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1. Two definitions—one space

It is safe to say that the Hardy space H^2 has been, and continues to be, the most influential Hilbert space of analytic functions. The structure theorems for the Hardy space are so complete, so elegant, and so connected to a diversity of problems in complex analysis and operator theory that much of the work on other Hilbert spaces of analytic functions, such as the Dirichlet and Bergman spaces, tries to imitate what happens with H^2 [1, 3, 6]. The same goes for the beautiful theorems of Smirnov and Beurling that classify the invariant subspaces for the shift operator $f(z) \mapsto zf(z)$ on H^2. In fact, one could argue that H^2 has had an oversized influence on function theory.

The primary reason for this success is that the Hardy space has two equivalent definitions and thus two powerful sets of tools at a researcher's disposal. In one sense, the Hardy space is defined as $H^2(m)$, the subspace of $L^2(m)$ (m is normalized Lebesgue measure on the unit circle $\mathbb{T}$), whose negative Fourier coefficients

$$\widehat{f}(n) = \int_{\mathbb{T}} f(\xi)\overline{\xi}^n dm(\xi), \quad n < 0,$$

vanish. In another sense, the Hardy space is defined as $H^2(\mathbb{D})$, the analytic functions f on the open unit disk $\mathbb{D}$ whose integral means

$$\int_{\mathbb{T}} |f(r\xi)|^2 dm(\xi)$$

are uniformly bounded in $r \in (0, 1)$. Parseval's theorem and a 1906 harmonic analysis result of Fatou show that when $f \in H^2(\mathbb{D})$, the radial limit function

$$f(\xi) := \lim_{r \to 1^-} f(r\xi)$$

exists for m-almost every $\xi \in \mathbb{T}$ and belongs to $H^2(m)$. On the other hand, a computation with power series shows that when $f \in H^2(m)$, its Cauchy integral

$$\int_{\mathbb{T}} \frac{f(\xi)}{1 - \overline{\xi} z} dm(\xi)$$

defines an analytic function on $\mathbb{D}$ belonging to $H^2(\mathbb{D})$. This identification of $H^2(m)$ with $H^2(\mathbb{D})$ becomes even more salient when one realizes that the Fourier coefficients of $f \in H^2(m)$ turn out to be the Taylor coefficients of $f \in H^2(\mathbb{D})$. So the success that H^2 has enjoyed since its initial discovery by Hardy in 1915 lies in the fact that one has the powerful tools of Lebesgue theory, when regarding H^2 as a subspace of $L^2(m)$, and the equally powerful tools of harmonic and complex analysis, when regarding H^2 as a set of analytic functions with bounded integral means. As it turns out, one can talk about Hardy spaces (notice the plural) since one can replace the parameter 2 in the above discussion with a $p \in [1, \infty)$ and the same theory goes through *mutatis mutandis*.

Nikolaï Nikolski (University of Bordeaux) has written an excellent text containing a selection of topics that explore how the two definitions of H^2 connect to a variety of analysis problems. Since H^2 has blossomed into a vast and influential subject with an impressive literature, Nikolski needed to make some choices as to where to put his efforts. Though he certainly covers the basic structure theorems for H^2, the old favorites many of us who work in the subject know well, the book places a special emphasis on the connections Hardy spaces make with operator theory, complex analysis, and applied mathematics. Nikolski wisely resists the urge to write the definitive book on H^p and even cites the poet Robert Browning's phrase "Less is more" as his guiding principle. We will survey four representative topics from Nikolski's book that demonstrate both the theory of Hardy spaces themselves and also the diversity of connections they make.

