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COMPACTIFYING MODULI SPACES

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Abstract. The boundary of some well known algebro-geometric moduli spaces is described by highlighting the recursive combinatorial properties in connection with tropical and non-Archimedean geometry.

Contents

1. Introduction

A characteristic of algebraic geometry is the fact that the sets parametrizing equivalence classes of a certain type of objects (for example, smooth projective curves of given genus up to isomorphism or hypersurfaces of given degree in projective space up to projective equivalence) are themselves endowed with a natural algebraic structure, and are called *moduli spaces*. In fact, the structure of a moduli space is largely governed by the geometric properties of the parametrized objects. This phenomenon, known and used for a long time, has been established on rigorous mathematical ground during the second half of the twentieth century when moduli theory flourished; see [18], [23], [28], [19], [3].

A moduli problem is thus a way of posing a classification problem. Loosely speaking, this amounts to considering a class $\mathcal{C}$ of objects to parametrize up to some equivalence relation and a notion of a *family* of objects, specifying how our objects are allowed to move within $\mathcal{C}$. A family is, essentially, a fibration of objects in $\mathcal{C}$ over an algebraic variety or scheme B, the base of the family. The precise definition depends on the particular moduli problem; the common point is that to any point b in B the fiber of the family over b is an object in $\mathcal{C}$. To say that a moduli problem admits a moduli space is to say that there is a natural algebraic structure on the set M of all equivalence classes on $\mathcal{C}$, which makes M into an algebraic variety or scheme (or a more sophisticated space, like a stack). The algebraic structure on M must satisfy some basic properties. As we said, the points of M are in bijective

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correspondence with the equivalence classes of objects in $\mathcal{C}$. Next, for every family of objects in $\mathcal{C}$ over a base B, the set theoretic map

$$\mu : B \longrightarrow M,$$

sending a point in B to the equivalence class of its fiber in the family, is a morphism of algebraic varieties or schemes (or stacks). One usually refers to μ as the *moduli map* of the family.

Without going into details, let us mention two subtle points. First of all, does a given moduli map determine the family uniquely? Second, is every morphism $B \to M$ the moduli map of some family of objects in $\mathcal{C}$? The answer to such questions is seldom positive. In case it is positive, we say that M is a *fine* moduli space.

Let us look at the geometric structure of our moduli spaces. It turns out that many of them are not complete as topological spaces, or, in algebro-geometric language, they are not projective. This reflects the fact that the objects they parametrize, the elements of $\mathcal{C}$, are bound to degenerate to objects that are not in $\mathcal{C}$. Constructing completions, or compactifications, for moduli spaces has been a major area of research in algebraic geometry since at least the mid-1950s. On the one hand, compactifications of moduli spaces are projective and, hence, more tractable in computations and applications. On the other hand, they usually come with a geometric description and are moduli spaces themselves. This provides new geometric insights and has been a constant source of inspiration for new progress.

In recent years a new interesting connection has been established between the compactification of certain moduli spaces and moduli spaces of polyhedral objects. In loose words, the "skeleton" of some compactified algebro-geometric moduli space M is expressed as the moduli space of the skeleta of the objects parametrized by M. The connection relies on the study of the boundary of the compactification and on its recursive, combinatorial properties, some of which have long been known but are now viewed from a new perspective.

We will survey this area by focusing on the moduli spaces of stable curves and of line bundles on curves. In the first case we have a clear picture of the above-mentioned connection; in the second, partial results are known, but a complete understanding of the connection with polyhedral geometry is not yet available.

2. Compactifying moduli spaces of smooth curves

2.1. **Moduli of smooth and stable curves.** The moduli problem for smooth (connected, projective) curves was perhaps the first moduli problem ever studied. A curve is a projective and connected variety over an algebraically closed field, unless we specify otherwise. Classically, over the field $\mathbb{C}$ a smooth curve is a compact, connected, orientable surface (a two-dimensional topological manifold) endowed with an algebraic structure, and our moduli problem was studied already by Riemann.

Now, $\mathcal{C}$ is the set of all smooth curves of genus g, and the equivalence is the isomorphism. A family is a flat, projective morphism $f : \mathcal{X} \to B$ such that for every point b in B, the fiber $f^{-1}(b)$ is a smooth curve of genus g.

The moduli space of smooth algebraic curves of genus g is an algebraic variety denoted by M_g. As we said, its structure captures some of the properties of the curves it parametrizes. In particular, if $g \geq 1$, it is not a projective variety, since

smooth curves of positive genus are often forced to specialize to singular ones. As we said in the introduction, it is quite important to find compactifications for M_g, and we shall now concentrate on this.

The best known compactification is a projective variety $\overline{M}_g$ whose boundary points (i.e., the points in $\overline{M}_g \smallsetminus M_g$) parametrize singular curves of a very simple type, called *stable* curves, and were introduced for the first time by P. Deligne, A. Meyer, and D. Mumford. A remarkable feature of stable curves (whose definition will come soon) is that they can be characterized inductively using smooth curves of lower genus. In order to do this, we need to extend our range a little and consider not only smooth curves, but smooth curves together with a set of n marked (and ordered) points. Our n has to be chosen so that every curve of genus $g \geq 0$ with n marked points has only finitely many automorphisms fixing the marked points. Thus, we need $n \geq 3$ if $g = 0$ and $n \geq 1$ if $g = 1$; hence, we assume from now on $n > 2 - 2g$.

We denote by $M_{g,n}$ the moduli space of smooth curves of genus g with n marked points. If $n = 0$, we just write M_g.

Example 2.1.1. The simplest cases are $g = 0$ and $n = 3$, which are almost trivial. Indeed $M_{0,3}$ is a point since any smooth curve of genus 0 is isomorphic to $\mathbb{P}^1$, and any two ordered triples of points in $\mathbb{P}^1$ are mapped to one another by an automorphism.

If $g = 0$ and $n = 4$, then $M_{0,4}$ has dimension 1, and, if we work over $\mathbb{C}$, it is not hard to see that $M_{0,4}$ can be identified with $\mathbb{C} \smallsetminus \{0, 1\}$.

The variety $M_{g,n}$, known to be irreducible of dimension $3g-3+n$, is not complete unless $g = 0$ and $n = 3$. We shall now describe its compactification $\overline{M}_{g,n}$ by the moduli space of *stable* n-pointed curves. A stable curve with n marked points is defined as a (connected, projective) curve having only nodes as singularities, plus n (ordered) smooth points on it, and having finitely many automorphisms fixing the marked points. We shall see a more explicit description soon.

