19E20

Suslin, Andrei; Voevodsky, Vladimir

Bloch-Kato conjecture and motivic cohomology with finite coefficients.

The arithmetic and geometry of algebraic cycles (Banff, AB, 1998), 117–189, *NATO Sci. Ser. C Math. Phys. Sci., 548, Kluwer Acad. Publ., Dordrecht*, 2000.

Let F be a field, m an integer prime to the characteristic of F, and Sm/F be the category of smooth schemes over F. The conjecture of Beilinson and Lichtenbaum for weight n states that the natural map $\mathbf{Z}/m(n) \to \tau_{\leq n}\mathbb{R}\alpha_*\mu_m^{\otimes n}$ is a quasi-isomorphism. Here $\mathbf{Z}/m(n)$ is the mod m motivic complex, and $\alpha\colon (\mathrm{Sm}/F)_{\text{ét}} \to (\mathrm{Sm}/F)_{\mathrm{Zar}}$ is the natural map. The main result of the paper is that the Bloch-Kato conjecture, i.e. the surjectivity of the norm residue homomorphism from Milnor K-theory to Galois cohomology $K_n^M(E)/m \to H^n(E, \mu_m^{\otimes n})$, is equivalent to the conjecture of Beilinson-Lichtenbaum. More precisely, assume that resolution of singularities holds over F and that the norm residue homomorphism in degree n is surjective for any extension E/F. Then the Beilinson-Lichtenbaum conjecture holds over F in weights at most n.

The authors start by reviewing the construction of motivic cohomology and the derived category of mixed motives [see V. Voevodsky, A. A. Suslin and E. M. Friedlander, *Cycles, transfers, and motivic homology theories*, Ann. of Math. Stud., 143, Princeton Univ. Press, Princeton, NJ, 2000; MR1764197]. The category $DM^-(F)$ is the full subcategory of the derived category of bounded above complexes of Nisnevich sheaves with transfers and homotopy invariant cohomology sheaves. The motivic complex $\mathbf{Z}(n)$ is a specific object of this category, and motivic cohomology of a variety X over F is defined to be $H^i_M(X, \mathbf{Z}(n)) = \mathrm{Hom}_{DM^-(F)}(M(X), \mathbf{Z}(n))$, where $M(X)$ is an object of $DM^-(F)$ naturally associated to X. This is in fact isomorphic to the hypercohomology in the cdh-topology of $\mathbf{Z}(n)$ on X, and if X is smooth over F, then it suffices to take the Nisnevich topology.

Several properties of these motivic cohomology groups are given, for example a Mayer-Vietoris exact sequence for open covers, a projective bundle formula, and a blow-up exact sequence. The natural isomorphism of graded rings $\bigoplus_n K_n^M(E) \cong \bigoplus_n H^n_M(E, \mathbf{Z}(n))$ gives an interpretation of the Bloch-Kato conjecture in terms of motivic cohomology.

In the second half of the paper the actual proof takes place. The theorem is easily reduced to the case of a field, and only injectivity is hard. The main idea is to use the motivic cohomology of the boundary of the r-simplex $\partial\Delta^r$ (a singular scheme) to shift degrees. More precisely, if S is the affine line $\mathbf{A}^1_E$ with the points 0 and 1 identified to the point p, then the map $H^i_M(E, \mathbf{Z}/m(n)) \to H^i(E, \mu_m^{\otimes n})$ is a direct summand of the map

$$H^{n+1}_M(\partial\Delta^{n-i+1}_E \times S, \mathbf{Z}/m(n)) \to H^{n+1}(\partial\Delta^{n-i+1}_E \times S, \mu^{\otimes n}).$$

Every class coming from $H^i_M(E, \mathbf{Z}/m(n))$ vanishes in some neighborhood U of the vertices, hence comes from the motivic cohomology with supports

$$H^{n+1}_T(\partial\Delta^{n-i+1}_E \times S, \mathbf{Z}/m(n)),$$

for T equal to $\partial\Delta^{n-i+1}_E \times S - U$. Finally, by purity and induction, the map is injective on the latter group.