2. Invariant subspaces

The shift operator plays an important representational role in operator theory. Starting with the spectral theorem, many linear transformations (operators) on Hilbert spaces are unitarily equivalent to a shift operator $f(z) \mapsto zf(z)$ on either an L^2 space (certain normal operators) or on a Hilbert space of analytic functions (certain subnormal operators). Furthermore, compressions of the shift to certain Hilbert spaces of analytic functions are used to model a wide class of contraction operators [11]. An early type of shift operator to be studied is $(M_\xi f)(\xi) = \xi f(\xi)$ on $L^2(m)$. This operator is called the shift as one can see by the following "shifting" action on Fourier series:

$$M_\xi\Big(\sum_{n\in\mathbb{Z}} \widehat{f}(n)\xi^n\Big) = \sum_{n\in\mathbb{Z}} \widehat{f}(n)\xi^{n+1}.$$

Notice how the sequence of Fourier coefficients are shifted one space to the right.

Of particular interest with the shift operator, or any operator, is the description of its invariant subspaces. With the shift M_ξ on $L^2(m)$, the invariant subspaces come in two basic types. A 1932 result of Wiener, connecting to stationary filters, says that if $\mathcal{E}$ is a closed subspace of $L^2(m)$ with $M_\xi\mathcal{E} = \mathcal{E}$, then there is a measurable set $A \subset \mathbb{T}$ such that $\mathcal{E} = \chi_A L^2(m)$ (the functions in $L^2(m)$ which vanish m-almost everywhere on $\mathbb{T} \setminus A$). Such subspaces $\mathcal{E} = \chi_A L^2(m)$ are called reducing subspaces for M_ξ since they are invariant for both M_ξ and its adjoint M_ξ^*. The nonreducing M_ξ-invariant subspaces (i.e., those for which $M_\xi\mathcal{E} \neq \mathcal{E}$) were characterized in 1964 by Helson as $\mathcal{E} = qH^2(m)$, where $q \in L^\infty(m)$ with $|q(\xi)| = 1$ for almost every $\xi \in \mathbb{T}$. Notice the appearance of the Hardy space as part of the solution to this $L^2(m)$ operator theory problem.

When $\mathcal{E} \subset H^2(m)$, it is automatically nonreducing and it turns out that $\mathcal{E} = qH^2(m)$, where $q \in H^2(m)$ with unimodular values on $\mathbb{T}$. Such q, which are not merely unimodular on $\mathbb{T}$ but also belong to $H^2(m)$, are called inner functions and can be alternatively characterized by the orthogonality condition $q \perp M_\xi^n q$, $n \geq 1$. Since $q \in H^2(m)$, it can, via Fourier series, be regarded as an analytic function on $\mathbb{D}$. We will discuss a beautiful formula for such functions in a moment.

This invariant subspace result, along with its generalization to classifying the M_ξ-invariant subspaces of $H^2(\mu)$, where μ is a positive measure on $\mathbb{T}$ and $H^2(\mu)$ is the closure of the analytic polynomials in $L^2(\mu)$, are covered, as are all of the proofs in Nikolski's book, with both mathematical efficiency and pedagogical care. Included in his discussion of $H^2(\mu)$, he also covers the 1916 gem of the Riesz brothers, which classifies the complex measures σ on $\mathbb{T}$ for which $\widehat{\sigma}(n) = 0$, $n < 0$, as $d\sigma = f dm$ for some $f \in H^1(m)$. This seemingly innocent technical result turns out to be very useful when discussing more advanced topics.

3. Factorization

Beurling and Smirnov were giants in the development of Hardy spaces and especially the finer points of the M_ξ-invariant subspaces of $H^2(m)$. One of their important explorations was the space $\mathcal{E}(f) := \bigvee\{M_\xi^n f : n \geq 0\}$, the M_ξ-invariant subspace generated by an $f \in H^2(m)$. The description of $\mathcal{E}(f)$ depends on the important fact, proved by Smirnov in 1928, that any $f \in H^2(m) \setminus \{0\}$ can be factored as $f = f_I f_O$, where $f_I \in H^2(m)$ is an inner function and $f_O \in H^2(m)$ is outer (i.e., $\mathcal{E}(f_O) = H^2$). The summary result of Beurling and Smirnov is

