

Discrete harmonic analysis, by Tullio Ceccherini-Silberstein, Fabio Scarabotti, and Filippo Tolli, Cambridge Studies in Advanced Mathematics, Vol. 172, Cambridge University Press, Cambridge, 2018, xiii+573 pp., ISBN 978-1-107-18233-2

What is discrete harmonic analysis? The book under review, by Tullio Ceccherini-Silberstein, Fabio Scarabotti, and Filippo Tolli, gives a comprehensive answer. It is a continuation of a series of wonderful books by the same authors, including *Harmonic analysis on finite groups: Representation theory, Gelfand pairs and Markov chains* [13] and *Representation theory and harmonic analysis of wreath products of finite groups* [14] (both published by Cambridge University Press), which witness the power of harmonic analysis on finite groups both in theory and in applications.

The study of mathematical models involving infinite objects, via their approximation by finite counterparts and leading to a discretization of these models, is one of the features of modern mathematics. This also reflects a “competition” and interaction between pure and applied mathematics. The title of the famous article of Louis Auslander and Richard Tolimieri that appeared in 1979 in the *Bulletin of the AMS* [6] asks, “Is computing with the finite Fourier transform pure or applied mathematics?” The book of Ceccherini-Silberstein, Scarabotti, and Tolli gives an exposition and a clarification of many questions raised by Auslander and Tolimieri, and much more.

In order to confirm the importance and popularity of discretization, let us recall a couple of examples. The statistical models on \mathbb{Z}^d , $d \geq 2$ (including the Ising model), are studied by means of their approximations in finite “subdomains” $[-n, n]^d$ and by passing to the limit for $n \rightarrow \infty$ (the van Hove limit). The spectral properties of the Laplacian on an infinite graph Γ can be retrieved from the spectral properties of a sequence $(\Gamma_n)_{n \in \mathbb{N}}$ of finite graphs approximating Γ . Similarly, the L^2 -Betti numbers of an infinite residually finite group G can be computed via a sequence $(G_n)_{n \in \mathbb{N}}$ of finite groups approximating G (the Lück approximation), and the Gibbs measures on such a group G are limits of Gibbs measures of the G_n ’s [34]. Finally, the classical Fourier transform on \mathbb{Z} or on the unit circle \mathbb{S}^1 can be approximated by the discrete Fourier transform (DFT for short) on the finite cyclic groups \mathbb{Z}_n . An additional advantage of this approach is the fact that the groups \mathbb{Z}_n are not only finite but self-dual (in the sense of Pontryagin), in contrast with \mathbb{Z} and \mathbb{S}^1 that are mutually dual.

Jean-Baptiste Joseph Fourier [27], at the turn of the nineteenth century, showed that representing a function as a sum of trigonometric functions (the exponentials $e^{2i\pi nx}$), namely its *Fourier series*, greatly simplifies the study of the heat transfer. The study of the decomposition process of a function into these trigonometric functions is called *Fourier analysis*, while the operation of building back the function from these pieces is known as *Fourier synthesis*. In parallel, the original concept of Fourier analysis has been extended to apply to more and more abstract and general situations: the general field is often known as *abstract harmonic analysis*.

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A generalization along these lines was started at the end of the nineteenth century with the fundamental work of Georg Frobenius, Issai Schur, and William Burnside, giving birth to the *representation theory* of (finite) groups. Later, with the influential works by Lev Pontryagin (1939), *Topological groups* [59] (leading to the Pontryagin duality theory for locally compact Abelian groups), and by André Weil (1940), *L'intégration dans les groupes topologiques et ses applications* [72] (with a comprehensive treatment of the Haar measure), the domain moved beyond the classical real world to the more general and abstract framework of topological groups (including the Lie groups). Books by Loomis, *An introduction to abstract harmonic analysis* [43], by E. Hewitt and K.A. Ross, *Abstract harmonic analysis* [38], and by J. Wolf, *Harmonic analysis on commutative spaces* [75], further develop this direction and witness the growing importance of these generalizations. In the latter book, the domains are the so-called commutative spaces, that is, homogeneous spaces G/K with G a locally compact group, equipped with a Haar measure, and K a compact subgroup of G such that the Banach algebra $L^1(K \backslash G/K)$ of summable bi- K -invariant functions on G is commutative with respect to convolution. Gelfand pairs are one of the byproducts of this model and will be discussed later.

Topics not touched in the present book, but which are related to abstract harmonic analysis, include the theory of invariant means on groups, which deals with the class of amenable groups introduced by J. von Neumann [55], A. Tarski [68, 69], and N. Bogolyubov [8], and groups with Kazhdan's property (T). The theory of invariant means is presented, for instance, in the classical book of F. Greenleaf, *Invariant means on topological groups and their applications* [31] (the book by Hewitt and Ross also contains a lot of material on this topic). The comprehensive discussion of the Banach–Tarski paradox that led von Neumann to the discovery of the class of amenable groups, the Tarski characterization of amenability, and detailed information about the structure of the class of amenable groups are presented in the book by G. Tomkowicz and S. Wagon, *The Banach–Tarski paradox* [70] (see also [12]). The contribution of Bogolyubov is explained in [33]. A wonderful exposition of the theory of groups with property (T) with a very accessible introduction to the theory of unitary representations of groups can be found in the book by B. Bekka, P. de la Harpe, and A. Valette, *Kazhdan's Property (T)* [7]. The notion of expander graphs, whose discussion constitutes a big part of the book under review, can be considered as a nonamenability property for an infinite family of finite graphs. We mention that the first explicit construction of expanders was exclusively based on the use of infinite groups with property (T) [50].

