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Introduction to Global Analysis. Minimal Surfaces in Riemannian Manifolds, by John Douglas Moore, Graduate Studies in Mathematics, Vol. 187, American Mathematical Society, Providence, RI, 2017, xiv+368 pp., $83.00, ISBN 978-1-4704-2950-8

A minimal surface is a critical point for the area functional. The simplest of these are area-minimizers that are models for soap films and have the least possible area among all competing surfaces. In general, minimal surfaces do not have to be minimizers, or even stable critical points, but could instead be unstable with positive index. The Belgian physicist Plateau experimented with soap films in the mid-1800s, initially by studying oil suspended in a water and alcohol solution and then later by dipping a wire into a soap solution to produce minimal surfaces. The fundamental existence question became known as the Plateau problem: Prove that an area-minimizer exists for every boundary curve. Moreover, based on these experiments, he observed several universal properties of soap films that were finally mathematically proven by Taylor a century later, [T].

As a mathematical field, minimal surfaces date back more than a century before Plateau to work of Euler and Lagrange in the mid-1700s and the beginning of the calculus of variations. Many of the techniques developed in the study of minimal surfaces have played key roles in geometry and differential equations. A prominent example is monotonicity and tangent cone analysis that originated in the regularity theory for minimal surfaces and is now used throughout geometry and differential equations. Other examples are estimates for nonlinear equations based on the maximum principle arising in Bernstein's classical work and even Lebesgue's definition of the integral that he developed in his thesis on the Plateau problem for minimal surfaces.

This book would serve as a rich and interesting textbook for a graduate course on variational methods for constructing parameterized minimal surfaces. There are several approaches to constructing minimal surfaces, depending on whether one thinks of the surface as the image of a map or as an unparameterized subset of the ambient space (see, e.g., [P] for a book in this second direction). The book under review, [Mj2], focuses on the parametric theory which is most powerful for curves and surfaces where there is a close relationship between area and energy. The point of view is to approach minimal surfaces as critical points for a functional on the infinite-dimensional space of surfaces, along the lines initiated by Morse in 1925 for geodesics, [Mo], [Sm], [W]. The Hessian of the area functional is the second variation operator, and the function is Morse precisely when there are no nontrivial Jacobi fields (i.e., the Hessian has no kernel at a critical point). The index of a minimal surface is the number of negative eigenvalues of the Hessian at the critical point, so area-minimizers have index 0 and are said to be stable. This book is very accessible, assuming only some familiarity with basic differential geometry of curves and surfaces, some complex analysis, and very basic linear elliptic PDE. It fits very well in the literature, complementing the more classical book of Osserman

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[O], Lawson's book [L], and [CM2]. See [Pe] for a survey of some of the interesting recent developments.

The most fundamental existence question for minimal surfaces is the classical Plateau problem in $\mathbf{R}^3$, where one searches for the surface with the least area for a given fixed boundary curve. This question was finally settled independently by Douglas and Rado around 1930; Douglas was awarded one of the first Fields Medals for this work. A key point for the Plateau problem is the link between area and energy for surfaces. Let Ω be a compact surface, and let $u : \Omega \to \mathbf{R}^3$ be a smooth map. If $E(u) = \int |\nabla u|^2$ and $A(u)$ denote the energy and area, respectively, of u, then the Cauchy–Schwarz inequality gives that

$$A(u) \leq \frac{1}{2} E(u) . \tag{1}$$

Moreover, equality holds if and only if u is conformal almost everywhere. The same estimate holds when u maps to a more general manifold. This point of view for curves played a crucial role in Morse's theory for geodesics, [Mi], [Mo].

A typical modern solution of the Plateau problem (see, e.g., [L] or [CM2]) uses this inequality, and it proceeds by finding the map u with the lowest possible energy among maps with the desired boundary and then showing that this map is conformal and, thus, also minimal. The reason this approach works is because there is a compactness theorem for the space of energy-minimizers with different boundary parameterizations given by the Courant–Lebesgue lemma.

Existence gets more complicated if we try the same approach to find minimal 2-spheres in a Riemannian manifold M as, in general, there is no corresponding compactness. Namely, one can produce a sequence of harmonic maps from the sphere with uniformly bounded energy that has no convergent subsequence. This lack of compactness comes from the conformal invariance and is known as bubbling. This challenge was finally overcome by Sacks and Uhlenbeck, [SU1], by considering a perturbed energy functional with a higher power (where existence is now easy) and then proving a crucial uniform estimate that allowed them to get a limiting solution to the original problem. Crucially, the Sacks–Uhlenbeck approach can also be used to produce higher index minimal spheres. The methods and ideas developed in a number of important directions, including analytic aspects of harmonic maps, [Sc], [CM1], and they played a key role in geometric results such as the sphere theorem of Micallef and Moore, [MMj]. The existence theory was extended to higher genus by Sacks and Uhlenbeck, [SU2], and Schoen and Yau, [ScY]. This fascinating and fundamental material is a major focus of Moore's book, [Mj2]. The minimal spheres that are produced might not be embedded (e.g., the map might not be injective); embeddedness is addressed by other (nonparametric) methods by Meeks, Simon, and Yau, [MSY].

Moore's book begins with the basic theory of analysis on infinite-dimensional manifolds aimed at infinite-dimensional Morse theory, including Palais–Smale Condition C and Birkhoff's minimax principle. Chapter 2 covers the Morse theory of geodesics in Riemannian manifolds, the second variation of energy, and Abraham's bumpy metrics, [A], where the length function is a Morse function on the space of curves. Chapter 3 studies the topology of spaces of mappings, including Sullivan's theory of minimal models, the cohomology of spaces of maps, Gromov dimension, and Postnikov towers. Chapter 4 turns to minimal surfaces from the parametric point of view, starting with the fundamentals and culminating with beautiful results of Sacks and Uhlenbeck, [SU1], and Micallef and Moore, [MMj], using the existence theory and the complexified second variation to prove the sphere theorem. The last chapter is on bumpy metrics for minimal surfaces. A metric is bumpy for surfaces if the area function is a Morse function on the infinite-dimensional space of surfaces; by a result of White, [W], there is a set of bumpy metrics of Baire category. Moreover, the book focuses on a subset of these bumpy metrics, still of Baire category, that has even stronger properties by the earlier work of Moore, [Mj1].

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