Thanks to a series of remarkable achievements (see [15], [21], [16]) we know that that $M_{g,n}$ is an open dense subset in $\overline{M}_{g,n}$, and that $\overline{M}_{g,n} \smallsetminus M_{g,n}$ is a union of strata described in terms of smooth curves of genus at most g with marked points. The strata parametrize curves of a fixed topological type and are described via an important combinatorial tool, the dual graph of a curve, which we shall introduce below.

The basic observation is that to give a curve X with exactly one node N is the same as to give the desingularization X^ν of X and the two branch points of X^ν that get identified in N. More generally, to give a curve having only nodes as singularities is the same as giving its desingularization (a disjoint union of smooth curves, one for each irreducible component of X) and a set of pairs of points, one for each node. To reconstruct X from the data of the smooth curves with marked points, one needs gluing data, which are encoded in the dual graph G_X of X.

G_X will be a so-called *vertex-weighted graph with legs* (i.e., every vertex is weighted by a nonnegative integer, which we call the *genus* of the vertex) and may have some leg (i.e., some half-edge) adjacent to it. For example, in Figure 1 we have a graph with three legs, labeled by l_1, l_2, l_3. In this paper the legs are always ordered and will be fixed by any automorphism. We draw vertices of genus 0 by an empty circle "$\circ$" and vertices of positive genus by a "$\bullet$", with sometimes the

FIGURE 1. On the left is a (weighted) graph with three legs. On the right is the underlying leg-free graph.

genus as subscript. We denote by $[G]$ the graph obtained from G by removing all legs.

To define the dual graph, let $(X; p_1, \ldots, p_n)$ be a nodal curve with n smooth points. Its dual graph G_X has as set of vertices, $V = V(G_X)$, the set of irreducible components of X, and as set of edges, $E = E(G_X)$, the set of nodes of X. An edge joins the (at most two) vertices/components on which the node lies. For every marked point, the graph has a leg attached to the vertex/component in which the marked point lies. Finally, every vertex v of the graph is assigned an integer $g(v)$ equal to the geometric genus of the corresponding component.

The (arithmetic) genus $g(X)$ of the curve X is expressed through the first Betti number $b_1(G_X)$ of its dual graph as

$$g(X) = b_1(G_X) + \sum_{v \in V} g(v) = g(G_X),$$

with $b_1(G_X) = |E| - |V| + c$, where c is the number of connected components of G_X (here, but not always in the rest of this paper, assumed to be 1). The number $g(G_X)$ is defined as the genus of the graph. For example, the graphs of the above picture have genus 2.

The requirement that X is stable translates into (and is equivalent to) a simple requirement on G, namely that vertices of weight 0 (resp., 1) have degree at least 3 (resp., 1). Such graphs are called *stable* and form a finite set denoted by $\mathcal{G}_{g,n}$. In Figure 1 the graph on the left is stable, whereas the graph on the right is not.

Now we can exhibit a stratification of $\overline{M}_{g,n}$. Let $G \in \mathcal{G}_{g,n}$ be a stable graph, let $M_G \subset \overline{M}_{g,n}$ be the locus of curves whose dual graph is G, and it is easy to see that M_G is never empty. We have

$$\overline{M}_{g,n} = \bigsqcup_{G \in \mathcal{G}_{g,n}} M_G. \tag{1}$$

Example 2.1.2. In the case where $g = 0$ and $n = 4$, encountered in Example 2.1.1, it is quite easy to list all stable graphs. Together with the graph having no edges and the four legs attached to its unique vertex, we have the three cases pictured in Figure 2. To each of them there corresponds a unique curve up to isomorphism, made of two smooth components (isomorphic to $\mathbb{P}^1$) intersecting in one node, and each component has two marked points. The three distributions of the four points give the three different isomorphism classes. It is not hard to show that $\overline{M}_{0,4} \cong \mathbb{P}^1$.

From now on, X will denote a connected, projective curve having only nodes as singularities, defined over an algebraically closed field k.

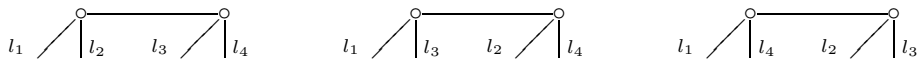


FIGURE 2. The three genus 0, stable graphs with four legs.

2.2. **The strata of $\overline{M}_{g,n}$.** Before analyzing the stratification (1), we study its strata M_G and describe them explicitly. Let $\mathrm{Aut}(G)$ be the automorphism group of G, and recall that elements of $\mathrm{Aut}(G)$ fix every leg of G. For example, the three graphs in Figure 2 have no automorphisms.

Recall what we said above about reconstructing a curve from the union of its (desingularized) components plus the marked points. We shall use it now to introduce the natural morphism

$$\pi : \prod_{v\in V} M_{g(v),\deg(v)} \longrightarrow \Big(\prod_{v\in V} M_{g(v),\deg(v)}\Big)/\mathrm{Aut}(G) \cong M_G, \tag{2}$$

where $\deg(v)$ is the degree of the vertex v. Let X be a curve in M_G, and consider a point in $\pi^{-1}(X)$. This will correspond to the disjoint union of $|V|$ stable pointed (smooth) curves, which maps to X birationally via a surjective *normalization* (or desingularization) morphism, ν:

$$\nu : \bigsqcup_{v\in V} (C_v; p_1, \ldots, p_{\deg(v)}) \longrightarrow X \tag{3}$$

with $(C_v; p_1, \ldots, p_{\deg(v)}) \in M_{g(v),\deg(v)}$. In fact, $\bigsqcup_{v\in V} C_v = X^\nu$ is the desingularization of X mentioned earlier. The map ν glues some pairs in $\bigsqcup_v \{p_1, \ldots, p_{\deg(v)}\}$ to the nodes of X. The p_i which are not glued to anything will correspond to the marked points of X. We think of (3) as a *presentation* of X. Now, $\mathrm{Aut}(G)$ acts on the gluing data, and we may have different presentations of the same curve; see Example 2.2.2.