$$\mathcal{E}(f) = f_I H^2.$$

Nikolski pauses at the end of this chapter to give some "bare handed" examples of inner functions (e.g., $f = (z-a)/(1-\overline{a}z)$, $a \in \mathbb{D}$) and outer functions (e.g., $f = 1 + g$, g analytic on $\mathbb{D}$ and $g(\mathbb{D}) \subset \mathbb{D}$). These examples give the reader an enhanced appreciation for the next chapter where Nikolski surveys the work of Riesz, Szegő, Herglotz, and Smirnov which uses the bounded integral means definition of H^2 to give concrete formulas for the inner and outer parts of f. These beautiful formulas say that any $f \in H^2(m) \setminus \{0\}$ can be written uniquely as

$$f(z) = \xi \prod_{n \geq 1} \frac{|z_n|}{z_n} \frac{z_n - z}{1 - \overline{z_n} z} \exp\Big(- \int_{\mathbb{T}} \frac{\xi + z}{\xi - z} d\mu(\xi)\Big) \exp\Big(\int_{\mathbb{T}} \frac{\xi + z}{\xi - z} \log|f(\xi)| dm(\xi)\Big),$$

where $\xi \in \mathbb{T}$, $z_n \in \mathbb{D}$ and satisfy $\sum_{n \geq 1}(1 - |z_n|) < \infty$, and μ is a positive measure on $\mathbb{T}$ with $\mu \perp m$. The last factor above is the outer factor f_O of f while the other factors form the inner factor f_I of f. The fact that $\log|f| \in L^1(m)$, needed for the convergence of the integral describing f_O, is also an important part of the result and can be used to specify the modulus of an outer function on $\mathbb{T}$. Notice how this also says something about the boundary zeros of an $f \in H^2(m)$: the set $\{\xi \in \mathbb{T} : f(\xi) = 0\}$ must have measure zero. Finally, the convergence of $\sum_{n \geq 1}(1 - |z_n|)$, called the Blaschke condition, characterizes the zeros (in $\mathbb{D}$) of H^2 functions.

One cannot underestimate the usefulness and ubiquity of the factorization formula $f = f_I f_O$. It not only drove the stunning success of Hardy spaces but also influenced the direction of the general theory of Hilbert spaces of analytic functions, for example the Bergman and Dirichlet spaces. Much of this work attempted to reproduce versions of this factorization formula, along with their own notions of inner ($f \perp z^n f$ for all $n \geq 1$) and outer (f is a cyclic vector for the shift), and the valiant attempts at a version of the Beurling/Smirnov theorem (a description of the shift invariant subspaces). Continued efforts to imitate these Hardy space results remain an active area of research.

The analytic continuation properties of the inner part of an $H^2(m)$ function, along with alternate characterizations of outer functions, are carefully covered in Nikolski's book and play an important role when discussing model spaces $H^2(m) \ominus qH^2(m)$. This is a vast field in itself with applications to contractions on Hilbert spaces [4, 8–10].

It is worth mentioning here as it is covered in Nikolski's book, that most of the theorems about $H^2(m)$ (invariant subspaces, boundary values, factorization) have analogues for the $H^p(m)$ classes, together with their important cousins, the Nevanlinna and Smirnov classes.

4. The past and future

Hardy spaces make a connection to discrete time stationary processes. A sequence $\{\mathbf{x}_n\}_{n\in\mathbb{Z}}$ in a Hilbert $\mathcal{H}$ is stationary if $\bigvee\{\mathbf{x}_n : n \in \mathbb{Z}\} = \mathcal{H}$ and the correlation matrix $(\langle \mathbf{x}_n, \mathbf{x}_k\rangle)_{n,k\in\mathbb{Z}}$ depends only on $n-k$. A representative example of a stationary process is when the Hilbert space is $L^2(\mu)$, where μ is a positive measure on $\mathbb{T}$, and the vectors $\mathbf{x}_n$ are the functions ξ^n, $z \in \mathbb{Z}$. To see this, observe that