Nowadays, the development of computers and technologies based on digital data has revealed the importance of discrete mathematics and combinatorics (including graph theory and number theory). The domain here is a finite (possibly very large), thus *discrete*, set (often equipped with the action of a finite group of symmetries) and the corresponding Fourier analysis is then called *discrete harmonic analysis*, briefly *DHA*. The book by Persi Diaconis, *Group representations in probability and statistics* (1988) [22], and the more recent book [13] by the authors of the monograph under review, present an exposition of the methods, developed by Diaconis and his collaborators, which use the representation theory of finite groups, especially of the symmetric groups, to estimate the rate of convergence to the stationary distribution of several diffusion processes (such as the Ehrenfest model, the Bernoulli–Laplace model, and the random matchings). The setting is again a

homogeneous space G/K with G a finite group and $K \leq G$ a subgroup, and the requirement is that the algebra of bi- K -invariant functions on G is commutative with respect to convolution. This is, in turn, equivalent to the permutation representation $\lambda_{G/K}$ (also called quasi-regular representation) associated with the action of G on the homogeneous space G/K being multiplicity-free, that is, decomposing into irreducible pairwise inequivalent subrepresentations. One then says that (G, K) constitutes a *Gelfand pair*. Diaconis observed that given a finite Markov chain (e.g., a random walk) invariant under the action of a finite group G , the associated transition operator (Markov operator) can be expressed as a convolution operator whose kernel can be written, at least in theory, as a “Fourier series” where the classical exponentials $e^{2i\pi nx}$ are replaced by the irreducible representations of the group G . This would, in theory, lead to the analysis of the convolution powers of the kernel, and therefore of the powers of the transition operator, in order to determine its asymptotic behavior. This program, however, is not easy to implement in general, because on the one hand, the irreducible representations of a given finite group, although known in theory, most often are, in practice, not suitable for concrete calculations, and, on the other hand, since the representations are not necessarily one dimensional, one cannot determine a complete diagonalization (in particular a *spectral analysis*) of the transition operator, but just a block decomposition of it. Fortunately, in the case of Gelfand pairs, this can be reduced to an essentially commutative analysis, even if the involved representations, the so-called *spherical representations*, still remain of higher dimension, and therefore leads to a complete spectral analysis yielding the aforementioned estimates for the asymptotic behavior of the given Markov process. One of the striking outcomes of this analysis is the discovery, by Diaconis, of the so-called *cut-off phenomenon* (for the Ehrenfest and Bernoulli–Laplace models): the total variation distance between the distribution of the system at time t and the stationary distribution remains close to 1 (that is, essentially remains maximal) for a long time, then it drops down to a small value quite suddenly and decreases to 0 exponentially fast. In other words, the transition from order to chaos is concentrated in a small neighborhood of the *critical time* $t = \frac{1}{4}n \log n$, where n denotes the number of particles of the system. As a side remark, it is interesting to mention that the averages of characters of spherical representations alluded to above, give rise to interesting *special functions*, such as the Krawtchouk and Hahn polynomials.

One often refers to DHA as to a collection of ancillary mathematical tools, methods, and results: for example, Winnie Li in *Number theory with applications* (1996) [42] uses DHA methods to treat Deligne’s solution to Weil’s conjectures in number theory as well as an explicit construction of Ramanujan graphs [21, 73], and M. Nathanson in *Elementary methods in number theory* (2000) [54] presents elementary methods *à la* Erdős to prove the prime number theorem. Several other applications, besides number theory, are in numerical analysis and combinatorics. However, DHA should deserve a more pure math theoretical recognition, and the book under review indeed offers such a possibility.

Finite cyclic groups constitute the domain of the *discrete Fourier transform*, briefly *DFT*. In practice, DFT converts a finite sequence of equally spaced samples of a function into a same-length sequence of equally spaced samples of the discrete-time Fourier transform, which is a complex-valued function of frequency. It has

fundamental applications to digital signal processing, image processing, and solving PDEs.

Formally, the DFT is a linear transformation $\mathcal{F}_n: \mathbb{C}^n \rightarrow \mathbb{C}^n$ that transforms a vector $f = (f(0), \dots, f(n-1)) \in \mathbb{C}^n$ to its Fourier transform $\hat{f} = \mathcal{F}_n(f)$. It is represented by the $n \times n$ matrix whose (k, ℓ) -entry is equal to $e^{2\pi i k \ell / n}$. The ℓ th component of \hat{f} is given by

$$\hat{f}(\ell) = \sum_{k=0}^{n-1} f(k) e^{2\pi i k \ell / n}.$$

The theoretical background for the DFT is the spectral analysis on the finite cyclic group \mathbb{Z}_n (this goes back to Schur; see also [47]). Surprisingly, such an elementary approach led to several great achievements such as the celebrated *Gauss reciprocity law* in number theory. Recall that it states that for odd prime numbers p and q

$$\left(\frac{p}{q}\right) \left(\frac{q}{p}\right) = \frac{\text{tr}(\mathcal{F}_{pq})}{\text{tr}(\mathcal{F}_p) \text{tr}(\mathcal{F}_q)},$$

where $\left(\frac{p}{q}\right)$ denotes the Legendre symbol. Another example showing the power of this approach is the formula for the quadratic Gauss sums which is discussed in detail in the book under review.

Let us mention that Gauss, while involved in astronomical studies and, more precisely, in calculations of asteroid orbits from a finite set of equally spaced observations, was led to the computation of the Fourier coefficients a_k and b_k of a function f represented by a standard Fourier series of bandwidth n ,

$$f(x) = \sum_{k=0}^m a_k \cos(2\pi k x) + \sum_{k=1}^m b_k \sin(2\pi k x),$$

where $m = n/2$ for n even and $m = (n-1)/2$ for n odd. Gauss first observed that the coefficients a_k and b_k can be computed by a DFT of length n using the values of $f(x)$ at equispaced sample points. He then showed that if $n = n_1 n_2$, then this computation can be realized by first computing n_1 DFTs of length n_2 and then computing these shorter DFTs. This is the basic idea of the *fast Fourier transform*, briefly *FFT*, an algorithm that, when n is a composite integer, allows us to substantially decrease the number of operations in the computation of \hat{f} . For instance, when n is of the form $n = m^k$, with $m \geq 2$ fixed, the number of such operations decreases from n^2 to $\mathcal{O}(n \log n)$ (for $k \rightarrow \infty$). It was implemented later by various researchers in different forms. We mention, in particular the paper by Danielson and Lanczos in the early 1940s [18] (see also [41] and [16]) who devised a practical scheme intended to reduce the computational complexity of the Fourier transform with applications to X-ray scattering from liquids.