Example 2.2.1. Let G be the stable graph in Figure 1. We have

$$M_G \cong M_{0,3} \times M_{0,3} \times M_{0,4} \times M_{1,1}$$

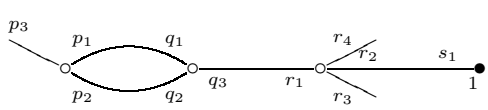
since the action of $\mathrm{Aut}(G)$ on the product is trivial. Indeed, the only nontrivial automorphism of G, interchanging the two edges on the left, acts trivially on $M_{0,3}$ which is a point. The stable curve of genus 2 associated to

$$\Big((C_0; p_1, p_2, p_3),\ (D_0; q_1, q_2, q_3),\ (E_0; r_1, r_2, r_3, r_4),\ (C_1; s_1)\Big)$$

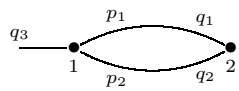
with $C_0 \cong D_0 \cong E_0 \cong \mathbb{P}^1$ and C_1 of genus 1, is given by identifying

$$p_1 = q_1, \qquad p_2 = q_2, \qquad q_3 = r_1, \qquad r_2 = s_1$$

so that the legs correspond to p_3, r_3, r_4. In these identifications we made a choice which, being the same for all curves parametrized by M_G, is irrelevant. Below is a representation of these identifications on the graph.



Example 2.2.2. Consider the graph G in the picture below, stable of genus 4 with one leg. We marked the identifications on the edges. Now $\mathrm{Aut}(G)$ contains the automorphism α, which interchanges the two edges. We shall see that α acts nontrivially.



In this case we have $M_G \cong \frac{M_{1,3}\times M_{2,2}}{\mathrm{Aut}(G)}$, and the action of α on $M_{1,3}\times M_{2,2}$ swaps the two marked points of $M_{2,2}$ and the first two marked points of $M_{1,3}$. Indeed, the following two elements in $M_{1,3}\times M_{2,2}$

$$\big((C_1;p,p',p_3),\ (C_2;q,q')\big), \qquad \big((C_1;p',p,p_3),\ (C_2;q',q)\big)$$

are conjugated by $\mathrm{Aut}(G)$ and give the same point of M_G. In other words, they are different presentations for the same curve.

Example 2.2.3. We now consider $M_{0,3}\times M_{0,3}$ and list the strata M_G presented by it, with $G\in\mathcal{G}_{g,n}$ for all g and n. It is easy to check that $g\le 2$. If $g=0$, we leave it to the reader to check that $n=4$ and to find the corresponding unique (up to labeling the legs) stable graph. The remaining cases are drawn in Figures 3 and 4.

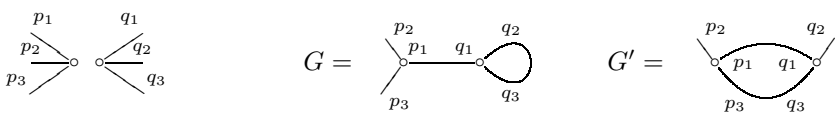


FIGURE 3. Presentation of M_G and $M_{G'}$ in $\overline{M}_{1,2}$ by $M_{0,3}\times M_{0,3}$

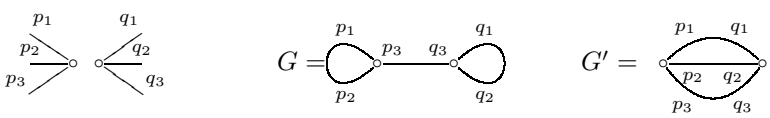


FIGURE 4. Presentation of M_G and $M_{G'}$ in $\overline{M}_2$ by $M_{0,3}\times M_{0,3}$

Let G be a stable graph of genus g with n legs; recall that $[G]$ denotes the graph obtained from G by removing all legs. To describe the inductive structure of M_G, we need some notation.

If S is a set of edges of G, the graph $G-S$ is the subgraph obtained by removing the edges in S, so that G and $G-S$ have the same vertices. We denote by $G-S^o$ the graph obtained by replacing every edge e in S by a pair of legs attached to the vertices adjacent to e. Hence $G-S^o$ has $n+2|S|$ legs and

$$[G-S^o]=G-S.$$

Consider the set of all such graphs

$$\mathcal{H}_G := \{H = G - S^o \quad \forall S \subset E(G)\}. \tag{4}$$

For any $H \in \mathcal{H}_G$ it is convenient to denote by S_H the set of edges of G such that $H = G - S_H^o$.

We have a lattice structure (i.e., a poset structure with a unique maximum and a unique minimum) on $\mathcal{H}_G$ induced by the inclusion, so that $H \geq H'$ if $[H] \supset [H']$. Of course, the maximum is G and the minimum $G - E^o$.

If $H \in \mathcal{H}_G$ is connected, H is a stable graph of genus $g - |S_H|$ with $n + 2|S_H|$ legs. Therefore, we can consider the locus M_H of curves in $M_{g-|S_H|,n+2|S_H|}$ having H as dual graph. More generally, if $H = H_1 \sqcup \cdots \sqcup H_c$ with H_i connected, then one easily checks that H_i is a stable graph for every i, and we set

$$M_H = M_{H_1} \times \cdots \times M_{H_c}.$$

Of course, if $H = G - E^o$, we recover $M_H = \prod_{v \in V} M_{g(v),\deg(v)}$, which we encountered earlier in (2). In fact, generalizing (2), for every $H \in \mathcal{H}_G$, we have a natural surjection

$$\pi_H : M_H \longrightarrow M_G,$$

which we call, again, a *presentation* of M_G. The map described in (2) is $\pi = \pi_{G-E^o}$, and it factors for any $H \in \mathcal{H}_G$ as

$$\pi : \prod_{v \in V} M_{g(v),\deg(v)} \longrightarrow M_H \xrightarrow{\pi_H} M_G.$$

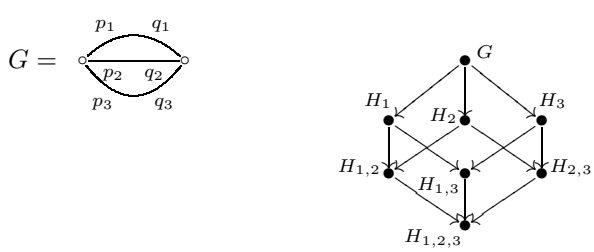
The set $\{\pi_H \ \forall H \in \mathcal{H}_G\}$, of all presentations of M_G, is in a natural bijection with $\mathcal{H}_G$ and hence has the lattice structure $\pi_H \geq \pi_{H'}$ if $H \geq H'$.

