BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 57, Number 2, April 2020, Pages 301–316
https://doi.org/10.1090/bull/1696
Article electronically published on March 3, 2020

SELECTED MATHEMATICAL REVIEWS

related to the paper in the previous section by
ARACELI BONIFANT AND JOHN MILNOR

MR0414561 (54 #2662) 14H20; 14H45

Zariski, Oscar

Le problème des modules pour les branches planes. (French)

Avec un Appendice de Bernard Teissier et une réimpression de "Characterization of plane algebroid curves whose module of differentials has maxuimum torsion" (Proc. Nat. Acad. Sci. U.S.A. **56** (1966), no. 3, 781–786), par Oscar Zariski.
École Polytechnique, *Paris*, 1973, iii+199 pp.

This book consists of notes from a course of Zariski at the École Polytechnique, plus an appendix by B. Teissier. The main problem is to study the space of isomorphism classes of plane branches (analytically irreducible curve germs) of given equisingularity type, i.e., with the same valuation semigroup Γ.

The first three chapters of the course review the basic material and introduce several important notions, such as short parametrization: every plane branch has a Puiseux parametrization of the form $(t^n, t^m + \sum_1^q a_{\nu_i} t^{\nu_i})$, where $\nu_1, \cdots, \nu_q$ are the integers in $\{m+1, \cdots, c\} - \Gamma$ ($c =$ conductor). If the characteristic is $(n; \beta_1, \cdots, \beta_g)$, then the $a_{\beta_i} \neq 0$; and certain of the a_{ν_i} may be eliminated after appropriate coordinate change. Thus, one can associate (in a non-unique way) to each curve of semigroup Γ a point in $\mathbf{C}^p - \bigcup_2^g H_i$, using the a_{ν_i} as coordinates; here, p is some integer $\leq q$, and the hyperplanes are removed because $a_{\beta_i} \neq 0$ $(i = 2, \cdots, g)$. The moduli space $M(\Gamma)$ is then the quotient of $\mathbf{C}^p - \bigcup H_i$ by the equivalence relative of analytic isomorphism of the corresponding branches. $M(\Gamma)$ is not separated in general; it is compact if and only if $g = 1$ or $g = 2$, $n = 4$, $m = 6$ (in the last case, $M(\Gamma)$ is a point). Further, if $g = 1$ (i.e., $(n, m) = 1$), the curve $y^n = x^m$ gives a point in the closure of every other point. That $M(\Gamma)$ is non-compact in other cases is established by finding analytic invariants of the parametrization (i.e., ${a_{\beta_3}}^{\beta_2 - \beta_1}$, if $g \geq 3$), which provide surjective maps of $M(\Gamma)$ onto $\mathbf{C} - \{0\}$. The basic technique is to change the parametrization via appropriate automorphisms, to see how the coefficients can change. In Chapter V, the complete structure of $M(\Gamma)$ is worked out for $(n, m) \in \{(2, m), (3, m), (4, 5), (5, 6), (6, 7)\}$. The last case is the most interesting; $M(\Gamma)$ consists, of six components (there are "six types of curves"), the one of largest dimension being an algebraic surface. Thus, the "generic" component has dimension 2. Chapter VI examines the generic component for $g = 1$ via equisingular deformations. The main result here is that the generic dimension is the dimension of the space of equisingular deformations for a "general" curve of type (n, m); this is computed if $m = n + 1$. At the end of the course an article of the author's is reproduced [Proc. Nat. Acad. Sci. U.S.A. **56** (1966), 781–786; MR0202716].

Teissier's appendix is based on the very pretty idea that every plane branch C of semi-group Γ appears as the general fibre in a one-parameter deformation of the monomial curve C^Γ (defined by $(t^{a_0}, t^{a_1}, \cdots, t^{a_g})$, where $a_0, \cdots, a_g$ are minimal generators for Γ). The idea is that $\mathcal{O} = \mathcal{O}_C$ can be filtered using the natural filtration on the integral closure, and the associated graded is the local ring $\mathcal{O}_\Gamma$ of C^Γ.

Now, $\mathcal{O}_\Gamma$ is a complete intersection with a $\mathbf{C}^*$-action, so its (smooth) miniversal deformation $X \to S$ may be written down, and S has a $\mathbf{C}^*$-action [H. C. Pinkham, *Deformations of algebraic varieties with* $\mathbf{G}_m$-*action*, Soc. Math. France, Paris, 1974; MR0376672]. An important theorem then states that the induced deformation over the "negatively weighted part" of S parametrizes deformations of $\mathcal{O}_\Gamma$ of constant semigroup, and there is a natural section picking out the singular point. Using the openness of versality (in a form stated in another appendix), one recovers the reviewer's result [Trans. Amer. Math. Soc. **193** (1974), 143–170] that every plane branch has a miniversal equisingular deformation, over a smooth base, with an equisingular section. In fact, this result is true for every irreducible curve for which C^Γ is a complete intersection (except that one is not sure whether "equisingular" should mean "constant semigroup" in the general case). As another application, Teissier gives a new proof of the two main theorems of Zaríski's 1966 article [op. cit.] generalizing them as well to C for which C^Γ is a complete intersection; that is, the dimension of the space of torsion differentials is $\leq c$ (conductor), with equality if and only if $C \approx C^\Gamma$. Finally, a "natural" compactification of M_Γ is given, as well as an interpretation of the generic component. An interesting analogy is drawn with Pinkham's work [op. cit.] on the relation between the positively weighted part of S and deformations of non-singular projective curves with a Weierstrass point of semigroup Γ.

{Reviewer's remark: The generic dimension of $M(\Gamma)$ for $g = 1$ has been computed by C. Delorme [C. R. Acad. Sci. Paris Sér. A-B **280** (1975), A1287-A1289].}

Jonathan M. Wahl

From MathSciNet, March 2020

MR0539411 (81c:57010) 57N10

Jaco, William H.; Shalen, Peter B.

Seifert fibered spaces in 3-manifolds.

Memoirs of the American Mathematical Society **21** (1979), *no.* 220, viii+192.

Some optimistic low-dimensional topologists believe that, in the not too distant future, the closed three-manifolds will be classified up to homeomorphism. If it works out this way, then the results contained in this carefully written, illuminating book will play an important role in the classification scheme.

The book begins with a complete set of the basic definitions needed in three-dimensional topology, including such terms as incompressible and boundary incompressible surfaces, sufficiently-large manifold, irreducible manifold, essential map, boundary parallelism, perfect embedding and hierarchy. The basic results on Seifert fiber spaces and Seifert pairs needed for their main theorem, "the mapping theorem", are developed, in addition to some material on Fuchsian groups and combinatorial group theory.

Chapter IV is devoted to a proof of the mapping theorem. Let f be a nondegenerate map of a Seifert pair (S, F) into a sufficiently-large 3-manifold pair (M, T). Then there exists a Seifert pair $(\Sigma, \Phi) \subset (M, T)$, well embedded in M, and a map f' of (S, F) into (M, T), homotopic to f as a map of pairs, such that $f'(S) \subset \Sigma$ and $f'(F) \subset \Phi$.