Thomas Geisser

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Voevodsky, Vladimir

Reduced power operations in motivic cohomology.

Publications Mathématiques. Institut de Hautes Études Scientifique (2003), *no.* 98, 1–57.

MR2031199 (2005b:14038b) 14F42; 12G05, 19D45, 19E15

Voevodsky, Vladimir

Motivic cohomology with Z/2-coefficients.

Publications Mathématiques. Institut de Hautes Études Scientifiques (2003), *no.* 98, 59–104.

The papers under review present Voevodsky's proof of the "Milnor conjecture", a remarkable achievement which marks the culmination of Voevodsky's program to extend Grothendieck's constructions of new "topologies", incorporate the philosophy of motives, and integrate into abstract algebraic geometry important techniques of homotopy theory. Voevodsky's work has inspired considerable further work by algebraic geometers and algebraic topologists, and holds great promise for dramatic new geometric results.

The fundamental theorem of Voevodsky states that if k is a field of characteristic different from 2 then the Galois cohomology groups $H^i(k,\mathbb{Z}/2)$ are generated by classes in $H^1(k,\mathbb{Z}/2)$. More precisely, J. Milnor [Invent. Math. **9** (1969/1970), 318–344; MR0260844] conjectured that the norm residue symbol determines an isomorphism

$$(1) \qquad K_*^M(k)\otimes\mathbb{Z}/2 \overset{\sim}{\longrightarrow} H^*(k,\mathbb{Z}/2),$$

where $K_*^M(k)$ is the Milnor K-theory of the field k defined as the quotient of the tensor algebra on the multiplicative group k^* by the ideal generated by elements of the form $a\otimes b$ with $a,b\in k^*,\ a+b=1$. Indeed, there is a conjectural generalization formulated by K. Kato [J. Fac. Sci. Univ. Tokyo Sect. IA Math. **27** (1980), no. 3, 603–683; MR0603953] (the so-called Bloch-Kato conjecture) of this Milnor conjecture (1) applicable to any prime number l which asserts that the norm residue homomorphism determines an isomorphism

$$(2) \qquad K_*^M(k)\otimes\mathbb{Z}/l \overset{\sim}{\longrightarrow} H^*(k,\mathbb{Z}/\mu_l^{\otimes *})$$

for any prime l. Voevodsky has written these two papers so that whenever possible the results are proved for all primes. In a forthcoming paper with D. Orlov and A. Vishik ["An exact sequence for Milnor's K-theory with applications to quadratic forms", preprint, `arxiv.org/abs/math/0101023`], Voevodsky uses his proof of the Milnor conjecture to prove a companion conjecture of Milnor's [op. cit.] relating $K_*^M(k)\otimes\mathbb{Z}/2$ to the sections of the natural filtration of the Witt ring of quadratic forms over k.

The first of these papers verifies existence and important properties of cohomology operations in motivic cohomology. Earlier work of Voevodsky and joint work of Voevodsky, A. A. Suslin and E. M. Friedlander [*Cycles, transfers, and motivic homology theories*, Ann. of Math. Stud., 143, Princeton Univ. Press, Princeton, NJ, 2000; MR1764197] established many good properties of motivic cohomology, and Voevodsky has shown [Int. Math. Res. Not. **2002**, no. 7, 351–355; MR1883180] that the motivic cohomology groups agree with the higher Chow groups of S. J. Bloch

[Adv. in Math. **61** (1986), no. 3, 267–304; MR0852815]. The relationship of motivic cohomology to algebraic K-theory for smooth varieties over a field closely parallels the relationship of singular cohomology to (complex) topological K-theory of a topological space. Indeed, Voevodsky's results are results about motivic cohomology, and these results translate directly into results concerning algebraic K-theory with finite coefficients [see, for example, E. M. Friedlander and A. A. Suslin, Ann. Sci. École Norm. Sup. (4) **35** (2002), no. 6, 773–875; MR1949356].