In the spirit of the title of the Auslander and Tolimieri paper, the authors revisited the DFT and one of the most important algorithms for computing it, namely the *Cooley-Tukey algorithm* [17] (which often is used as a synonym of FFT; see [16, 71] as well as the nice survey by Maslen and Rockmore [52]), by establishing a relation with the representation theory of the finite Heisenberg groups (see also [9]). The original methods for FFT (which, as was already mentioned, in fact goes back to Gauss) have been generalized by several authors (e.g., Good [29], Rader [60], Rose [62], and Winograd [74]) yielding a matrix and tensor product approach of a purely algebraic flavor. A big part of the book under review deals with the tensor

product stuff and helps the reader understand what was done in those papers and more.

Also, a generalization of the DFT to any finite Abelian group is possible, yielding to the discrete versions of the classical *Poisson formula* and the *uncertainty principle*. The uncertainty principle is a classical result of harmonic analysis asserting that given a nonzero function f on a finite Abelian group A , then f or its Fourier transform \widehat{f} have a large support.

This fact is similar to the well-known phenomenon, discovered by Werner Heisenberg in the context of quantum mechanics, that establishes a limit to the precision with which certain pairs of physical properties of a particle, such as position x and momentum p , can be determined.

There are two versions of this uncertainty principle: multiplicative and additive.

The multiplicative version states that for a function f on finite Abelian group A one has the inequality

$$|\text{supp}(f)| \cdot |\text{supp}(\widehat{f})| \geq |A|,$$

where $\text{supp}(\cdot)$ denotes the support.

The additive uncertainty principle is established for the cyclic groups \mathbb{Z}_p of prime order,

$$|\text{supp}(f)| + |\text{supp}(\widehat{f})| \geq |\mathbb{Z}_p| + 1 = p + 1.$$

This lower bound was observed first by Roy Meshulam (unpublished) [53] who realized that it was equivalent to nonsingularity of all square submatrices of the Fourier matrix over \mathbb{Z}_p , the latter being a famous theorem of Chebotarëv. This version of the uncertainty principle was independently (re)discovered by T. Tao [66], whose proof, together with his new proof of Chebotarëv's theorem, are presented in the book. This result constitutes a small piece of the fascinating theory called *additive combinatorics* that Tao and other researchers developed during the last two decades (see [67]). There is a hope that this and other results in this direction could help to solve the long-standing problem concerning the existence of “good” cyclic codes. A recent paper of S. Evra, A. Lubotzky, and E. Kowalski [26] is a step in that direction (see also the paper by Goldstein, Guralnick, and Isaacs [28]).

It was already known to Euclid that the prime numbers are infinitely many, and the ancient proof is accessible to a high school student. There are several other interesting proofs (including the topological one by Hillel Furstenberg in *Proofs from The Book* (1998) by Martin Aigner and Günther Ziegler [1]). Euler improved on this result by showing that the series of reciprocals of prime numbers diverges (see again, the book by Aigner and Ziegler, where Erdős's proof [25] is reproduced). The celebrated *Dirichlet theorem on primes in arithmetic progressions* [64] constitutes yet another improvement: every arithmetic progression $(a + bk)_{k \in \mathbb{N}}$ with a and b coprime (integers) contains infinitely many primes. Its proof is a classical and beautiful application of the character theory of finite Abelian groups (see, for instance, the book *Fourier analysis: An introduction* by Elias Stein and Rami Shakarchi [64]). We mention that one of the most important results recently established in number theory is the *Green–Tao theorem* [30] stating that the prime numbers contain arbitrarily long arithmetic progressions: this can be seen as a sort of “reciprocal” of the Dirichlet theorem.

Finite fields, also called *Galois fields*, possess interesting character theory for additive and (removing the zero) multiplicative structures viewed as Abelian groups. The *Hasse–Davenport identity* [19], relating *Gauss sums* (analogues of the *gamma*

function) over a finite field and a finite extension, can be reformulated and proved in this context. This in turn leads to the *Hua–Vandiver–Weil theorem* [39, 73] establishing a bound for the number of solutions of a polynomial equation over a finite field, and to another important result such as the Weil conjectures on the zeta function. (Rationality was proved by Bernard Dwork (1960) [24], the functional equation by Alexander Grothendieck (1965) [37], and the analogue of the Riemann hypothesis by Pierre Deligne (1974) [21].)

Kloosterman sums, named after the Dutch mathematician Henrik Kloosterman who introduced them in 1926, are a particular kind of exponential sums, a finite ring analogue of the Bessel functions. Let \mathbb{F}_q (where $q = p^n$ for some prime p and $n \in \mathbb{N}$) denote the Galois field with q elements. A multiplicative character ν of the extension \mathbb{F}_{q^2} (that is, a homomorphism $\nu: \mathbb{F}_{q^2}^* \rightarrow \mathbb{T}$) is said to be decomposable provided there exists a multiplicative character ψ of \mathbb{F}_q such that $\nu(\alpha) = \psi(\alpha\bar{\alpha})$ for all $\alpha \in \mathbb{F}_{q^2}$ (note that $\alpha\bar{\alpha} \in \mathbb{F}_q$). Then, with a pair (χ, ν) consisting of a nontrivial additive character χ of \mathbb{F}_q and an indecomposable multiplicative character ν of \mathbb{F}_{q^2} , one associates the so-called *generalized Kloosterman sum* $j = j_{\chi, \nu}: \mathbb{F}_q^* \rightarrow \mathbb{C}$, which is defined by setting

$$j(x) = \frac{1}{q} \sum_{\substack{w \in \mathbb{F}_{q^2}^* \\ w\bar{w}=x}} \chi(w + \bar{w})\nu(w)$$