The mapping theorem is used in the proofs of many of the subsequent results, including the torus and annulus theorems. Torus theorem: Let (M, T) be an irreducible, sufficiently-large 3-manifold pair. Suppose that $f\colon (S^1 \times S^1, \varnothing) \to (M, T)$

is a map of pairs. If f is nondegenerate, then either there exists an embedding $g\colon (S^1 \times S^1, \varnothing) \to (M, T)$ which is nondegenerate, or $T = \partial M$ and M is homeomorphic to a special Seifert fibered 3-manifold. Annulus theorem: Let (M, T) be an irreducible, sufficiently-large 3-manifold pair. Suppose that $f\colon (S^1 \times I, S^1 \times \partial I) \to (M, T)$ is a map of pairs. If f is nondegenerate, then there exists an embedding $g\colon (S^1 \times I, S^1 \times \partial I) \to (M, T)$ which is nondegenerate. Furthermore, if $f|S^1 \times \partial I$ is an embedding, then we may choose g so that $g|S^1 \times \partial I = f|S^1 \times \partial I$.

These theorems are refinements of the torus and annulus theorems originally announced by Waldhausen.

The theorems that are most important for the classification of sufficiently-large 3-manifolds are the characteristic pair theorem and the splitting theorem proved in Chapter V. Definition: A characteristic pair for a compact, irreducible 3-manifold pair (M, T) is a perfectly-embedded Seifert pair $(\Sigma, \Phi) \subset (M, T)$ such that if f is any essential, nondegenerate map of an arbitrary Seifert pair (S, F) into (M, T), then f is homotopic, as a map of pairs, to a map f' such that $f'(S) \subset \Sigma$ and $f'(F) \subset \Phi$. Characteristic pair theorem: Every sufficiently-large 3-manifold pair (M, T) has a characteristic pair. Moreover, if (Σ, Φ) and (Σ', Φ') are two characteristic pairs for (M, T), then there is a homeomorphism $J\colon M \to M$, isotopic to the identity $(\mathrm{rel}(\partial M) - T)$, such that $J(\Sigma) = \Sigma'$ and $J(\Phi) = \Phi'$. Splitting theorem: Let M be a compact, sufficiently-large, irreducible, boundary-irreducible 3-manifold. Then there exists a two-sided, incompressible 2-manifold $W \subset M$, unique up to ambient isotopy, having the following three properties: (a) The components of W are annuli and tori, and none of them is boundary-parallel in M. (b) Each component of $(\sigma_W(M), \sigma_{\partial W}(\partial M))$ is either a Seifert pair or a simple pair. (c) W is minimal with respect to inclusion among all two-sided 2-manifolds in M having properties (a) and (b). Moreover, W may be taken to be a reduction of $\mathrm{Fr}_M(\Sigma)$, where (Σ, Φ) is a characteristic pair for $(M, \partial M)$.

The splitting theorem says, for example, that a closed irreducible sufficiently-large orientable 3-manifold splits along certain canonical tori into pieces that are Seifert fiber spaces and pieces that are atoroidal sufficiently-large irreducible 3-manifolds with boundaries that are tori. According to recent results of Thurston these latter have metrics with constant negative curvature in which they are complete (i.e. they have "hyperbolic" structure).

Finally, there are several theorems about 3-manifold groups. Here are two. Theorem: Let M be a sufficiently-large, irreducible, orientable, compact, connected 3-manifold. Let p and q be integers, and let a and b be elements of $\pi_1(M)$ such that $ab^pa^{-1} = b^q$. Then either $b = 1$ or $p = \pm q$. Theorem: Let M be an atoroidal, irreducible, 3-manifold. Let G be a two-generator subgroup of $\pi_1(M)$. Then either (i) G is free of some rank ≤ 2, or (ii) G is free abelian of rank 2, or (iii) G has finite index in $\pi_1(M)$.

There are more theorems in this book than we have stated here and some of the theorems we have stated are given by the authors in greater generality. This book should be in the library of anyone doing research in 3-dimensional topology.

Hugh M. Hilden

From MathSciNet, March 2020

MR0551744 57N10

Johannson, Klaus

Homotopy equivalences of 3-manifolds with boundaries. (English)

Lecture Notes in Mathematics, 761.

Springer, *Berlin*, 1979, ii+303 pp., \$18.00, ISBN 3-540-09714-7

Much of what we now know about the structure and classification of 3-manifolds has its origins in the work of Haken and Waldhausen in the 1960s and particularly in the result of F. Waldhausen [Ann. of Math. (2) **87** (1968), 56–88; MR0224099] that a boundary-preserving homotopy equivalence between two 3-manifolds is homotopic to a homeomorphism provided the manifolds are compact, orientable, irreducible, boundary irreducible and sufficiently large in the sense of containing a 2-sided incompressible surface. Such manifolds are called Haken manifolds. They are $K(\pi, 1)$ spaces, and thus homotopy equivalences correspond to isomorphisms of their fundamental groups. So in the case of closed manifolds the homeomorphisms correspond precisely to the isomorphisms of the corresponding fundamental groups. However, in the case of nonempty boundary, the situation is more complicated. The boundary-preserving homotopy equivalences correspond to the isomorphisms of the fundamental groups which preserve the peripheral subgroup structure—a difficult property to handle. Moreover, exotic homotopy equivalences (which cannot be deformed to preserve the boundary) are known to exist. Examples of exotic homotopy equivalences between 2-manifolds are easily constructed by "flipping" at an arc, and this idea extends in dimension three to flipping at an annulus. The homotopy equivalence between the knot spaces of the square knot and the granny knot is of this form. It turns out that these examples are typical of the exotic homotopy equivalences between Haken manifolds, and a fairly complete account of all the homotopy equivalences between Haken manifolds can be given. This is the subject of the work under review.

The background for this work comes from the brief announcement by Waldhausen [Proceedings of the International Symposium on Topology and its Applications (Herceg-Novi, 1968), pp. 331–332, Savez Društ. Mat. Fiz. Astronom., Belgrade, 1969], which contained statements of two theorems, known as the annulus and torus theorems. These assert that if a 3-manifold M admits a map $f\colon (A, \partial A) \to (M, \partial M)$ of an annulus [a torus] which is essential in the sense that $f_*\colon \pi_1(A) \to \pi_1(M)$ is monic and f is not homotopic, $\operatorname{rel} \partial A$, to a map into ∂M then M admits an essential embedding of an annulus [an annulus or a torus]. These theorems had clear relevance to the analysis of homotopy equivalences between 3-manifolds—particularly knot spaces—and considerable activity ensued in the effort to understand them. In the process, a significant generalization, the characteristic submanifold theorem, was developed by the author and, independently and by rather different means, by W. Jaco and P. Shalen [Mem. Amer. Math. Soc. **21** (1979), no. 220; MR0539411]. This theorem gives the existence and uniqueness, up to isotopy, in a Haken 3-manifold M of a codimension-zero submanifold V (the characteristic submanifold) whose components are either Seifert fibered spaces meeting ∂M in a union of fibers or 1-bundles meeting ∂M in the corresponding 0-sphere bundles and which satisfies the enclosing theorem: Every essential map of an annulus, a torus or, more generally, a Seifert fibered space or 1-bundle into M can be deformed into V. While the point of this work is in the application of the characteristic submanifold theorem to the analysis of homotopy equivalences, it should be mentioned that it has important applications in other areas—in particular, in conjunction with the work of Thurston on geometric structures on 3-manifolds. As it turns out, it is the components of V and of $\overline{M - V}$ which admit geometric structures.