If $\pi_H \geq \pi_{H'}$, we have a map $\pi_{H',H} : M_{H'} \to M_H$ and a factorization

$$\pi_{H'} : M_{H'} \xrightarrow{\pi_{H',H}} M_H \xrightarrow{\pi_H} M_G.$$

In conclusion, the recursive structure of the stratum M_G is described by the lattice $\mathcal{H}_G$.

Example 2.2.4. In the next picture we have a stable graph G and the lattice $\mathcal{H}_G$ which, as we said, can be viewed as representing the recursive structure of M_G. We denote by H_* the graph obtained from G by removing the edges with labels in $*$.



2.3. **The stratification of $\overline{M}_{g,n}$.** We now go back to $\overline{M}_{g,n}$ and recall that we have $\overline{M}_{g,n} = \bigsqcup_{G \in \mathcal{G}_{g,n}} M_G$. This decomposition has some properties reflecting how curves degenerate. Indeed, consider a family of curves in a stratum M_G degenerating to a curve X_0 in a different stratum M_{G_0} so that G_0 is the dual graph of X_0. How are G and G_0 related? Of course, every node of the curves in M_G must specialize to a node of X_0, i.e., an edge of G_0. And the remaining nodes of X_0 (those that are not specializations of nodes of the curves in M_G) form a well defined set of edges S_0 of G_0. Then the graph G is obtained from G_0 by contracting to a vertex every edge of S_0. And the converse turns out to be true: if G is obtained from G_0 contracting a set of edges, we have that M_{G_0} lies in the closure of M_G.

We denote (weighted) edge-contractions as follows

$$\gamma : G_0 \longrightarrow G = G_0/S_0,$$

where $S_0 \subset E(G_0)$. Observe that the genus of a vertex of G_0/S_0 is defined as the genus of its preimage in G_0, so that edge-contractions preserve genus and stability.

The simplest example is the contraction of every edge of G_0. Then G is the graph with one vertex of genus g, no edges, and n legs, so that G is the dual graph of a smooth curve of genus g, i.e., $M_G = M_{g,n}$. Here the contraction $G_0 \to G$ corresponds to a family of smooth curves specializing to a curve in M_{G_0}.

Now, we define a partial order on $\mathcal{G}_{g,n}$ by setting $G_0 \geq G$ if there exists a contraction $G_0 \to G = G_0/S_0$. The poset $\mathcal{G}_{g,n}$ is graded, and the rank function $\rho : \mathcal{G}_{g,n} \to \mathbb{Z}_{\geq 0}$ defined by $\rho(G) = |E(G)|$ is such that $\rho(G) = \operatorname{codim} M_G$.

Example 2.3.1. The picture of the poset $\mathcal{G}_{2,0}$ is in Figure 5, with the horizontal levels marking the ranks and the minimum at the top.

So, the top vertex corresponds to $M_2 \subset \overline{M}_2$. The two stata below correspond to the two codimension-1 strata of $\overline{M}_2$, parametrizing curves with exactly one node. The two bottom strata correspond to two points whose recursive presentation is in Figure 4.

The graded poset structure on $\mathcal{G}_{g,n}$ corresponds to the poset structure on the strata $\{M_G \ \forall G \in \mathcal{G}_{g,n}\}$ of (1) induced by inclusion of closures. More precisely, we have the following theorem.

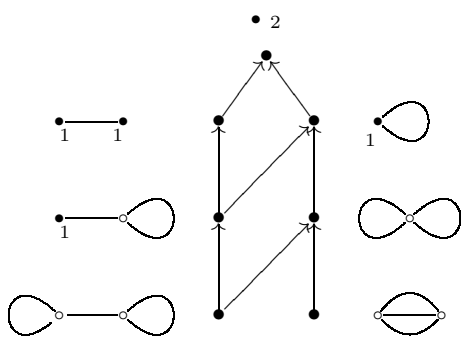


FIGURE 5. The poset $\mathcal{G}_{2,0}$, or the dual graph of $\overline{M}_2$

Theorem 2.3.2. *The decomposition* (1) *is a* graded stratification *of* $\overline{M}_{g,n}$ *by the poset* $\mathcal{G}_{g,n}$*, i.e., the following properties hold.*

(1) $M_G \cap \overline{M_{G'}} \neq \emptyset \Leftrightarrow M_G \subset \overline{M_{G'}} \Leftrightarrow G \geq G'$.
(2) M_G *is irreducible and locally closed.*
(3) *The map* $\mathcal{G}_{g,n} \to \mathbb{Z}$ *sending* G *to* $\operatorname{codim} M_G$ *is a rank function.*

Example 2.3.3. We now describe $\mathcal{G}_{0,5}$. We say that two graphs have the same type if they differ only by the labeling on the legs. Then in $\mathcal{G}_{0,5}$ we have three types of graphs: the graph with no edges and the two types below.

For the type on the left, we have $\binom{5}{2} = 10$ different graphs. For the type on the right, we have 15 different graphs (the number of partitions of shape $l_1 l_2 | l_3 | l_4 l_5$, up to symmetry), corresponding to the top vertices of Figure 6.

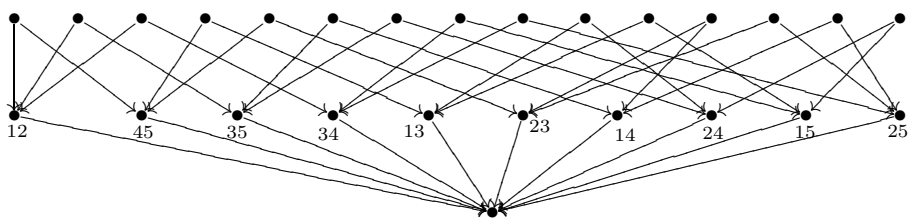


FIGURE 6. The poset $\mathcal{G}_{0,5}$ of strata of $\overline{M}_{0,5}$

2.4. **Skeleta and tropical curves.** Now we want to view $\mathcal{G}_{g,n}$ as a category whose objects are stable graphs and whose arrows are generated by isomorphisms and edge-contractions. We shall construct a cone complex out of the category $\mathcal{G}_{g,n}$, as follows. To a graph G we associate a cone