for all $x \in \mathbb{F}_q^*$. Such objects play a crucial role in the Piatetski-Schapiro treatment of the representation theory of the group $G = \mathrm{GL}(2, \mathbb{F}_q)$ [57], the general linear group over \mathbb{F}_q . In [57] all irreducible representations of G are explicitly described and divided into two types, namely, the *parabolic* and the *cuspidal* representations. The former are obtained by inducing the irreducible characters χ of the Borel subgroup B consisting of all upper-triangular matrices. Such characters χ are uniquely determined by pairs (ψ_1, ψ_2) of multiplicative characters of the ground field \mathbb{F}_q . Then, if $\psi_1 \neq \psi_2$, the corresponding induced representation $\mathrm{Ind}_B^G \chi$ is irreducible (and, in fact, independent of the ordering of the pair (ψ_1, ψ_2)), whereas if $\psi_1 = \psi_2$, then $\mathrm{Ind}_B^G \chi$ decomposes into the sum of two irreducible representations. The representations obtained in this way are pairwise nonequivalent and are called parabolic. The outstanding irreducible representations of G , all higher dimensional, are called cuspidal and can be determined as follows. Consider the subgroup $U \leq G$ of *unipotent matrices* (which is isomorphic to the additive group of \mathbb{F}_q). If χ is a fixed nontrivial character of U , then the induced representation $\mathrm{Ind}_U^G \chi$ is multiplicity free and contains all higher-dimensional irreducible representations (those which are not parabolic are the cuspidal). Piatetski-Schapiro [57], using the Kloosterman sums, gave an explicit description of these cuspidal representations.

Expanders are families of finite “sparse” graphs of uniformly bounded degree (e.g., regular) and have strong connectivity properties. Their initial motivation is revealed in telecommunications: they serve as practical models for economical robust networks. Other significant applications have been found nowadays in several branches of computer science and technology, including designing algorithms, error correcting codes, extractors, pseudo-random generators, sorting networks, and cryptography. From a more theoretical viewpoint, expanders have been used in relation with several important results in computational complexity theory. Thus, it is not surprising that their idea was implemented mathematically rigorously at the Institute of Transmission of Information (ITI) (Moscow, 1960s). The first precise

definition was given by M. Pinsker in 1973 [58] but, in implicit form, expanders already appeared in a 1967 paper of Barzdin and Kolmogorov [40]. M. Gromov and L. Guth [36] give the following characterization of the main result from [40]:

This result and its proof contain several interesting geometric ideas.

The most important idea is the discovery of expanders. Kolmogorov and Barzdin essentially observed that a random graph is an expander.

Now, to be more specific, a family of expanders is a sequence $(\Gamma_n)_{n \in \mathbb{N}}$ of finite, connected, undirected graphs $\Gamma_n = (V_n, E_n)$ (possibly with loops and multiple edges) of uniformly bounded degree (e.g., d -regular for a fixed integer d (not depending on n)) such that $|V_n| \rightarrow \infty$, and in which every subset $A \subset V_n$ of the vertices that is not “too large” ($|A| \leq |V_n|/2$) has a “large” boundary. The latter may be expressed as follows: given a nonempty subset $A \subset V$ of vertices of a connected graph $\Gamma = (V, E)$, we denote by $\partial A := \{e = \{a, v\} \in E : a \in A, v \in V \setminus A\}$ the *boundary* of A and call the positive number

$$h(\Gamma) = \min_{\substack{A \subset V \\ 0 < |A| \leq |V|/2}} \frac{|\partial A|}{|A|}$$

the *isoperimetric constant* of Γ (also called the *Cheeger constant*). The expanding property is then $h(\Gamma_n) \geq h$ for all $n \in \mathbb{N}$, where $h > 0$ is a fixed (desirably large) constant. For d -regular connected graphs, there is an equivalent reformulation of the expanding property in terms of the spectrum of the adjacency matrix $M(\Gamma)$ of a graph Γ : if $\mu_0(\Gamma) = d$ and $\mu_1(\Gamma)$ denote the first and the second eigenvalues of $M(\Gamma)$, their difference $\delta(\Gamma) := \mu_0(\Gamma) - \mu_1(\Gamma)$ is called the *spectral gap* of Γ . Then a sequence $(\Gamma_n)_{n \in \mathbb{N}}$ of finite graphs (with $|V_n| \rightarrow \infty$) is expanding if and only if $\delta(\Gamma_n) \geq \delta$ for all $n \in \mathbb{N}$, where $\delta > 0$ is a fixed (desirably large) constant. For d -regular graphs, the link between these two viewpoints is offered by the Alon–Milman [3] inequality

$$\frac{d - \mu_1(\Gamma)}{2} \leq h(\Gamma)$$

and the Dodziuk [23] inequality

$$h(\Gamma) \leq \sqrt{2d(d - \mu_1(\Gamma))}$$

(these are the discrete analogues of the Cheeger–Buser inequalities for Riemannian manifolds [10, 11, 15]). Moreover, an upper-bound for the spectral gap of any family of finite d -regular graphs $d \geq 2$, as above, is provided by the celebrated Alon–Boppana inequality [2, 56, 63]:

$$\liminf_{n \rightarrow \infty} \mu_1(\Gamma_n) \geq 2\sqrt{d-1}.$$

In [35] Andrzej Żuk and the author of this review considered the compact metric space of all rooted, connected graphs with uniformly bounded vertex degrees. Given any such graph Γ , the adjacency operator $M(\Gamma)$ on the Hilbert space of all complex-valued functions on the vertex set (where the inner product is weighted with the vertex degrees) is self-adjoint. We proved that the spectral measure (spectral resolution) of $M(\Gamma)$ depends continuously on Γ . This provided various useful applications, including among others a generalization of the Alon–Boppana theorem.