The main result is the classification theorem: Let M_1, M_2 be Haken 3-manifolds with boundary and let V_1, V_2 be their characteristic submanifolds; then any homotopy equivalence $f\colon M_1 \to M_2$ can be deformed to a map $f_1\colon M_1 \to M_2$ which takes $\overline{M_1 - V_1}$ homeomorphically to $\overline{M_2 - V_2}$ and takes V_1 to V_2 by a homotopy equivalence. As an immediate corollary: If M_1 and M_2 are simple ($\equiv V_1$ and V_2 are trivial) then every homotopy equivalence $M_1 \to M_2$ is homotopic to a homeomorphism. The analysis is extended to a study of the mapping class group $H(M)$ of isotopy classes of self-homeomorphisms of M. For M a simple 3-manifold, $H(M)$ is shown to be finite. For M a general Haken manifold the analysis splits into a study of the subgroups of $H(V)$ and $H(M-V)$ of elements which extend to M, the first of which can be handled by explicit calculations, and it is shown that there is a normal subgroup of finite index in $H(M)$ generated by a finite set of Dehn twists along annuli and tori. It is also shown that there are only finitely many manifolds in the homotopy type of any Haken manifold.

The basic idea behind the proof of the classification theorem is the introduction of the concept of boundary pattern. Roughly speaking, a boundary pattern for an n-manifold ($n \leq 3$) is a collection of $(n-1)$-manifolds in ∂M, the intersection of any k of which is an $(n-k)$-manifold. When splitting a 3-manifold along an incompressible surface [or a 2-manifold along a curve] the copies of the surface [curve] become part of the resulting boundary pattern. When one deals with the (simpler) cut open manifold, one keeps track of the boundary pattern so as to be able to transfer information back to the original manifold. Thus one uses arguments by induction on a hierarchy for a Haken manifold done in the category of manifolds with boundary pattern. In fact everything is done in this category—including the major results, though we have suppressed this from our discussion for simplicity. Much of the classical material from 2- and 3-dimensional manifold theory is reviewed in the context of manifolds with boundary pattern; so, for example, there is a loop theorem with boundary pattern. This introduces a considerable amount of technical difficulty, but the approach has been fruitful in one important area of 3-manifold topology and may well be fruitful in others. There are now, however, proofs of the classification theorem which do not use boundary patterns [Jaco, *Lectures on 3-manifold topology*, Amer. Math. Soc., Providence, R.I., 1980; MR0565450].

The book contains proofs of the characteristic submanifold theorem, the enclosing theorem (including the annulus and torus theorems) as well as the classification theorem and its consequences discussed above. The introduction contains a nice summary of the main ideas as well as a detailed outline of the chapters.

John Hempel

From MathSciNet, March 2020

MR0855294 (88g:57014) 57N10

Thurston, William P.

Hyperbolic structures on 3-manifolds. I. Deformation of acylindrical manifolds.

Annals of Mathematics. Second Series **124** (1986), *no.* 2, 203–246.

This is the first in a long-awaited series of papers in which the author plans to give the proof of his theorem that the interior of an atoroidal Haken 3-manifold admits a complete hyperbolic structure. At the end of the series, he also plans to give the proof that his Geometrisation Conjecture holds for compact irreducible 3-orbifolds with a singular set which is neither empty nor 0-dimensional.

Let M be a Haken 3-manifold. Then M contains an incompressible surface F and cutting M along F yields a Haken manifold M_1 (which may not be connected). One can repeat this cutting procedure to eventually obtain M_n, which is a disjoint union of 3-balls. The minimum value of n taken over all such cutting sequences is called the length of M. The author's hyperbolisation result for Haken 3-manifolds is proved by induction on the length, as are most results about Haken manifolds. For the induction step, one has a Haken manifold M_1, whose interior admits a complete hyperbolic structure, and one wants to show that the manifold M obtained from M_1 by glueing two copies of F_1 in ∂M_1 also admits a hyperbolic structure. The property that M is atoroidal is crucial here. The argument consists of analysing the space of deformations of the hyperbolic structure on M_1, called $\mathrm{AH}(M_1)$, and showing that there is a deformation such that one can glue the new structure to obtain a hyperbolic structure on M. The existence of such a deformation is equivalent to the existence of a fixed point for a certain map from $\mathrm{AH}(M_1)$ to itself. If $\mathrm{AH}(M_1)$ is compact, it is less difficult to show that the required fixed point exists.

In the paper under review, the author shows that if M is an acylindrical 3-manifold then $\mathrm{AH}(M)$ is compact, where acylindrical means that ∂M is incompressible and that any proper map of the annulus A to M, which injects $\pi_1(A)$, is properly homotopic into ∂M. This is a generalization of Mostow's rigidity theorem, which asserts that if M has finite volume, than $\mathrm{AH}(M)$ consists of at most one point. The author's arguments are basically geometric. He uses the uniform injectivity of pleated surfaces, a result which he proves after giving a brief introduction to the theory of pleated surfaces.

There is also a largely algebraic proof that $\mathrm{AH}(M)$ is compact due to J. W. Morgan and P. Shalen [same journal (2) **120** (1984), no. 3, 401–476; MR0769158].

G. Peter Scott

From MathSciNet, March 2020

MR0972700 (90k:14058) 14N10; 14C17

Aluffi, Paolo

The enumerative geometry of plane cubics. I. Smooth cubics.

Transactions of the American Mathematical Society **317** (1990), *no.* 2, 501–539.

Let the base field be algebraically closed and of characteristic different from 2 or 3. The author computes the ten characteristic numbers of smooth plane cubics. Such characteristic numbers are the numbers $N(n_p, n_l)$ of nonsingular cubics passing through n_p general points and simultaneously tangent to n_1 lines in general position, where $n_p + n_l = 9$. In order to effect such computations, the author compactifies the $\mathbf{P}^9$ parametrizing all plane cubics via a sequence of five blow-ups over this $\mathbf{P}^9$, each blow-up having a smooth center. This blow-up sequence begins with the center $B_0 \subset \mathbf{P}^9$ that parametrizes cubics that are triple lines (i.e., the image of the Veronese triple embedding of $\check{\mathbf{P}}^2$ in $\mathbf{P}^9$). Beyond this stage, the idea is to use subsequent blow-ups in order to separate the so-called line conditions. (These are the proper transforms of the hypersurfaces in $\mathbf{P}^9$ that represent cubics tangent to a given line.) The fifth blow-up, denoted $\tilde{V}$, is then the desired smooth variety parametrizing complete cubics in the sense that the intersection of all the line conditions may be seen to be empty (Proposition 5.3). Then the characteristic number $N(n_p, n_l)$ is simply $\int \tilde{P}^{n_p} \cdot \tilde{L}^{n_l}$ where $\tilde{P}$ is a general point condition (a divisor in $\tilde{V}$) and $\tilde{L}$ a general line condition (also a divisor). In §5, the author also uses this construction to give numbers built from codimension-2 conditions representing cubics tangent to a given line at a given point. These results, as well as those for the characteristic numbers, have classical roots in the work of Maillard and Zeuthen.