$$\sigma_G = \mathbb{R}_{\geq 0}^{E(G)}.$$

To an arrow (a composition of contractions and isomorphisms)

$$\gamma : G_0 \longrightarrow G$$

we associate an injective morphism of cones

$$\iota_\gamma : \sigma_G \hookrightarrow \sigma_{G_0}.$$

If γ is the contraction of $S_0 \subset E(G_0)$, then ι_γ identifies σ_G with the face of σ_{G_0}, where the coordinates of the edges in S_0 are zero. If γ is an isomorphism, then ι_γ corresponds to the bijection $E(G) \to E(G_0)$ induced by γ. We can thus construct the colimit over $\mathcal{G}_{g,n}$ and define it to be the *skeleton* of $\overline{M}_{g,n}$

$$\Sigma(\overline{M}_{g,n}) := \varinjlim(\sigma_G, \iota_\gamma). \tag{5}$$

By definition, the above space is a topological space and a so-called *generalized cone complex*. Let us give it an explicit geometric interpretation, so let Γ be a

point in it. Then there exists a stable graph G such that Γ lies in the interior of σ_G. Hence Γ can be identified with G, and with a set of positive real numbers $\{x_e,\ x_e \in \mathbb{R}_{>0},\ e \in E(G)\}$, we think of x_e as the length of the edge e. Then Γ is a so-called *stable tropical curve*, i.e., a stable graph whose edges have a length (a positive real number). By construction, two different points in $\Sigma(\overline{M}_{g,n})$ correspond to nonisomorphic curves, and to every stable tropical curve there corresponds a point in $\Sigma(\overline{M}_{g,n})$.

Tropical curves have been studied as objects of independent interest, and, as such, they have a moduli space, $M_{g,n}^{\mathrm{trop}}$, parametrizing isomorphism classes of stable tropical curves of genus g with n marked points, where a marked point of a tropical curve is a leg of the graph. As in the algebraic case, the n legs are ordered and must be fixed by any automorphism. By its very construction, $M_{g,n}^{\mathrm{trop}}$ is a topological space and has the structure of a generalized cone complex; see [22], [8].

Our previous analysis on the geometric interpretation of $\Sigma(\overline{M}_{g,n})$ enables us to conclude, at least set-theoretically, that the skeleton of $\overline{M}_{g,n}$ can be identified with the moduli space of tropical curves of genus g with n legs:

$$\Sigma(\overline{M}_{g,n}) \cong M_{g,n}^{\mathrm{trop}}.$$

It turns out that the above is not only a set-theoretic bijection, but it is an isomorphism of generalized cone complexes.

It is clear that the spaces above are not compact, but it is not hard to compactify them. We do that by extending our cones σ_G to $\overline{\sigma}_G = (\mathbb{R}_{\geq 0} \cup \{\infty\})^E$ endowed with the compact topology. This implies that we allow our tropical curves to have edges of infinite lengths. We thus obtain the compactified version of the above isomorphism (see [1]),

$$\overline{\Sigma}(\overline{M}_{g,n}) \cong \overline{M}_{g,n}^{\mathrm{trop}}. \tag{6}$$

2.5. **Curves over valuation fields and analytifications.** The isomorphism (6) is actually the reflection of a deep connection between algebraic and tropical curves. In loose words, a tropical curve encodes the local data of a family of smooth algebraic curves specializing to a singular one.

In algebraic geometry, local problems are often studied via local rings and valuation fields, whose associated algebraic schemes are the analogues of small balls in complex geometry. Let us set up the notation. We shall denote by K a valuation field and by $R \subset K$ its valuation ring. The valuation of K is a homomorphism, $v : K^* \to \mathbb{R}$, from the multiplicative group K^* to the additive group $\mathbb{R}$. We have $R = v^{-1}(\mathbb{R}_{\geq 0})$ and $M = v^{-1}(0)$, so that M is the unique maximal ideal of R. We shall always assume that the residue field $k = R/M$ is algebraically closed. Now, the scheme $\operatorname{Spec} R$ associated to R has a unique closed point, corresponding to the ideal M, called *special* and denoted by s. The scheme $\operatorname{Spec} K$ is open and dense in $\operatorname{Spec} R$ and is referred to as the *generic point*. It is useful to think of $\operatorname{Spec} K$ as having a limiting point built in itself: the special point s of $\operatorname{Spec} R$. In analogy with complex geometry, $\operatorname{Spec} R$ corresponds to a small ball about the origin, s corresponds to the origin, and $\operatorname{Spec} K$ corresponds to the complement of the origin.

For the rest of this section we assume that K is also complete with respect to a non-Archimedean valuation. Without going into technical details, let us mention that this has the effect of making the analysis as local as possible, enabling us to better handle limits.

Let $\mathcal{X}_K$ be a smooth (connected, projective) curve over K. As $\operatorname{Spec} K$ has a limiting point, one can ask whether $\mathcal{X}_K$ has a limiting curve, i.e., if it can be completed over $\operatorname{Spec} R$ by adding a curve over s. By the moduli properties of M_g, we know that to $\mathcal{X}_K$ there is associated a canonical moduli map $\mu_K : \operatorname{Spec} K \to M_g$ mapping $\operatorname{Spec} K$ to the isomorphism class of $\mathcal{X}_K$. On the other hand we have a compactification $\overline{M}_g$ of M_g; therefore, μ_K extends to a morphism

$$\mu_R : \operatorname{Spec} R \longrightarrow \overline{M}_g.$$

The image $\mu_R(s)$ of the special point is thus a stable curve X over the residue field of R (which we assume is algebraically closed). In this way we associate to $\mathcal{X}_K$ the stable curve X. Now, $\overline{M}_g$ is not a fine moduli space, hence there does not necessarily exist a family of curves over $\operatorname{Spec} R$ such that the fiber over s is X and the fiber over $\operatorname{Spec} K$ is $\mathcal{X}_K$. But, by the Stable Reduction Theorem, such a family exists for the base change $\mathcal{X}_{K'}$ of $\mathcal{X}_K$ to some finite extension $K \subset K'$ of non-Archimedean fields. In other words, denoting by R' the valuation ring of K', there does exist a family over $\operatorname{Spec} R'$ having $\mathcal{X}_{K'}$ as a generic fiber and X as a special fiber.