Given a finite, connected, d -regular graph $\Gamma = (V, E)$, let $d = \mu_0 \geq \mu_1 \geq \dots \geq \mu_{|V|-1}$ denote the elements in the spectrum of M_Γ and set

$$\mu(\Gamma) = \max\{|\mu_i| : |\mu_i| \neq d, i = 1, 2, \dots, |V| - 1\}.$$

Then one says that Γ is a *Ramanujan graph* provided that $\mu(\Gamma) \leq 2\sqrt{d-1}$. In view of the Alon–Boppana inequality, a family $(\Gamma_n)_{n \in \mathbb{N}}$ of Ramanujan d -regular graphs with $|V_n| \rightarrow \infty$ constitutes a family of expanders which is optimal from a spectral viewpoint. Toshikazu Sunada [65] observed that a regular graph is Ramanujan if and only if its Ihara zeta function satisfies an analogue of the Riemann hypothesis.

As it was already mentioned, existence of families of expanders can be deduced using the *probabilistic method* (in addition to [40] see, e.g., [5, 44]), while an explicit construction is much more involved: it was Gregory Margulis in the early 1970s who, using Kazhdan property (T), provided such a construction [50]. More recently, the *zig-zag product*, introduced by Reingold, Vadhan, and Wigderson in 2000 [61] resulted in a useful method, purely graph theoretical, to build families of expanders as iterated zig-zag products.

On the other hand, an explicit construction of Ramanujan graphs was obtained by Lubotzky, Phillips, and Sarnak [46] and, independently, by Margulis [51] in 1988 who constructed, for every fixed $d = p + 1$ with p a prime, an infinite family of d -regular Ramanujan graphs. We refer to the book by Davidoff, Sarnak, and Valette [20] for a comprehensive and elementary introduction to this topic based, among other things, on methods of representation theory, and to the book of A. Lubotzky [45]. We emphasize that, although the construction of general expander families is, nowadays, achieved by rather elementary methods, the proof of the Ramanujan property still requires sophisticated methods from number theory (such as Deligne’s work on the Weil conjectures) and representation theory (in particular of $\mathrm{PSL}(2, \mathbb{F}_q)$). Recently, Marcus, Spielman, and Srivastava [48, 49], combining the probabilistic method with new techniques, managed to show that, for every integer $d \geq 3$, there are infinitely many Ramanujan d -regular graphs.

Now, the classical Ramanujan graphs are finite graphs, but one may be interested in d -regular Ramanujan graphs without the finiteness condition. In this case, the spectrum of the adjacency matrix is contained in the interval $[-2\sqrt{d-1}, 2\sqrt{d-1}]$ which represents the spectrum of the d -regular tree, and moreover, in the case of $d = 2k$ is even, it represents the spectrum of the Cayley graph of a free group F_k of rank k with respect to a free basis (which is a $2k$ -regular tree). Any $2k$ -regular graph Γ can be realized as a Schreier graph $\Gamma(F_k, H, S)$, where $H \leq F_k$ is a subgroup, $S \subset F_k$ is a free basis, whose vertex set is $G/H = \{gH : g \in G\}$, the set of left cosets, and the edges are the pairs (gH, sgH) with $g \in F_k$ and $s \in S \cup S^{-1}$. The value $\alpha_H = \limsup_{n \rightarrow \infty} \sqrt[n]{h_n}$, where $h_n = |\{g \in H : d_S(g, 1_G) \leq n\}|$ denotes the cardinality of the relative H -ball of radius n (with respect to geodesic distance) centered at the identity element, is called the *cogrowth* (or *relative growth*) of H and ranges in the interval $[1, 2k - 1]$. The main result of [32] says that the graph $\Gamma(F_k, H, S)$ (or, equivalently, the quotient group F_k/H if H is normal) is amenable if and only if $\alpha_H = 2k - 1$, while, under the assumption that the graph is infinite, $\Gamma(F_k, H, S)$ is Ramanujan if and only if $\alpha_H \leq \sqrt{2k - 1}$. This yields an easy combinatorial way to construct infinite Ramanujan graphs.

The book under review is a very good introduction to the above-mentioned topics. In a self-contained way (it requires just elementary undergraduate rudiments of algebra and analysis and some mathematical maturity) it leads the reader to

cutting-edge research. It begins with an elementary but quite complete treatment of finite Abelian groups, including their automorphism group. Then character theory of these groups is used in order to establish the basic properties of the DFT, including Tao's uncertainty principle. In Chapter 3 a proof of the Dirichlet theorem on arithmetic progressions is given. Chapter 4 is based on the paper of Auslander and Tolimieri [6]: a complete spectral analysis of the DFT is given and it is used to establish Gauss's law of quadratic reciprocity. This is not the shortest path to Gauss's result, but it is a quite natural application of the spectral theory of the DFT (the authors suggest another proof in an exercise in the chapter on finite fields). Chapter 5 examines the FFT algorithm from a quite abstract algebraic point of view. In particular, it contains a systematic study of the connections between the Kronecker product of matrices and the stride permutations. Chapter 6 begins the examination of finite fields: their structure, their automorphisms, the properties of the norm and the trace, as well as quadratic extensions, are described in an elementary manner in full detail. These notions are repeatedly used in most of the subsequent sections. Chapter 7 contains an exposition on character theory of finite fields: in this setting, both additive and multiplicative characters arise, and the study of their interactions leads to some of the deepest results in number theory, e.g., the Weil conjectures, just to mention the most famous example. Chapter 8 is an introduction to graph theory, with emphasis on various notions of products. Again, the authors begin from scratch but then lead the reader to modern aspects of the theory, such as the zig-zag product introduced by Reingold, Vadhan, and Wigderson [61]. In Chapter 9, the notion of a family of expanders is discussed, giving various constructions (due to Margulis [50, 51]; Reingold, Vadhan, and Wigderson [61]; and Alon, Schwartz, and Schapira [4]). The harmonic analysis on finite Abelian groups and finite fields plays a fundamental role in these constructions. Also, the notion of a Ramanujan graph is discussed and two proofs of the Alon–Boppana–Serre inequalities are given. Chapters 10 and 11 constitute a standard but quite detailed introduction to the representation theory of finite groups, from the point of view of harmonic analysis. Chapter 12 discusses two families of examples: the Heisenberg groups and the affine groups over either finite fields or integers mod n . Following again the paper by Auslander and Tolimieri [6], the authors discuss the deep relationships between representation theory of the Heisenberg group, the DFT, and the FFT. Chapter 13 is an original description of the commutant (Hecke algebra) of $\text{Ind}_K^G \chi$, where χ is a one-dimensional representation of a subgroup $K \leq G$. The authors analyze the commutative case, giving a new generalization of the theory of Gelfand pairs and their spherical functions. Chapter 14 is an exposition of the paper by Piatetski-Schapiro on the representation theory of $\text{GL}(2, \mathbb{F}_q)$. The authors have done a good job using the theory developed in Chapter 13, which clarifies many points and shed light on the original approach in [57].