Given a suitable variety $\tilde{V}$ of complete plane curves of degree d, the techniques described above could in principle be used to calculate characteristic numbers of smooth curves of any given degree. Indeed, the author, in a forthcoming paper ["Two characteristic numbers for smooth plane curves of any degree", same journal, to appear], computes $N(n_p, n_l)$, for $n_l = 2d-1$, $2d$ and $n_p + n_l = d(d+3)/2$. The technical details and the intricacies of the relevant intersection theory are very involved, thus making it difficult to obtain further characteristic numbers.

Susan J. Colley

From MathSciNet, March 2020

MR1103035 (92f:14055) 14N10; 14C17

Aluffi, Paolo

The enumerative geometry of plane cubics. II. Nodal and cuspidal cubics.

Mathematische Annalen **289** (1991), *no.* 4, 543–572.

The author calculates the characteristic numbers for six families of nodal and cuspidal plane cubics (i.e., the number of cubics in a particular family that are tangent to l general lines and pass through p general points, where $l + p$ is the dimension of the variety parametrizing the given family). The spirit of the work is similar to the author's computation of the characteristic numbers of smooth plane cubics [Part I, Trans. Amer. Math. Soc. **317** (1990), no. 2, 501–539; MR0972700]: a family of "complete cubics" is constructed via a sequence of five blowups of the $\mathbf{P}^9$ that parametrizes all cubics. The characteristic numbers are then obtained using some deftly applied intersection theory.

An interesting and important feature of the author's approach is that it gives a way of calculating characteristic numbers without resorting to finding the characteristic numbers of families that are degenerations of the family under consideration. This stands in contrast to the classical work of Zeuthen and Maillard and its contemporary, rigorous treatment in the work of S. L. Kleiman and R. Speiser [in *Proceedings of the 1984 Vancouver Conference in Algebraic Geometry*, 227–268, Amer. Math. Soc., Providence, RI, 1986; MR0846022; in *Algebraic geometry* (Sundance, UT, 1986), 156–196, Lecture Notes in Math., 1311, Springer, Berlin, 1988;

MR0951646; "Enumerative geometry of smooth plane cubics", in *Proceedings of the 1989 Zeuthen Symposium*, Contemp. Math., Amer. Math. Soc., Providence, RI, to appear] and J. M. Miret and S. Xambó-Descamps [in *Algebraic curves and projective geometry* (Trento,1988), 195–234, Lecture Notes in Math., 1389, Springer, Berlin, 1989; MR1023399]. Thus the author provides a truly twentieth-century verification of the nineteenth-century results.

Susan J. Colley

From MathSciNet, March 2020

MR1896179 (2003f:14056) 14L30; 32M12, 57S20

Luna, D.

Variétés sphériques de type A.

Publications Mathématiques Institut de Hautes Études Scientifiques (2001), *no.* 94, 161–226.

All varieties will be defined over an algebraically closed field of characteristic zero. Let G be a connected reductive group acting on a variety X. If one aims at classifying G-varieties one first needs a good notion for the "size" of X. In the past, several notions have been proposed, like the dimension of X or the dimension of its orbit space, but most fail to produce an interesting theory.

Already in 1959, I. M. Gel′fand and M. I. Graev [Trudy Moskov. Mat. Obšč. **8** (1959), 321–390; addendum; ibid. **9** (1959), 562; MR0126719] noticed that the dimension of X/B may be a good measure, where B is a Borel subgroup of G. They called this number the class of X. More precisely, since X/B is in most cases not a variety, the class of X is defined to be the codimension of a generic B-orbit.

This concept lay dormant until the fundamental paper of Luna and T. Vust [Comment. Math. Helv. **58** (1983), no. 2, 186–245; MR0705534], where it was rediscovered and called the complexity of X. In that paper it was demonstrated that G-varieties of complexity d behave roughly like ordinary varieties of dimension d together with some extra discrete structure.

In this scheme, G-varieties of complexity zero are singled out. These varieties have (by definition) an open B-orbit and are called spherical. This time, this concept includes many if not most of the interesting examples that occur "in nature". For example, all generalized flag varieties are spherical and so are all symmetric varieties (the complexifications of symmetric spaces). Moreover, research in the last twenty years has shown that much of the beautiful theory of symmetric and flag varieties can be carried over to spherical varieties.

On the other hand, spherical varieties are so numerous that it was (and maybe still is) doubtful that one would ever get a hand on all of them. According to the philosophy above, spherical varieties should be completely describable by some sort of discrete data. Nevertheless, it is not even known whether spherical varieties may appear in nontrivial families.

The paper under review is a breakthrough in this direction. It solves the above problems for groups of type A, i.e., where G is a quotient of a product of $\mathbf{G}_m$'s and SL's. More precisely, the author classifies spherical varieties for groups of type A in terms of combinatorial structures called "spherical data".

Of course, the paper is rooted in previous work. First, if the open orbit of X is given then X can be described by a combinatorial object called "colored fan" (a generalization of a fan used to classify toric varieties [see G. Kempf et al., *Toroidal* *embeddings. I*, Lecture Notes in Math., 339, Springer, Berlin, 1973; MR0335518]). This is one of the main results of the aforementioned paper by Luna and Vust and essentially reduces the classification task to "X homogeneous".

Moreover, several important subclasses of spherical varieties had been classified beforehand, most notably: (i) for X affine and homogeneous by M. Krämer [Compositio Math. **38** (1979), no. 2, 129–153; MR0528837] (G simple), I. V. Mikityuk [Mat. Sb. (N.S.) **129(171)** (1986), no. 4, 514–534, 591; MR0842398] and M. Brion [Compositio Math. **63** (1987), no. 2, 189–208; MR0906369] (remaining cases); (ii) for X of rank one by D. N. Akhiezer [Ann. Global Anal. Geom. **1** (1983), no. 1, 49–78; MR0739893]; see also [M. Brion, in *Group actions and invariant theory (Montreal, PQ, 1988)*, 31–41, Amer. Math. Soc., Providence, RI, 1989; MR1021273] (the rank of a spherical variety is the codimension of a generic $U = (B,B)$-orbit); (iii) for X of rank two by B. Wasserman [Transform. Groups **1** (1996), no. 4, 375–403; MR1424449].

These classifications are used extensively in the paper under review.