The construction and essential properties of these cohomology operations are challenging to verify. The proofs are much more than a mere translation of corresponding results in algebraic topology. In the first paper, Voevodsky establishes results needed for the proof of the Milnor conjecture: construction of the Steenrod p-th power operations P^i, their relationship to the Bockstein operation, the Cartan formula, and the Adem relations. These are established in the context of the pointed motivic homotopy category $H_\bullet(k)$ considered by F. Morel and Voevodsky [Inst. Hautes Études Sci. Publ. Math. No. 90 (1999), 45–143 (2001); MR1813224]. Indeed, this extension of the category of k-varieties is essential for the formulation as well as proof of many results (e.g., Thom isomorphism and suspension isomorphism, as well as the representability of motivic cohomology by "Eilenberg-Mac Lane objects"). Other properties (uniqueness of P^i; the identification of the ring of all stable cohomology operations) not needed for Voevodsky's proof of the Milnor conjecture as presented in the second paper are not proved here.

The second paper provides Voevodsky's proof of the Milnor conjecture, referring freely to earlier papers by Voevodsky, M. Rost, and Suslin and Voevodsky for important subsidiary results as well as to the preceding paper on motivic cohomology operations. In some sense, one can view this paper as presenting the "master plan", with details to be found elsewhere. For example, no reference is given to the fact that Milnor K-group $K_M^n(k)\otimes\mathbb{Z}/l$ of a field k can be viewed as the (Zariski) mod-l motivic cohomology $H^n_{\mathrm{Zar}}(k,\mathbb{Z}/l(n))$ of k, and only a brief sketch is given of the fact that the Galois cohomology $H^n(k,\mu_l^{\otimes n})$ can be viewed as the (étale) mod-l motivic cohomology $H^n_{\text{ét}}(k,\mathbb{Z}/l(n))$ of k. The proof of the cohomological interpretation of $K_M^n(k)\otimes\mathbb{Z}/l$ was given by Bloch in [op. cit.] (in the context of his higher Chow groups); a proof of the second is outlined by Voevodsky with a reference to [V. Voevodsky, C. Mazza and C. Weibel, "Lectures on motivic cohomology. I", `math.rutgers.edu/~weibel/motiviclectures.html`] for a detailed proof. Voevodsky proves, as conjectured by A. A. Beilinson [in *K-theory, arithmetic and geometry (Moscow, 1984–1986)*, 1–25, Lecture Notes in Math., 1289, Springer, Berlin, 1987; MR0923131] and S. Lichtenbaum [in *Number theory, Noordwijkerhout 1983 (Noordwijkerhout, 1983)*, 127–138, Lecture Notes in Math., 1068, Springer, Berlin, 1984; see MR 85i:11001 MR0756089], that the natural map determines an isomorphism

$$H^p_{\mathrm{Zar}}(k,\mathbb{Z}/l(q)) \xrightarrow{\sim} H^p_{\text{ét}}(k,\mathbb{Z}/l(q)),\quad p\le q, \tag{3}$$

for $l=2$; in particular, he affirms the Milnor conjecture (1). As shown earlier by Suslin and Voevodsky [in *The arithmetic and geometry of algebraic cycles (Banff, AB, 1998)*, 117–189, Kluwer Acad. Publ., Dordrecht, 2000; MR1744945] and then extended by T. Geisser and M. N. Levine [J. Reine Angew. Math. **530** (2001), 55–103; MR1807268], one need only prove the surjectivity assertion of the Bloch-Kato conjecture (2) to conclude via an inductive argument the Beilinson-Lichtenbaum isomorphism (3).