On the one hand, of course, this involves the choice of K' and will therefore give different families if we vary that. But we can still ask what data do not depend on the choice of K'. We have already said that the curve X will not change, hence its dual graph will not change either. But also, the local geometry of the family at the nodes of X will be independent on the choice of K', and this is precisely what will enable us to define a tropical curve, denoted by $\Gamma_{\mathcal{X}_K}$, associated to $\mathcal{X}_K$.

Our $\Gamma_{\mathcal{X}_K}$ will have as underlying graph the dual graph G_X of X. To define the length of its edges, pick one of them $e \in E(G_X)$. Then e corresponds to a node of X, which we shall denote by e again. Consider a field extension $K \subset K'$ as above, so that there is family

$$\mathcal{X}' \longrightarrow \operatorname{Spec} R'$$

whose special fiber is X and whose fiber over $\operatorname{Spec} K'$ is the base change of $\mathcal{X}_K$, i.e., $\mathcal{X}'_{K'} = \mathcal{X}_K \times_{\operatorname{Spec} K} \operatorname{Spec} K'$. Then, locally at the node e, the equation of $\mathcal{X}'$ is $xy = f_e$ for some f_e in the maximal ideal of R'. We set the length of e to be equal to $v'(f_e)$ where v' is the valuation of K'. We have thus defined the tropical curve $\Gamma_{\mathcal{X}_K}$, and this definition turns out to be independent of the choice of K' or of the local equation of $\mathcal{X}$ at e; see [29].

We shall say that the tropical curve $\Gamma_{\mathcal{X}_K}$ is the skeleton, or the tropicalization, of $\mathcal{X}_K$.

Now, the construction we just described fits into a more general picture involving the theory of analytifications of algebraic schemes (developed by V. Berkovich) and its connection to tropical geometry. By the theory developed in [6], to every algebraic scheme T there corresponds an analytic space, the analytification T^{an}, and this correspondence is functorial and profound.

We then introduce $\overline{M}_{g,n}^{\mathrm{an}}$, the analytification of $\overline{M}_{g,n}$. A point in $\overline{M}_{g,n}^{\mathrm{an}}$ corresponds, up to base change, to a stable curve over an algebraically closed field K complete with respect to a non-Archimedean valuation.

The analytic space $\overline{M}_{g,n}^{\mathrm{an}}$ has quite a complicated structure, but, by general results, it retracts onto a simpler subspace. Indeed, to every space T with a boundary and a toroidal structure, one associates the (extended) *Berkovich skeleton* $\overline{\Sigma}(T) \subset T^{\mathrm{an}}$ which is a generalized, extended cone complex onto which the

analytification T^{an} retracts; see [27]. Now, $\overline{M}_{g,n}$ does not have a proper toroidal structure associated to its boundary $\overline{M}_{g,n} \smallsetminus M_{g,n}$, but the stack associated to it, written $\overline{\mathcal{M}}_{g,n}$, does. This enables us to construct the Berkovich skeleton of $\overline{\mathcal{M}}_{g,n}$. Now, recall definition (5) and its compactified version $\overline{\Sigma}(\overline{M}_{g,n})$. How does it compare with the Berkovich skeleton of the stack $\overline{\mathcal{M}}_{g,n}$? It turns out that they are naturally isomorphic, and hence we shall identify them. This allows us to apply Berkovich's theory to $\overline{\Sigma}(\overline{M}_{g,n})$, so that we have a retraction ρ,

$$\overline{\Sigma}(\overline{M}_{g,n}) \subset \overline{M}_{g,n}^{\mathrm{an}} \xrightarrow{\rho} \overline{\Sigma}(\overline{M}_{g,n}).$$

To give a geometric interpretation of the map ρ, we recall the isomorphism $\overline{\Sigma}(\overline{M}_{g,n}) \cong \overline{M}_{g,n}^{\mathrm{trop}}$ introduced in (6). Composing it with ρ we have the following.

Theorem 2.5.1. *The following* tropicalization *map*

$$\mathrm{trop} : \overline{M}_{g,n}^{\mathrm{an}} \xrightarrow{\rho} \overline{\Sigma}(\overline{M}_{g,n}) \xrightarrow{\cong} \overline{M}_{g,n}^{\mathrm{trop}}$$

is a continuous, surjective map that sends the class of a stable curve $\mathcal{X}_K$ over the (*algebraically closed, complete, non-Archimedean*) *field K to its* skeleton $\Gamma_{\mathcal{X}_K}$.

See [1]. Concluding in loose words: the skeleton of $\overline{M}_{g,n}$ is the moduli space of skeleta of stable curves.

The existence of analogous "correspondence" results in other situations has been investigated, and is under investigation, as of this writing. An important case in which it has been proved to hold is the compactification of the space of admissible covers, constructed in [20]. For details, we refer to the original paper [13], where the authors also relate their results to the ones for moduli space of stable curves described above, proving that the various correspondence results are consistent with one another.

3. Compactifying Jacobians and Néron models

3.1. **Moduli of line bundles on curves.** Line bundles on curves and, more generally, on abstract algebraic varieties capture the information about the projective models of the varieties, i.e., their mappings in projective spaces. Here we focus on line bundles on curves, and on their moduli spaces. Motivated by our earlier discussion, we restrict to stable curves although some of what we are going to explain holds in greater generality. For example, if we release the condition of stability by assuming our curves have at most nodes as singularities, all the results described before section 3.4 continue to hold.

The set of isomorphism classes of line bundles on a curve X forms a fine moduli space, denoted by $\mathrm{Pic}(X)$ and called the *Picard scheme* of X; see [17], [23]. The isomorphism class of a line bundle corresponds to the linear equivalence class of a Cartier divisor. If the line bundle L corresponds to the divisor $\sum_{i=1}^m n_i p_i$, where p_i are smooth points of X and $n_i \in \mathbb{Z}$, we write $L = \mathcal{O}(\sum_{i=1}^m n_i p_i)$. Also, $\mathrm{Pic}(X)$ is a group under the tensor product of line bundles, and it has infinitely many connected, irreducible components. The component containing the trivial line bundle (or the zero divisor) is called the *Jacobian* of X and is denoted by J_X. More precisely, J_X is the moduli space for line bundles of multidegree $(0, 0, \ldots, 0)$ (i.e., of degree 0 on every irreducible component of X) so that, with the notation of the introduction, the set $\mathcal{C}$ consists of line bundles of multidegree $(0, 0, \ldots, 0)$ on X, and the equivalence

is an isomorphism over X. A family over a base B, written

$$\mathcal{L} \longrightarrow X \times B \xrightarrow{f} B,$$

is given by a line bundle $\mathcal{L}$ on $X \times B$ whose restriction to every fiber of f has multidegree $(0, 0, \ldots, 0)$.