The paper also uses a reduction step as follows. Assume $X = G/H$ is homogeneous. The complement of the open B-orbit consists of irreducible divisors $D_1, \dots, D_s$ (these are often called colors). The normalizer $N_G(H)$ acts on X and permutes the D_i. The kernel $\overline{H}$ of this permutation action is called the spherical closure of H. In the last two paragraphs of the paper the classification problem is reduced to "H spherically closed", i.e., $\overline{H} = H$.

Assuming now $\overline{H} = H$, the author can invoke a theorem of the reviewer [J. Amer. Math. Soc. **9** (1996), no. 1, 153–174; MR1311823] (important previous work by Brion [J. Algebra **134** (1990), no. 1, 115–143; MR1068418] is acknowledged here) to the effect that G/H has a unique equivariant compactification $\overline{X}$ which is a wonderful variety. By definition, this is a smooth projective G-variety with a G-invariant normal crossing divisor $E_1 \cup \dots \cup E_r$ such that the 2^r intersections $E_{i_1} \cap \dots \cap E_{i_t}$ are precisely the closures of the G-orbits. The bulk of the paper (§3–§5) is devoted to the classification of wonderful varieties for groups of type A.

At that point we can describe the combinatorial structure attached to a wonderful variety. We start with the parabolic subgroup P which is the stabilizer of the open B-orbit. Consider a P-semi-invariant rational function f on $\overline{X}$. This means there is a character $\chi_f \in X^*(P)$ with $f(p^{-1}x) = \chi_f(p)f(x)$ for all $p \in P$, $x \in \overline{X}$. Let Ξ be the group of characters obtained this way. If $\chi_f = 1$ then f is P-invariant, hence constant. This shows that χ_f determines f uniquely up to a constant.

Consider the sets $\mathcal{C} = \{\overline{D}_1, \dots, \overline{D}_s\}$ of closures of colors and $\mathcal{B} = \{E_1, \dots, E_r\}$ of boundary components. Then $\mathcal{B} \cup \mathcal{C}$ is the set of all P-invariant irreducible divisors in $\overline{X}$. Hence, the principal divisor (f) attached to a P-semi-invariant rational function f is a linear combination of elements of $\mathcal{B} \cup \mathcal{C}$. This way we get a homomorphism

$$\psi\colon \Xi \to \mathbb{Z}^{\mathcal{B}} \oplus \mathbb{Z}^{\mathcal{C}} \colon \chi_f \mapsto (f).$$

As it turns out, the composed map $\Xi \to \mathbb{Z}^{\mathcal{B}}$ is an isomorphism (in this step the smoothness of $\overline{X}$ is decisive). Let $\Sigma \subset \Xi \subset X^*(P)$ correspond to $\mathcal{B}$ under this isomorphism (elements of Σ are called spherical roots). Then ψ reduces to an integral $s \times r$-matrix A describing the map $\mathbb{Z}^{\mathcal{B}} \cong \Xi \to \mathbb{Z}^{\mathcal{C}}$.

Now we have a triple (P, Σ, A) where P is a parabolic subgroup of G, Σ is a finite subset of $X^*(P)$ and A is a matrix with integral coefficients. This triple is basically the spherical system attached to $\overline{X}$. The main result of the paper is: If G is of type A then a wonderful G-variety is uniquely determined by its spherical system, and, amazingly, there is a complete axiomatic characterization of those systems which correspond to wonderful varieties.

This is not how the result is stated in the paper, though. Before reading the paper under review one should also consult the paper [in *Algebraic groups and Lie groups*, 267–280, Cambridge Univ. Press, Cambridge, 1997; MR1635686] by the same author in which the tools for the present paper have been forged. Among other things, it is shown that Σ determines a large part of the matrix A. Therefore, the main theorem is stated in terms of more "economical" data.

Also, the set Σ itself is by no means arbitrary. As a matter of fact, only finitely many characters of B can be spherical roots. It is important to know which ones, and at this point one of the aforementioned classifications (namely Akhiezer's) enters in a crucial way.

It turns out that the conditions on a spherical system are so stringent that they allow only finitely many possibilities. This proves that a group of type A can act on only finitely many wonderful varieties, a fact which seems to be unknown for other groups.

For the proof, the author establishes five reduction techniques called "localization", "quotient", "parabolic induction", "decomposition as fiber product", and "projective fibration" for both the class of wonderful varieties and the class of spherical systems. Thus, the author ends up with non-reducible objects on both sides which he calls primitive. Then the classification is finished off by enumerating the primitive types (there are 29 cases to consider) and showing that they are in bijection.

The power of the theory described above has already led to the solution of two longstanding problems, at least for groups of type A. The first is Brion's conjecture on the normality of the Demazure embedding. This has now been solved for groups of type A by Luna himself [J. Algebra **258** (2002), no. 1, 205–215]. The other is a conjecture by the reviewer on the algebra structure of smooth affine spherical varieties, which has been solved by R. Camus ["Variétés sphériques affines lisses", thèse de doctorat, Inst. Fourier, Grenoble, 2001]. This implies in turn a conjecture of T. Delzant on multiplicity-free Hamiltonian manifolds [Ann. Global Anal. Geom. **8** (1990), no. 1, 87–112; MR1075241], again all for groups of type A.

The author expresses the hope that the results of the paper will carry over to all connected reductive groups. The uniqueness part may be used verbatim, while the axiomatic characterization of spherical systems certainly has to be refined.

Friedrich Knop

From MathSciNet, March 2020

MR2348953 (2009h:37090) 37F10; 30D05, 37-02

Milnor, John

On Lattès maps.

Dynamics on the Riemann sphere, 9–43, *Eur. Math. Soc., Zürich*, 2006.

In the summary the author writes that this is "[a]n exposition of the 1918 paper of S. Lattès [C. R. Acad. Sci. Paris **166** (1918), 26–28; JFM 46.0522.01], together with its historical antecedents, and its modern formulations and applications". In particular, the author provides a complete classification of Lattès's maps up to conformal conjugacy, providing a complete set of invariants.

A rational map $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ of degree at least 2 of the Riemann sphere is called a Lattès map if $f = \Theta \circ L \circ \Theta^{-1}$, where Θ is a holomorphic map $\Theta\colon \mathbb{T} \to \widehat{\mathbb{C}}$, $\mathbb{T} = \mathbb{C}/\Lambda$ is a torus for a rank-2 lattice $\Lambda \subset \mathbb{C}$, and $L(z) = az + b$ is an affine self-map of $\mathbb{T}$.

More explicitly, $f = L/G_n\colon \mathbb{T}/G_n \to \mathbb{T}/G_n$, where G_n is the group of n-th roots of unity acting on $\mathbb{T}$ by rotation around a base point, with n equal to either 2, 3, 4, or 6, and L commutes with a generator of G_n.

Θ can be expressed in terms of elliptic functions; for example, it is the Weierstrass function $\wp$ for $n = 2$. Every Lattès map is postcritically finite. Moreover (in the absence of exceptional points) it is characterized by having a parabolic orbifold or can be linearized by a flat metric (with singularities at postcritical points), i.e. f is affine in this metric. This is the only case where the measure of maximal entropy is absolutely continuous with respect to Lebesgue measure [A. Zdunik, Invent. Math. **99** (1990), no. 3, 627–649; MR1032883].