REFERENCES

- [1] M. Aigner and G. M. Ziegler, *Proofs from The Book*, 5th ed., Springer-Verlag, Berlin, 2014. Including illustrations by Karl H. Hofmann. MR3288091
- [2] N. Alon, *Eigenvalues and expanders*, *Combinatorica* **6** (1986), no. 2, 83–96, DOI 10.1007/BF02579166. Theory of computing (Singer Island, Fla., 1984). MR875835
- [3] N. Alon and V. D. Milman, λ_1 , *isoperimetric inequalities for graphs, and superconcentrators*, *J. Combin. Theory Ser. B* **38** (1985), no. 1, 73–88, DOI 10.1016/0095-8956(85)90092-9. MR782626

- [4] N. Alon, O. Schwartz, and A. Shapira, *An elementary construction of constant-degree expanders*, *Combin. Probab. Comput.* **17** (2008), no. 3, 319–327, DOI 10.1017/S0963548307008851. MR2410389
- [5] N. Alon and J. H. Spencer, *The probabilistic method*, 3rd ed., Wiley-Interscience Series in Discrete Mathematics and Optimization, John Wiley & Sons, Inc., Hoboken, NJ, 2008. With an appendix on the life and work of Paul Erdős. MR2437651
- [6] L. Auslander and R. Tolimieri, *Is computing with the finite Fourier transform pure or applied mathematics?*, *Bull. Amer. Math. Soc. (N.S.)* **1** (1979), no. 6, 847–897, DOI 10.1090/S0273-0979-1979-14686-X. MR546312
- [7] B. Bekka, P. de la Harpe, and A. Valette, *Kazhdan's property (T)*, *New Mathematical Monographs*, vol. 11, Cambridge University Press, Cambridge, 2008. MR2415834
- [8] N.N. Bogolyubov, *On some ergodic properties of continuous transformation groups*, *Nauch. Zap. Kiev Univ. Phys.-Mat. Sb.* **4:3** (1939), 45–53 (see also N.N. Bogolyubov, *Selected works, vol 1*, Naukova Dumka, Kiev 1969. pp. 561–569).
- [9] D. Bump, P. Diaconis, A. Hicks, L. Miclo, and H. Widom, *An exercise(?) in Fourier analysis on the Heisenberg group* (English, with English and French summaries), *Ann. Fac. Sci. Toulouse Math.* (6) **26** (2017), no. 2, 263–288, DOI 10.5802/afst.1533. MR3640891
- [10] P. Buser, *Über eine Ungleichung von Cheeger* (German), *Math. Z.* **158** (1978), no. 3, 245–252, DOI 10.1007/BF01214795. MR0478248
- [11] P. Buser, *A note on the isoperimetric constant*, *Ann. Sci. École Norm. Sup. (4)* **15** (1982), no. 2, 213–230. MR683635
- [12] P. de la Harpe, R. I. Grigorchuk, and T. Chekerini-Sil'berstaĭn, *Amenability and paradoxical decompositions for pseudogroups and discrete metric spaces* (Russian, with Russian summary), *Tr. Mat. Inst. Steklova* **224** (1999), no. Algebra. Topol. Differ. Uravn. i ikh Prilozh., 68–111; English transl., *Proc. Steklov Inst. Math.* **1(224)** (1999), 57–97. MR1721355
- [13] T. Ceccherini-Silberstein, F. Scarabotti, and F. Tolli, *Harmonic analysis on finite groups: Representation theory, Gelfand pairs and Markov chains*, *Cambridge Studies in Advanced Mathematics*, vol. 108, Cambridge University Press, Cambridge, 2008. MR2389056
- [14] T. Ceccherini-Silberstein, F. Scarabotti, and F. Tolli, *Representation theory and harmonic analysis of wreath products of finite groups*, *London Mathematical Society Lecture Note Series*, vol. 410, Cambridge University Press, Cambridge, 2014. MR3202374
- [15] J. Cheeger, *A lower bound for the smallest eigenvalue of the Laplacian*, *Problems in analysis* (Papers dedicated to Salomon Bochner, 1969), Princeton Univ. Press, Princeton, N. J., 1970, pp. 195–199. MR0402831
- [16] J. W. Cooley, *Lanczos and the FFT: a discovery before its time*, *Proceedings of the Cornelius Lanczos International Centenary Conference* (Raleigh, NC, 1993), SIAM, Philadelphia, PA, 1994, pp. 3–9. MR1298218
- [17] J. W. Cooley and J. W. Tukey, *An algorithm for the machine calculation of complex Fourier series*, *Math. Comp.* **19** (1965), 297–301, DOI 10.2307/2003354. MR0178586
- [18] G. C. Danielson and C. Lanczos, *Some improvements in practical Fourier analysis and their application to X-ray scattering from liquids*, *J. Franklin Inst.* **233** (1942), 365–380, 435–452, DOI 10.1016/S0016-0032(42)90767-1. MR0006233
- [19] H. Davenport and H. Hasse, *Die Nullstellen der Kongruenzzetafunktionen in gewissen zyklischen Fällen* (German), *J. Reine Angew. Math.* **172** (1935), 151–182, DOI 10.1515/crll.1935.172.151. MR1581445
- [20] G. Davidoff, P. Sarnak, and A. Valette, *Elementary number theory, group theory, and Ramanujan graphs*, *London Mathematical Society Student Texts*, vol. 55, Cambridge University Press, Cambridge, 2003. MR1989434
- [21] P. Deligne, *La conjecture de Weil. I* (French), *Inst. Hautes Études Sci. Publ. Math.* **43** (1974), 273–307. MR0340258
- [22] P. Diaconis, *Group representations in probability and statistics*, *Institute of Mathematical Statistics Lecture Notes—Monograph Series*, vol. 11, Institute of Mathematical Statistics, Hayward, CA, 1988. MR964069
- [23] J. Dodziuk, *Difference equations, isoperimetric inequality and transience of certain random walks*, *Trans. Amer. Math. Soc.* **284** (1984), no. 2, 787–794, DOI 10.2307/1999107. MR743744
- [24] B. Dwork, *On the rationality of the zeta function of an algebraic variety*, *Amer. J. Math.* **82** (1960), 631–648, DOI 10.2307/2372974. MR0140494
- [25] P. Erdős, *Über die Reihe $\sum \frac{1}{p}$* , *Mathematica, Zutphen B*, **7** (1938), 1–2.