$$\langle \mathbf{x}_n, \mathbf{x}_k\rangle_{L^2(\mu)} = \int_{\mathbb{T}} \xi^{n-k} d\mu(\xi)$$

depends only on $n-k$. A 1939 theorem of Kolmogorov says that this representative example is, via the spectral theorem, canonical. The problem of optimal prediction is to compute

$$\inf\Big\{\|\mathbf{x}_n - \mathbf{x}\| : \mathbf{x} \in \bigvee\{\mathbf{x}_k : k < n\}\Big\},$$

which measures the dependence of the future on the past. By Kolmogorov's theorem and the fact that the correlation matrix depends only on $n-k$, one can compute the optimal prediction by computing what is known as the Szegő infimum,

$$\inf\{\|1 - zg\|_{L^2(\mu)} : g \in H^2(\mu)\}.$$

A conglomeration of theorems developed over a period of time by Szegő (1920), Verblunski (1936), and Kolmogorov (1941), show that if $\mu = w\,dm + \mu_s$ is the Radon–Nikodym decomposition of μ into its absolutely continuous and singular parts (with respect to m), then either there is no $f \in H^2(m) \setminus \{0\}$ for which $w = |f|^2$, in which case the Szegő infimum is zero, or there is a unique outer function $f \in H^2(m)$ satisfying $w = |f|^2$, in which case the Szegő infimum is equal to $|f(0)| > 0$. The summary version of the theorem is thus the formula

$$\inf\{\|1 - zg\|_{L^2(\mu)} : g \in H^2(\mu)\} = \exp\Big(\int_{\mathbb{T}} \log w\, dm\Big).$$

5. Connection to the Riemann hypothesis

There is also a fascinating connection that Hardy spaces make to the Riemann hypothesis, and Nikolski uses this an opportunity to bring in the Hardy spaces of the upper half-plane into his book. Just like the Hardy spaces of the disk, which can be defined as the $L^2(m)$ functions whose negative Fourier coefficients vanish, the Paley–Wiener theorem can be used to define $H^2(\mathbb{C}_+)$, where $\mathbb{C}_+$ is the upper half-plane, as the $L^2(\mathbb{R})$ functions whose Fourier transform vanishes on $(-\infty, 0)$.

As with the disk case, there is an alternate definition of $H^2(\mathbb{C}_+)$ involving uniform boundedness of integral means. There is also a version of the Wiener–Helson–Beurling theorem for $H^2(\mathbb{C}_+)$. Indeed, the shift operator (multiplication by the independent variable) is not defined on $L^2(\mathbb{R})$. However, the family of operators $\{f(x) \mapsto e^{isx} f(x) : s \in \mathbb{R}\}$ is defined, and one can consider the closed subspaces E of $L^2(\mathbb{R})$ which remain invariant under this family. If E is a closed subspace of $L^2(\mathbb{R})$ with $e^{isx}E \subset E$ for all $s \in \mathbb{R}$, then $E = \chi_A L^2(\mathbb{R})$ for some measurable $A \subset \mathbb{R}$. If $e^{isx}E \subset E$ for all $s \geq 0$ and there is an $s_0 > 0$ with $e^{is_0x}E \neq E$, then

$E = QH^2(\mathbb{C})$ for some $Q \in L^\infty(\mathbb{R})$ with $|Q(x)| = 1$ for almost every $x \in \mathbb{R}$. If $\{0\} \neq E \subset H^2(\mathbb{C}_+)$ and $e^{isx}E \subset E$ for all $s \geq 0$, then $E = QH^2(\mathbb{C}_+)$ where Q is an inner function on $\mathbb{C}_+$.