For $\phi\colon \mathbb{C} \to \mathbb{C}/\Lambda$ and $\theta = \Theta \circ \phi$, the group of affine maps (rigid Euclidean motions) g of $\mathbb{C}$ such that $\theta = \theta \circ g$ is a crystallographic group $\tilde{G}_n$. The author presents all possible related tilings of $\mathbb{C}$ and their ramification indices at singularities of the corresponding orbifold geometry.

A Lattès map is called flexible if one can vary Λ and L continuously so as to obtain other Lattès maps which are not conformally conjugate to it. The only flexible Lattès maps have $n = 2$ and L' real. Then there exist invariant line fields. Another characterization: multipliers of all periodic orbits are integers. Lattès maps can be with or without postcritical fixed points, depending on b, e.g. $\Lambda = \mathbb{Z}^2$ and $L(z) = z + 1/2$.

Several recent developments are discussed. In particular, integrable maps [see also G. M. Levin and F. Przytycki, Proc. Amer. Math. Soc. **125** (1997), no. 7, 2179–2190; MR1376996], characterization by multipliers at periodic orbits, and higher dimensional analogues. A theory by Schröder, preceding Lattès, is recalled. Finally, degree-2 Lattès maps are discussed in more detail. "A beautifully symmetric example", one of whose cases is $f(z) = 6z/(z^3 - 2)$ in appropriate coordinates, is analyzed. This is a beautiful expository article.

Feliks Przytycki
From MathSciNet, March 2020

MR2483934 (2010d:14036) 14H10; 14C20, 14E08, 14E30
Farkas, Gavril
The global geometry of the moduli space of curves.
Algebraic geometry—Seattle 2005. Part 1, 125–147, *Proc. Sympos. Pure Math., 80, Part 1, Amer. Math. Soc., Providence, RI*, 2009.

The article under review is a comprehensive account of the current state of knowledge (as of 2005/2006) of the ample and effective cones of the moduli spaces $\overline{\mathcal{M}}_{g,n}$ of stable curves of genus g with n marked points. From the point of view of birational geometry, these cones are fundamental objects. The F-conjecture of Gibney, Keel, and Morrison predicts that the ample cone is dual to the cone spanned by the irreducible components of the locus of stable (marked) curves with $3g - 4 + n$ nodes. In particular, their conjecture, which has been confirmed in numerous cases, predicts that the Mori cone $\mathrm{NE}(\overline{\mathcal{M}}_{g,n})$, comprising formal linear combinations of irreducible curves with real coefficients modulo numerical equivalence, is polyhedral, spanned by the classes of rational curves. The state of affairs for the effective cone $\mathrm{Eff}(\overline{\mathcal{M}}_{g,n})$ is more complicated, even when $n = 0$. Indeed, Farkas, Khosla, and Popa have shown the Slope Conjecture of Harris and Morrison, which predicts that $\mathrm{Eff}(\overline{\mathcal{M}}_g)$ is spanned by Brill–Noether divisors, to be false. Still, the author proposes that a weakened version of the Slope Conjecture continues to hold, namely, that a uniform lower bound on the slopes of effective divisors on $\overline{\mathcal{M}}_g$ exists which is independent of g. If true, this would highlight a major difference between $\mathcal{M}_g$ and the moduli space $\mathcal{A}_g$ of genus-g abelian varieties. A proof of the (weak) Slope Conjecture would also yield a novel approach to the Schottky problem of determining equations for the Torelli embedding of $\mathcal{M}_g$ in $\mathcal{A}_g$.

Ample divisors. The study of ample divisors on $\overline{\mathcal{M}}_g$ dates back to the work of Arbarello and Harris from the late 1980s, who proved that the $\mathbb{Q}$-divisor class $a\lambda - \delta_0 - \cdots - \delta_{\lfloor g/2 \rfloor}$ is ample if and only if $a > 11$. Shortly afterwards, Faber determined the defining inequalities for $\mathrm{Ample}(\overline{\mathcal{M}}_3)$, as a subspace of the span of λ, δ_0, and δ_1. His work provided the first clue that F-curves might generate $\mathrm{NE}(\overline{\mathcal{M}}_{g,n})$.

A. Gibney, S. Keel and I. Morrison [J. Amer. Math. Soc. **15** (2002), no. 2, 273–294; MR1887636] showed that $\mathrm{NE}(\overline{\mathcal{M}}_{g,n})$ is the sum of the cone spanned by F-curves and $\mathrm{NE}(\overline{\mathcal{M}}_{0,g+n})$. (Here the "flag map" that attaches elliptic tails at g marked points realizes $\overline{\mathcal{M}}_{0,g+n}$ as a subvariety of $\overline{\mathcal{M}}_{g,n}$.) In so doing, they reduced the determination of the cone of curves of $\overline{\mathcal{M}}_{g,n}$ to a problem about rational curves. Their result also has important applications to the study of regular morphisms from $\overline{\mathcal{M}}_{g,n}$ to other projective varieties. It also allows one to show directly any divisor class $a\lambda - \sum_i b_i\delta_i$ that intersects all F-curves non-negatively, and such that $b_i \geq b_0$ for all $1 \leq i \leq \lfloor g/2 \rfloor$, is nef.

The author mentions three other interesting problems related to the determination of the ample cone of the moduli space. The first among these asks for a characterization of all semi-ample line bundles on the moduli space, for example. Keel has given examples of non-nef semi-ample line bundles on $\overline{\mathcal{M}}_{g,1}$, for all $g > 2$. On the other hand, for $g \leq 2$ or $n = 0$, there are no known examples of semi-ample line bundles on $\overline{\mathcal{M}}_{g,n}$ that fail to be nef.

Another natural problem is that of determining the nef cone of (suitably chosen compactification of) $\mathcal{A}_g$. Shepherd-Barron and Hulek–Sankaran computed the nef cone of the first and second Voronoĭ compactifications, respectively.