Voevodsky's effort is dedicated to proving a higher-order version of the "Hilbert theorem 90". In Voevodsky's terminology, k satisfies $H90(n,l)$ if the l-adic étale cohomology group $H^{n+1}_{\text{ét}}(k, \mathbb{Z}_{(l)}(n))$ vanishes. For $n = 1$, one can interpret this vanishing as a restatement of the classical Hilbert theorem 90, and for $n = 2$ this is essentially the famous result of A. S. Merkur′ev and Suslin [Izv. Akad. Nauk SSSR Ser. Mat. **46** (1982), no. 5, 1011–1046, 1135–1136; MR0675529], which we may view as the confirmation of the Bloch-Kato conjecture in weight 2. One readily sees that $H90(n,l)$ implies the surjectivity assertion of Bloch-Kato in weight n. The main result of Voevodsky is that $H90(n,2)$ is valid for any field k and any $n \geq 0$. (For k of characteristic l, $H90(n,l)$ was established by Geisser and Levine [op. cit.].)

Voevodsky proceeds to prove $H90(n,2)$ by induction on n; thus, one begins by assuming the validity of $H90(n-1,2)$; in fact, he assumes $H90(n-1,l)$ for an arbitrary prime l and proceeds quite far towards the proof of the general Bloch-Kato conjecture before restricting to the case $l = 2$. Voevodsky makes the observation that $H90(n-1,l)$ implies (2) for weights $q < n$ (and this choice of prime l) as well as a version of the Hilbert theorem 90 for K^M_q with $q < n$. Using "classical" techniques of Galois cohomology, Voevodsky then shows that these two conditions imply the vanishing of $H^n_{\text{ét}}(k, \mathbb{Z}/l)$ provided that k satisfies two conditions: (i) k has no extensions of degree prime to l; and (ii) $K^M_n(k)$ is l-divisible. Reasonably straightforward arguments reduce the required cohomological vanishing of $H90(n,l)$ to the vanishing of $H^n_{\text{ét}}(k, \mathbb{Z}/l)$ in this case, so that it remains to prove that we can pass from our given field k to a field extension K/k satisfying these two conditions as well as the injectivity

$$H^{n+1}_{\text{ét}}(k, \mathbb{Z}_{(l)}(n)) \hookrightarrow H^{n+1}_{\text{ét}}(K, \mathbb{Z}_{(l)}(n)) \tag{4}$$

of the induced map. Condition (i) for a field extension K/k satisfying (4) can be easily arranged using a transfer argument. The heart of the proof is showing that Condition (ii) can also be arranged for K/k satisfying (4) when $l = 2$.

In order to arrange the 2-divisibility of $K^M_n(k)$, Voevodsky chooses a symbol $\underline{a} = (a_1, \ldots, a_n)$ representing a generator of $K^M_n(k)$ and takes K to be the function field of the associated norm quadric $Q_{\underline{a}}$. It is well known that the class in $K^M_n(k)$ associated to $\underline{a}$ is divisible by l. The challenge is to prove (3) for K/k. Up to this point, the proof has been largely inspired by the proof of Merkur′ev and Suslin for the Bloch-Kato conjecture in weight 2 [op. cit.]. Voevodsky proceeds to investigate the motivic cohomology of the Čech simplicial scheme $\mathcal{X}_{\underline{a}}$ associated to the norm quadric $Q_{\underline{a}}$. He employs his motivic cohomology operations and the vanishing of "Margolis homology" of a closely related simplicial scheme to prove that $H90(n-1,2)$ implies that $H^{n+1}(\mathcal{X}_{\underline{a}}, Z_{(2)}) = 0$. Now Voevodsky invokes results of Rost ["On the spinor norm and $A_0(X, K_1)$ for quadrics", preprint, 1988, `www.mathematik.uni-bielefeld.de/~rost/spinor.html`; "Some new results on the Chow groups of quadrics", preprint, 1990, `www.mathematik.uni-bielefeld.de/~rost/chowqudr.html`; J. Ramanujan Math. Soc. **14** (1999), no. 1, 55–63; MR1700870] concerning the motive of the norm quadric $Q_{\underline{a}}$ to obtain the necessary injectivity by relating the motive of $\mathcal{X}_{\underline{a}}$ to that of the field k.

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