If X is nonsingular, then the Jacobian J_X is projective and is thus an *abelian variety*, i.e., a projective algebraic group. Moreover, if X has positive genus, J_X encodes important information about X itself. For example, fixing a point x_0 in X we can map every $x \in X$ to the line bundle $\mathcal{O}(x - x_0)$, and this gives an injective morphism, $X \hookrightarrow J_X$. Our curve X can thus be realized as a subvariety of its Jacobian, and in fact it generates the whole of J_X as a group.

On the other hand, if X is allowed to have nodes, J_X easily fails to be projective; let us see how. Consider $G_X = (V, E)$, the dual graph of X. As we saw in (3), the desingularization X^ν of X is the disjoint union of the desingularizations of its components, and we have a birational morphism

$$\nu : X^\nu = \bigsqcup_{v \in V} C_v \longrightarrow X.$$

Then we have an exact sequence of algebraic groups,

$$0 \longrightarrow (k^*)^{b_1(G_X)} \longrightarrow J_X \xrightarrow{\nu^*} J_{X^\nu} = \prod_{v \in V} J_{C_v} \longrightarrow 0, \tag{7}$$

where ν^* is simply the pullback of line bundles via the map ν (and $b_1(G_X)$ the first Betti number). In this sequence, the product on the right is a product of Jacobians of smooth curves, which are all abelian varieties, hence the product is also an abelian variety. The kernel of ν^* accounts for the gluing data needed to specify a line bundle on X from a line bundle on X^ν. For instance, suppose X is irreducible with one node N, so that $b_1(G_X) = 1$ and the kernel of ν^* is k^*. A line bundle L on X^ν determines a line bundle on X once the datum of how to identify the fibers of L over the two branch points of N is given. This datum is an isomorphism between two one-dimensional k-vector spaces, hence it corresponds to an element of k^*.

By the exact sequence, J_X is complete if and only if $b_1(G_X) = 0$ (i.e., the dual graph of X, regardless of its weights, is a tree); if that is the case, X is said to be of *compact type*. Now, for any $g \geq 1$ there exist plenty of curves that are not of compact type for example irreducible singular curves.

We are thus in the situation described in the introduction, and we want to construct compactifications for J_X. This problem is classical and presents various aspects. As we said earlier, the noncompleteness of J_X reflects the fact that families of line bundles on X *degenerate*, i.e., do not admit a line bundle on X as a limit. On the other hand, if X is viewed, as it often happens, as the limit of a family of smooth curves, then families of line bundles on these smooth curves also degenerate. We want a good compactification of J_X to account for both types of degenerations.

3.2. **Line bundles on families of curves.** As we said, we want to view our curve X as a limit of smooth curves, and we do that in the following way. We denote (here and in the rest of this paper) by

$$X \hookrightarrow \mathcal{X} \longrightarrow \operatorname{Spec} R \tag{8}$$

a family of curves over the spectrum of a discrete valuation ring R with X as a special fiber and a smooth generic fiber. A discrete valuation ring can be thought of as a one-dimensional valuation ring. So, in (8) we assume the generic fiber $\mathcal{X}_K$ to be a nonsingular curve over the quotient field K of R. We shall always make the harmless (for us) assumption that $\mathcal{X}_K$ has a K-rational point.

The Jacobian $J_{\mathcal{X}_K}$ of $\mathcal{X}_K$ is a smooth projective variety over K and the moduli space of line bundles of degree 0 on $\mathcal{X}_K$. How are $J_{\mathcal{X}_K}$ and J_X related? The answer is, roughly speaking, that J_X is a (noncomplete) limit of $J_{\mathcal{X}_K}$ over $\operatorname{Spec} R$. Indeed, together with the existence of the Jacobian for a fixed curve, the general theory gives the existence of a *relative Jacobian*,

$$J_{\mathcal{X}/R} \longrightarrow \operatorname{Spec} R, \tag{9}$$

for any family $\mathcal{X} \to \operatorname{Spec} R$ as above. The morphism (9) has $J_{\mathcal{X}_K}$ as fiber over $\operatorname{Spec} K$ and J_X as special fiber, and it is thus a so-called *model* of $J_{\mathcal{X}_K}$ over R. It is smooth and separated, but not projective in general.

Now, as we said, a natural requirement for a good compactification of J_X is to control not only degenerations of line bundles on X but also degenerations of line bundles on $\mathcal{X}_K$ as $\mathcal{X}_K$ degenerates to X. Therefore, we want it to appear as a (complete) limit of $J_{\mathcal{X}_K}$ over $\operatorname{Spec} R$, which is why we are now looking at models of $J_{\mathcal{X}_K}$ over R.

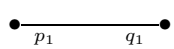
Of course, we also want a good compactification of J_X to have a moduli description extending that of J_X. From this perspective there is a fundamental model of $J_{\mathcal{X}_K}$ over R to consider, which is the relative degree 0 Picard scheme,

$$\operatorname{Pic}^0_{\mathcal{X}/R} \longrightarrow \operatorname{Spec} R. \tag{10}$$

The generic (resp., special) fiber of the above morphism is the moduli space of line bundles of degree 0 on $\mathcal{X}_K$ (resp., on X). The special fiber, written $\operatorname{Pic}^0(X)$, is quite big if X is not irreducible. Indeed we have a decomposition into connected components, $\operatorname{Pic}^0(X) = \bigsqcup_{|\underline{d}|=0} \operatorname{Pic}^{\underline{d}}(X)$, where $\underline{d} \in \mathbb{Z}^V$ and $\operatorname{Pic}^{\underline{d}}(X)$ is the locus parametrizing line bundles multidegree $\underline{d}$. Notice that $J_X = \operatorname{Pic}^{(0,0,\dots,0)}(X)$, and we have (noncanonical) isomorphisms $J_X \cong \operatorname{Pic}^{\underline{d}}(X)$; see section 4.1.