A third problem is that of furnishing modular interpretations of intermediate models encountered when "running" the (log) minimal model program for $\overline{\mathcal{M}}_g$. More precisely, for any real number $0 \leq \alpha \leq 1$, set

$$\overline{\mathcal{M}}_g^{\mathrm{can}}(\alpha) := \mathrm{Proj}\left(\bigoplus_{n\geq 0} H^0(\overline{\mathcal{M}}_g, n(K_{\overline{M}_g} + \alpha\delta))\right)$$

where $\delta := \sum_{i=0}^{\lfloor g/2 \rfloor} \delta_i$ denotes the total boundary divisor. Here $K_{\overline{M}_g}$ denotes the canonical class of the moduli stack. Work of B. Hassett, D. Hyeon, and Y. Lee [B. Hassett and D. Hyeon, Trans. Amer. Math. Soc. **361** (2009), no. 8, 4471–4489; MR2500894; D. Hyeon and Y. Lee, "Log minimal model program for the moduli space of stable curves of genus three", preprint, `arxiv.org/abs/math/0703093`] has yielded a partial understanding of what the various models $\overline{\mathcal{M}}_g^{\mathrm{can}}(\alpha)$ are, and how they are related to one another. Since this article appeared, M. Simpson ["On log canonical models of the moduli space of stable pointed genus zero curves", Ph.D.

dissertation, Rice Univ., Houston, TX, 2008] and D. I. Smyth ["Modular compactifications of $M_{1,n}$", preprint, `arxiv.org/abs/0808.0177`] have obtained complete descriptions of the analogous modular compactifications for $\overline{\mathcal{M}}_{0,n}$ and $\overline{\mathcal{M}}_{1,n}$, respectively. Smyth ["Towards a classification of modular compactifications of the moduli space of curves", preprint, `arxiv.org/abs/0902.3690v1`] has far-reaching results in the general setting of $\overline{\mathcal{M}}_{g,n}$. Since this article appeared, work of Simpson [op. cit.], V. Alexeev and D. Swinarski ["Nef divisors on $\overline{M}_{0,n}$ from GIT", preprint, `arxiv.org/abs/0812.0778`], and M. Fedorchuk and Smyth ["Ample divisors on moduli spaces of weighted pointed rational curves, with applications to log MMP for $\overline{M}_{0,n}$", preprint, `arxiv.org/abs/0810.1677`] has provided a complete description of the analogous birational models for $\overline{\mathcal{M}}_{0,n}$. In addition, Smyth ["Modular compactifications of $M_{1,n}$", op. cit.] has announced results for $\overline{\mathcal{M}}_{1,n}$.

Effective divisors. A fundamental invariant attached to an effective divisor on $\overline{\mathcal{M}}_g$ is its slope. Whenever the divisor $D \equiv a\lambda - \sum_{i=0}^{\lfloor g/2 \rfloor} b_i\delta_i$ in question is the closure of a divisor on $\mathcal{M}_g$, its slope is the quantity

$$s(D) = a/\min_i b_i.$$

Accordingly, the slope $s(\overline{\mathcal{M}}_g)$ of the moduli space is defined to be the infimum of the slopes of all effective divisors on $\overline{\mathcal{M}}_g$. The slope of the moduli space is intimately related to its birational type; indeed, the calculation of the canonical class of $\overline{\mathcal{M}}_g$ due to Harris and Mumford, coupled with the fact that the Hodge class λ is big and nef, implies that $\overline{\mathcal{M}}_g$ is of general type whenever $s(\overline{\mathcal{M}}_g) < 13/2$. Accordingly, it is important to understand which effective divisors on $\overline{\mathcal{M}}_g$ are of minimal slope. By explicitly computing the classes of non-Brill–Noether–Petri general curves in $\overline{\mathcal{M}}_g$ with the help of the limit linear series technology they had developed, Eisenbud and Harris deduced that $\overline{\mathcal{M}}_g$ is of general type whenever $g \geq 24$.

By contrast, the Slope Conjecture predicts that the Kodaira dimension of $\overline{\mathcal{M}}_g$ is $-\infty$ for all $g < 23$. Whenever $g \leq 16$, this is known to be true. Indeed, Severi showed that $\overline{\mathcal{M}}_g$ is unirational whenever $g \leq 10$. Work of M.-C. Chang and Z. Ran [Invent. Math. **76** (1984), no. 1, 41–54; MR0739623] and E. Sernesi [Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **8** (1981), no. 3, 405–439; MR0634856] yielded unirationality for $11 \leq g \leq 13$. A. Verra [Compos. Math. **141** (2005), no. 6, 1425–1444; MR2188443] recently proved that $\mathcal{M}_{14}$ is unirational. Verra uses Mukai's result that a general canonical genus-8 curve is a linear section of $\mathbb{G}(2,6)$ in its Plücker embedding. In this way, he deduces that the Hilbert scheme of degree-14, genus-8 curves in $\mathbb{P}^6$ is unirational; the rest of the argument is Brill–Noether theory.

A. Bruno and Verra [in *Projective varieties with unexpected properties*, 51–65, Walter de Gruyter, Berlin, 2005; MR2202246] have also recently proved that $\overline{\mathcal{M}}_{15}$ is unirational; Chang and Ran [J. Differential Geom. **34** (1991), no. 1, 267–274; MR1114463] proved that $\overline{\mathcal{M}}_{16}$ has Kodaira dimension $-\infty$ around 1990. Around 2000, the author [Math. Ann. **318** (2000), no. 1, 43–65; MR1785575] proved that $\overline{\mathcal{M}}_{23}$ has Kodaira dimension at least 2. Here, the author sketches a proof that $\overline{\mathcal{M}}_{22}$ is of general type. His method involves constructing new examples of divisors of low slope associated to linear series with codimension-1 syzygetic behavior.

Most of the latter half of the paper is devoted to a discussion of recent results (largely due to the author) related to the construction of syzygetic counterexamples to the original Slope Conjecture. The author's approach involves constructing maps of vector bundles over a space of (limit) linear series, which in turn projects naturally to a partial compactification of $\mathcal{M}_g$, and stipulating that these maps have suitable ranks. When these ranks are chosen appropriately, the projected images of the resulting determinantal schemes extend to determinantal divisors on $\overline{\mathcal{M}}_g$. In this way, the author obtains "Koszul" divisors on $\overline{\mathcal{M}}_g$ which recover all known counterexamples to the Slope Conjecture. Indeed, the fact that $\overline{\mathcal{M}}_{22}$ is of general type results from the existence of a divisor D_{22} on $\mathcal{M}_{22}$ associated to curves C equipped with complete g^6_{25}'s associated to line bundles L for which the multiplication maps

$$\mathrm{Sym}^2 H^0(C,L) \to H^0(C,L^{\otimes 2})$$

fail to be injective. Moreover, the author notes that the analogous locus $D_{23} \subset \mathcal{M}_{23}$ corresponding to curves with g^6_{26}'s should also be a divisor with slope less than $13/2$ (which would imply that $\overline{\mathcal{M}}_{23}$ is of general type), but for the moment he is unable to prove that D_{23} is properly contained in $\mathcal{M}_{23}$. The difficulty here comes from the fact that the expected codimension of D_{23} inside of the space of linear series $\mathcal{G}^6_{26}$ is 3 (instead of 2 in the case of $D_{22} \subset \mathcal{G}^6_{25}$). Note that the author proved around 10 years ago that the Kodaira dimension of $\overline{\mathcal{M}}_{23}$ is at least 2.