- [26] S. Evra, E. Kowalski, and A. Lubotzky, *Good cyclic codes and the uncertainty principle*, Enseign. Math. **63** (2017), no. 3-4, 305–332, DOI 10.4171/LEM/63-3/4-4. MR3852174
- [27] J. Fourier, *Théorie analytique de la chaleur* (French), Éditions Jacques Gabay, Paris, 1988. Reprint of the 1822 original. MR1414430
- [28] D. Goldstein, R. M. Guralnick, and I. M. Isaacs, *Inequalities for finite group permutation modules*, Trans. Amer. Math. Soc. **357** (2005), no. 10, 4017–4042, DOI 10.1090/S0002-9947-05-03927-9. MR2159698
- [29] I. J. Good, *The interaction algorithm and practical Fourier analysis*, J. Roy. Statist. Soc. Ser. B **20** (1958), 361–372. MR0102888
- [30] B. Green and T. Tao, *The primes contain arbitrarily long arithmetic progressions*, Ann. of Math. (2) **167** (2008), no. 2, 481–547, DOI 10.4007/annals.2008.167.481. MR2415379
- [31] F. P. Greenleaf, *Invariant means on topological groups and their applications*, Van Nostrand Mathematical Studies, No. 16, Van Nostrand Reinhold Co., New York-Toronto, Ont.-London, 1969. MR0251549
- [32] R. I. Grigorchuk, *Symmetrical random walks on discrete groups*, Multicomponent random systems, Adv. Probab. Related Topics, vol. 6, Dekker, New York, 1980, pp. 285–325. MR599539
- [33] R. Grigorchuk and P. de la Harpe, *Amenability and ergodic properties of topological groups: from Bogolyubov onwards*, Groups, graphs and random walks, London Math. Soc. Lecture Note Ser., vol. 436, Cambridge Univ. Press, Cambridge, 2017, pp. 215–249. MR3644011
- [34] R. I. Grigorchuk and A. M. Stëpin, *Gibbs states on countable groups* (Russian), Teor. Veroyatnost. i Primenen. **29** (1984), no. 2, 351–354. MR749922
- [35] R. I. Grigorchuk and A. Żuk, *On the asymptotic spectrum of random walks on infinite families of graphs*, Random walks and discrete potential theory (Cortona, 1997), Sympos. Math., XXXIX, Cambridge Univ. Press, Cambridge, 1999, pp. 188–204. MR1802431
- [36] M. Gromov and L. Guth, *Generalizations of the Kolmogorov-Barzdin embedding estimates*, Duke Math. J. **161** (2012), no. 13, 2549–2603, DOI 10.1215/00127094-1812840. MR2988903
- [37] A. Grothendieck, *Formule de Lefschetz et rationalité des fonctions L* (French), Séminaire Bourbaki, Vol. 9, Soc. Math. France, Paris, 1995, pp. Exp. No. 279, 41–55. MR1608788
- [38] E. Hewitt and K. A. Ross, *Abstract harmonic analysis. Vol. I: Structure of topological groups, integration theory, group representations*, 2nd ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 115, Springer-Verlag, Berlin-New York, 1979. MR551496
- [39] L. K. Hua and H. S. Vandiver, *Characters over certain types of rings with applications to the theory of equations in a finite field*, Proc. Nat. Acad. Sci. U.S.A. **35** (1949), 94–99, DOI 10.1073/pnas.35.2.94. MR0028895
- [40] A.N. Kolmogorov and Ya.M. Barzdin, *On the realization of networks in three-dimensional space*, in Selected Works of Kolmogorov, Volume 3, ed. Shiryaev, A. N., Kluwer Academic Publishers, Dordrecht, 1993.
- [41] C. Lanczos, *Applied analysis*, Prentice Hall, Inc., Englewood Cliffs, N. J., 1956. MR0084175
- [42] W. C. W. Li, *Number theory with applications*, Series on University Mathematics, vol. 7, World Scientific Publishing Co., Inc., River Edge, NJ, 1996. MR1390759
- [43] L. H. Loomis, *An introduction to abstract harmonic analysis*, D. Van Nostrand Company, Inc., Toronto-New York-London, 1953. MR0054173
- [44] A. Lubotzky, *Discrete groups, expanding graphs and invariant measures*, Progress in Mathematics, vol. 125, Birkhäuser Verlag, Basel, 1994. With an appendix by Jonathan D. Rogawski. MR1308046
- [45] A. Lubotzky, *Expander graphs in pure and applied mathematics*, Bull. Amer. Math. Soc. (N.S.) **49** (2012), no. 1, 113–162, DOI 10.1090/S0273-0979-2011-01359-3. MR2869010
- [46] A. Lubotzky, R. Phillips, and P. Sarnak, *Ramanujan graphs*, Combinatorica **8** (1988), no. 3, 261–277, DOI 10.1007/BF02126799. MR963118
- [47] J. H. McClellan and T. W. Parks, *Eigenvalue and eigenvector decomposition of the discrete Fourier transform*, IEEE Trans. Audio Electroacoust. **AU-20** (1972), no. 1, 66–74. MR0399751
- [48] A. W. Marcus, D. A. Spielman, and N. Srivastava, *Interlacing families I: Bipartite Ramanujan graphs of all degrees*, Ann. of Math. (2) **182** (2015), no. 1, 307–325, DOI 10.4007/annals.2015.182.1.7. MR3374962