Certainly, $H^2(\mathbb{C}_+)$ is an important Hilbert space of analytic functions and plays a role in a variety of problems such as determining the span of a family of exponential functions in $L^2(\mathbb{R})$. The space $H^2(\mathbb{C}_+)$ also connects to the Riemann hypothesis via an approximation problem of Nyman and Báez-Duarte. Here one defines the function

$$\varphi(x) := \frac{1}{x} - \lfloor \frac{1}{x} \rfloor, \quad x > 0,$$

where $\lfloor a \rfloor$ is the integer part of a real number. Using Hardy space theory, along with the Paley–Wiener theory and the Mellin transform, one has the following equivalent statements:

- the Riemann hypothesis is true (the zeros of the Riemann zeta function that lie in the critical strip $\{0 < \Re z < 1\}$ lie in the critical line $\{\Re z = 1/2\}$);
- $\chi_{(0,1)}$ belongs to the closed linear span in $L^2(0, \infty)$ of the family of dilations $\{\varphi(tx) : t > 1\}$;
- $\chi_{(0,1)}$ belongs to the closed linear span in $L^2(0, \infty)$ of $\{\varphi(nx) : n \in \mathbb{N}\}$.

The last two equivalent conditions are known as dilation completeness problems.

Though the Hardy space of the polydisk $\mathbb{D}^n$ is briefly mentioned, Nikolski spends considerable time in establishing a relationship between the Hardy space of the Hilbert multidisk $\mathbb{D}_2^\infty$ and the Riemann hypothesis. The above dilation completeness problems are related to the problem of determining the $f \in H^2(\mathbb{D})$ for which the closed linear span of $\{f(z^n) : n \geq 1\}$ is $H^2(\mathbb{D})$. Moreover, the onto isometry

$$U : zH^2(\mathbb{D}) \to H^2(\mathbb{D}_2^\infty), \quad U\Big(\sum_{n \geq 1} \widehat{f}(n) z^n\Big) = \sum_{n \geq 1} \widehat{f}(n) \zeta^{\alpha(n)}, \quad \zeta \in \mathbb{D}_2^\infty,$$

where $\alpha(n) = (\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_s, 0, 0, 0, \ldots)$, $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$ is the prime factorization of n, and $\zeta^{\alpha(n)} = \zeta_1^{\alpha_1} \zeta_2^{\alpha_2} \zeta_3^{\alpha_3} \cdots \zeta_s^{\alpha_s}$, connects these completeness problems with the cyclic vectors for M_ζ on $H^2(\mathbb{D}_2^\infty)$. Nikolski documents the contributions to this circle of ideas from Neuwirth, Ginsberg, Newman, and Wintner.

6. Comments

Though Niokolski's book places special emphasis on the connections H^2 makes with Hilbert space problems, there are connections to Banach spaces. For example, along with a discussion of H^p, there is a treatment of bases in Banach spaces.

A critic of this book might argue that some of the "old favorites" such as growth rates for H^p functions and their derivatives, extremal functions, the Hausdorff–Young inequalities, the corona theorem, maximal functions, interpolating sequences, maximal ideal spaces for H^∞, duality in $H^p(m)$, and bounded and vanishing mean oscillation are not covered in the main body of the text. This reviewer is not one of them. These favorites are readily available in popular texts on Hardy spaces [2, 5, 7, 9, 10], some of them by Nikolski himself. In this book, Nikolski has given us a wonderful survey of some different topics to consider. Some of the auxiliary topics appear but are part of Nikolski's detailed and very scholarly notes (to original sources) and exercises (with solutions).

Nikolski often puts the reader *in medias res* and, despite his "Less is more" precept, one will still have a healthy amount of material put in front of them.

Function-theoretic operator theory is a large field, and there is little time to dilly-dally. Get up to speed quickly, constantly learn unfamiliar things, and do a little digging in the literature to learn more. To help the reader through this material, Nikolski is both an experienced educator and writer and knows how to present the material, efficiently, though quickly at times, so the student can learn as well as appreciate the subject.

Nikolski also gives us plenty of historical vignettes of the main figures in the development of Hardy spaces and, especially for the student, gives several appendices for those needing some gentle reminders of measure theory, complex analysis, Hilbert spaces, Banach spaces, and operator theory.

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