Since this article was written, there have been new developments in the search for lower bounds on $s(\overline{\mathcal{M}}_g)$. In two papers ["Covers of elliptic curves and the lower bound for slopes of effective divisors on $\overline{\mathcal{M}}_g$", preprint, `arxiv.org/abs/0704.3994`; "Simply branched covers of an elliptic curve and the moduli space of curves", preprint, `arxiv.org/abs/0806.0674`], D. Chen has obtained combinatorial formulas for the slopes of moving curves (i.e., curves whose deformations cover an open subset of $\overline{\mathcal{M}}_g$) obtained from covers of elliptic curves. Note that Harris and Morrison had done analogous calculations for moving curves arising from simply branched covers of $\mathbb{P}^1$, and their computer experimentation suggested $s_g \geq O(1/g)$. I. Coskun, J. D. Harris and J. M. Starr [Canad. J. Math. **61** (2009), no. 1, 109–123; MR2488451] studied canonical curves and obtained sharp bounds on s_g for $g \leq 6$. Fedorchuk ["Linear sections of the Severi variety and moduli of curves", preprint, `arxiv.org/abs/0710.1623`] has obtained recursive formulas for the slopes of moving curves in moduli arising from Severi varieties, and obtained some new bounds when $g \leq 20$. In each of the preceding cases, the g-asymptotics of the formulas obtained are difficult to analyze. On the other hand, via a descendant integral calculation on $\overline{\mathcal{M}}_{g,n}$, R. Pandharipande ["Descendent bounds for effective divisors on the moduli space of curves", preprint, `arxiv.org/abs/0805.0601v1`] has deduced the exact asymptotic $s_g \geq \frac{60}{g+4}$.

The author concludes with a discussion of what is currently known about the Kodaira dimension of $\overline{\mathcal{M}}_{g,n}$ for $n \geq 1$, as well as a number of interesting questions related to hyperbolicity properties of $\overline{\mathcal{M}}_g$. For example, the author proposes determining a useful lower bound for $\gamma_g := \inf\{g(\Gamma) : \Gamma \subset \overline{\mathcal{M}}_g$ is a curve passing through a general point $[C] \in \overline{\mathcal{M}}_g\}$.

Ethan G. Cotterill

From MathSciNet, March 2020

MR3619765 57R18; 58A40

Watts, Jordan

The differential structure of an orbifold.

The Rocky Mountain Journal of Mathematics **47** (2017), *no.* 1, 289–327.

In this paper, the author proves that the underlying set of an orbifold equipped with the ring of smooth real-valued functions determines the orbifold atlas. A consequence of his work is the following main result.

Theorem 1. Given an orbifold, its atlas can be constructed out of invariants of the differential structure.

The main idea behind the proof of this theorem is to construct an appropriate orbifold, which has three ingredients: the topology, the orbifold stratification and the order of points at codimension-2 strata. We will see that all three ingredients are invariants of the orbifold differential structure.

As for the stratification of an orbifold, the author reviews stratified spaces in view of subcartesian spaces. See [T. Lusala and J. Śniatycki, Canad. Math. Bull. **54** (2011), no. 4, 693–705; MR2894519; J. Śniatycki, *Differential geometry of singular spaces and reduction of symmetry*, New Math. Monogr., 23, Cambridge Univ. Press, Cambridge, 2013 (Chapter 4); MR3112888] for more details. Also, [M. J. Pflaum, *Analytic and geometric study of stratified spaces*, Lecture Notes in Math., 1768, Springer, Berlin, 2001; MR1869601] contains a general introduction to stratified spaces. The discussions of the stratified spaces give us the following results.

Theorem 2. Let X be an orbifold. Then, the orbifold stratification is given by the set of orbits $\mathcal{O}_X$ induced by $\mathrm{vect}(X)$.

This theorem says that the orbifold stratification is induced by the family of vector fields on the orbifold. It uses some vector field theory on subcartesian spaces developed by Śniatycki [op. cit.].

Corollary 1. The orbifold stratification is an invariant of the orbifold differential structure.

The order of points at codimension-2 strata is discussed in Section 5, and readers may need some familiarity with orbifold covering space theory, including a notion of *good* and *bad orbifolds*. However, this article contains a brief summary of the theory based on [W. P. Thurston, *Three-dimensional geometry & topology*, Geom. Cent., Univ. Minnesota, Minneapolis, MN, 1991 (Chapter 13)]. In addition, this section discusses the following interesting theorem in Section 5.

Theorem 3. Let X be a connected orbifold. Then, a presentation for the orbifold fundamental group can be constructed using the topology, stratification, and the order of points in codimension-2 strata.

This theorem was proved by [A. Haefliger and Quach Ngoc Du, Astérisque No. 116 (1984), 98–107; MR0755164]. In the paper under review, the author presents an algorithm that offers an alternative to the Seifert–van Kampen theorem for computing an orbifold's fundamental group. Moreover, this section contains two theorems on invariance. Roughly speaking, the first one states that the codimension of the germ of f at 0 is invariant under diffeomorphism, where 0 is a critical point of $f \in C^\infty(\mathbb{R}^n)$. The second theorem says that the order of any point $x \in X$ is an invariant of the orbifold differential structure.

From here, the author applies a method proved by Haefliger and Quach Ngoc Du [op. cit.] to reconstruct the local isotropy groups. Further, he uses an inductive argument on the dimension of the orbifold to reconstruct the charts.

The author devotes the last two sections to expressing the main theorem in terms of a functor. In particular, the last theorem stated below requires familiarity with the category **Diffeol** of diffeological spaces. Related work on diffeologies may be found in [P. Iglesias-Zemmour, Y. Karshon and M. Zadka, Trans. Amer. Math. Soc. **362** (2010), no. 6, 2811–2831; MR2592936; P. Iglesias-Zemmour, *Diffeology*, Math. Surveys Monogr., 185, Amer. Math. Soc., Providence, RI, 2013; MR3025051].

Theorem 4. There is a functor F from the weak 2-category of effective proper étale Lie groupoids with bibundles as arrows to differential spaces that is essentially injective on objects.

Note that *essentially injective* means that given two objects G and H such that $F(G) \cong F(H)$, we have $G \simeq H$, where $\simeq$ indicates *Morita equivalence*.

Theorem 5. There is a functor G from the weak 2-category of effective proper étale Lie groupoids with bibundles as arrows to diffeological spaces that is essentially injective on objects.

In the reviewer's opinion this article is remarkably well organized and well thought-out. The author selects terrific examples which are modeled on $\mathbb{C} \cong \mathbb{R}^2$. More specifically, the author begins with the following familiar example to give us an idea of orbifolds. These are used to illustrate a concept of the stratification and the invariant theorems. Further, the same one is used to show us how the orbifold fundamental group algorithm works. He repeatedly uses the same example throughout this article, which makes it easier to follow the theories presented here.

Example 1. Let $\mathbb{D}_k$ be the dihedral group of order $2k$. The group is generated by maps x and y acting on $\mathbb{C}$, where $x\,z \mapsto \overline{z}$ and $y\,z \mapsto e^{2\pi i/k}\overline{z}$. Further, the cyclic group of order k denoted by $\mathbb{Z}_k$ is generated by xy. Then, $\mathbb{R}^2/\mathbb{D}_k$ and $\mathbb{R}^2/\mathbb{Z}_k$ are orbifolds.

Ryo Ohashi

From MathSciNet, March 2020