- [49] A. W. Marcus, D. A. Spielman, and N. Srivastava, *Interlacing families IV: Bipartite Ramanujan graphs of all sizes*, 2015 IEEE 56th Annual Symposium on Foundations of Computer Science—FOCS 2015, IEEE Computer Soc., Los Alamitos, CA, 2015, pp. 1358–1377. MR3473375
- [50] G. A. Margulis, *Explicit constructions of expanders* (Russian), *Problemy Peredači Informacii* **9** (1973), no. 4, 71–80. MR0484767
- [51] G. A. Margulis, *Explicit group-theoretic constructions of combinatorial schemes and their applications in the construction of expanders and concentrators* (Russian), *Problemy Peredachi Informatsii* **24** (1988), no. 1, 51–60; English transl., *Problems Inform. Transmission* **24** (1988), no. 1, 39–46. MR939574
- [52] D. K. Maslen and D. N. Rockmore, *The Cooley-Tukey FFT and group theory*, *Notices Amer. Math. Soc.* **48** (2001), no. 10, 1151–1160. MR1861656
- [53] R. Meshulam, private communication.
- [54] M. B. Nathanson, *Elementary methods in number theory*, Graduate Texts in Mathematics, vol. 195, Springer-Verlag, New York, 2000. MR1732941
- [55] J. von Neumann, *Zur allgemeinen Theorie des Masses*, *Fund. Math.* **13** (1929), 73–116 and 333 (= Collected works, vol. I, pp. 599–643).
- [56] A. Nilli, *Tight estimates for eigenvalues of regular graphs*, *Electron. J. Combin.* **11** (2004), no. 1, Note 9, 4. MR2056091
- [57] I. Piatetski-Shapiro, *Complex representations of $GL(2, K)$ for finite fields K* , *Contemporary Mathematics*, vol. 16, American Mathematical Society, Providence, R.I., 1983. MR696772
- [58] M.S. Pinkser, *On the complexity of a concentrator*, 7th International Teletraffic Conference (1973), 318/1–318/4.
- [59] L. Pontrjagin, *Topological groups*, Princeton University Press, Princeton, N.J., 1939 1958. Translated from the Russian by Emma Lehmer; (Fifth printing, 1958). MR0090766
- [60] Ch.M. Rader, *Discrete Fourier transforms when the number of data samples is prime*, *Proc. IEEE* **56** (1968), 1107–1108.
- [61] O. Reingold, S. Vadhan, and A. Wigderson, *Entropy waves, the zig-zag graph product, and new constant-degree expanders*, *Ann. of Math. (2)* **155** (2002), no. 1, 157–187, DOI 10.2307/3062153. MR1888797
- [62] D. J. Rose, *Matrix identities of the fast Fourier transform*, *Linear Algebra Appl.* **29** (1980), 423–443, DOI 10.1016/0024-3795(80)90253-0. MR562772
- [63] J.-P. Serre, *Répartition asymptotique des valeurs propres de l'opérateur de Hecke T_p* (French), *J. Amer. Math. Soc.* **10** (1997), no. 1, 75–102, DOI 10.1090/S0894-0347-97-00220-8. MR1396897
- [64] E. M. Stein and R. Shakarchi, *Fourier analysis*, Princeton Lectures in Analysis, vol. 1, Princeton University Press, Princeton, NJ, 2003. An introduction. MR1970295
- [65] T. Sunada, *L-functions in geometry and some applications*, *Curvature and topology of Riemannian manifolds (Katata, 1985)*, *Lecture Notes in Math.*, vol. 1201, Springer, Berlin, 1986, pp. 266–284, DOI 10.1007/BFb0075662. MR859591
- [66] T. Tao, *An uncertainty principle for cyclic groups of prime order*, *Math. Res. Lett.* **12** (2005), no. 1, 121–127, DOI 10.4310/MRL.2005.v12.n1.a11. MR2122735
- [67] T. Tao and V. Vu, *Additive combinatorics*, Cambridge Studies in Advanced Mathematics, vol. 105, Cambridge University Press, Cambridge, 2006. MR2289012
- [68] A. Tarski, *Sur les fonctions additives dans les classes abstraites et leurs applications au problème de la mesure*, *C.R. Séances Soc. Sci. Lettres Varsovie, Cl III* **22** (1929), 114–117.
- [69] A. Tarski, *Algebraische Fassung des Massproblems*, *Fund. Math.* **31** (1938), 47–66.
- [70] G. Tomkowicz and S. Wagon, *The Banach-Tarski paradox*, 2nd ed., *Encyclopedia of Mathematics and its Applications*, vol. 163, Cambridge University Press, New York, 2016. With a foreword by Jan Mycielski. MR3616119
- [71] C. Van Loan, *Computational frameworks for the fast Fourier transform*, *Frontiers in Applied Mathematics*, vol. 10, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1992. MR1153025
- [72] A. Weil, *L'intégration dans les groupes topologiques et ses applications* (French), *Actual. Sci. Ind.*, no. 869, Hermann et Cie., Paris, 1940. [This book has been republished by the author at Princeton, N. J., 1941.] MR0005741
- [73] A. Weil, *Numbers of solutions of equations in finite fields*, *Bull. Amer. Math. Soc.* **55** (1949), 497–508, DOI 10.1090/S0002-9904-1949-09219-4. MR29393

- [74] S. Winograd, *On computing the discrete Fourier transform*, Math. Comp. **32** (1978), no. 141, 175–199, DOI 10.2307/2006266. MR0468306
- [75] J. A. Wolf, *Harmonic analysis on commutative spaces*, Mathematical Surveys and Monographs, vol. 142, American Mathematical Society, Providence, RI, 2007. MR2328043

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