These two models of $J_{\mathcal{X}_K}$ are related by a natural inclusion $J_{\mathcal{X}/R} \subset \operatorname{Pic}^0_{\mathcal{X}/R}$, and neither of them is complete, unless X is of compact type. But even when X is of compact type, they present some problems, described in the next example.

Example 3.2.1. Let $\mathcal{X} \to \operatorname{Spec} R$ be a family of curves as above, and assume that $\mathcal{X}$ is nonsingular. Let the special fiber X have the following dual graph



(such a family surely exists). So, X is of compact type, with one node and two irreducible components of positive genus, C_1 and C_2, and it is obtained by gluing the two marked curves $(C_1; p_1)$ and $(C_2; q_1)$:

$$X = (C_1 \sqcup C_2)/p_1 = q_1.$$

We pick a line bundle $\mathcal{L}_K$ on $\mathcal{X}_K$ of degree 0; since $\mathcal{X}$ is nonsingular, $\mathcal{L}_K$ admits an extension to a line bundle $\mathcal{L}$ on $\mathcal{X}$. We choose $\mathcal{L}_K$ so that the restriction of $\mathcal{L}$ to X is the line bundle $L = \mathcal{O}(p - q)$, where $p \in C_1$ and $q \in C_2$ are two smooth

points of X. Notice that L has degree 0 and multidegree $(1,-1)$. The moduli map associated to $\mathcal{L}$

$$\mu_{\mathcal{L}} : \operatorname{Spec} R \to \operatorname{Pic}^0_{\mathcal{X}/R}$$

maps the special point to the connected component $\operatorname{Pic}^{(1,-1)}(X)$ of $\operatorname{Pic}^0(X)$. Therefore, although the image of $\operatorname{Spec} K$ under $\mu_{\mathcal{L}}$ lies in the Jacobian of $\mathcal{X}_K$, the image of the special point does not lie in J_X. Hence, even if J_X is complete, the relative Jacobian fails to parametrize degenerations of line bundles on $\mathcal{X}_K$.

Now let us show that the morphism (10) is not separated; i.e., loosely speaking, a line bundle on $\mathcal{X}_K$ can have more than one limit. To be more precise, let $\mu_{\mathcal{L}_K} : \operatorname{Spec} K \to J_{\mathcal{X}_K}$ be the moduli map of $\mathcal{L}_K$; it suffices to exhibit an extension $\operatorname{Spec} R \to \operatorname{Pic}^0_{\mathcal{X}/R}$ of $\mu_{\mathcal{L}_K}$ different from the map $\mu_{\mathcal{L}}$ defined above. Since $\mathcal{X}$ is nonsingular, C_1 is a Cartier divisor of multidegree $(-1,1)$, and $\mathcal{L}' := \mathcal{L}(C_1)$ is a line bundle of degree 0 on the fibers whose restriction to $\mathcal{X}_K$ coincides with the restriction of $\mathcal{L}$. But the restriction of $\mathcal{L}'$ to X satisfies

$$\mathcal{L}'_{|C_1} = \mathcal{O}(p - p_1), \qquad \mathcal{L}'_{|C_2} = \mathcal{O}(-q + q_1).$$

Hence the restriction of $\mathcal{L}'$ to X has multidegree $(0,0)$. Therefore $\mu_{\mathcal{L}'}$ maps the special point of $\operatorname{Spec} R$ to J_X, and it is obviously different from $\mu_{\mathcal{L}}$.

The problem described in this example is the existence and uniqueness of an extension for a map of the form $\mu_{\mathcal{L}_K} : \operatorname{Spec} K \to J_{\mathcal{X}_K}$, where $\mathcal{L}_K$ is a degree 0 line bundle on $\mathcal{X}_K$. For the relative Jacobian the existence of the extension can fail. On the other hand, for the relative Picard scheme the existence holds if $\mathcal{X}$ is nonsingular but the uniqueness can fail.

3.3. **The Néron model of the Jacobian.** We shall now describe a third model for $J_{\mathcal{X}_K}$ for which the problems illustrated in the previous example do not occur. The issue, as we saw, was the existence and uniqueness of an extension for maps of the form $\mu_{\mathcal{L}_K}$. We now approach it using Néron models, and their defining mapping property. The existence of Néron models holds in more general situations than ours; see [24]. For us the following special case is enough.

Theorem 3.3.1. *Let K be a discrete valuation field. Let $\mathcal{X}_K$ be a smooth curve over K, and let $J_{\mathcal{X}_K}$ be its Jacobian. Then there exists a smooth and separated model of $J_{\mathcal{X}_K}$, the* Néron model

$$N(J_{\mathcal{X}_K}) \longrightarrow \operatorname{Spec} R,$$

satisfying the following mapping property. *For any smooth scheme $Y_R \to \operatorname{Spec} R$ and any morphism $\phi_K : Y_K \to J_{\mathcal{X}_K}$ (with Y_K the fiber of Y_R over K) there exists a unique extension of ϕ_K to a morphism $\phi_R : Y_R \to N(J_{\mathcal{X}_K})$.*

We just mention that the Néron model is a group scheme, and it extends the group structure of its generic fiber $J_{\mathcal{X}_K}$. Moreover, the Néron model is natural because its mapping property determines it uniquely, but it is known not to commute with ramified base change.

Now, in the setup of Example 3.2.1, we can apply the mapping property of Theorem 3.3.1 to $Y = \operatorname{Spec} R$ and $\phi_K = \mu_{\mathcal{L}_K}$, obtaining that the map $\mu_{\mathcal{L}_K}$ extends uniquely to a map μ_R from $\operatorname{Spec} R$ to $N(J_{\mathcal{X}_K})$.

In general, to give a geometric interpretation to the extension μ_R and to the Néron model itself, we assume that the curve $\mathcal{X}_K$ in the theorem admits a model

$\mathcal{X} \to \operatorname{Spec} R$ with a stable curve X as special fiber and nonsingular total space $\mathcal{X}$. We denote by N_X the special fiber of the Néron model $N(J_{\mathcal{X}_K}) \to \operatorname{Spec} R$. We shall see that under these assumptions N_X is a finite disjoint union of components isomorphic to the Jacobian of X, and such union is indexed by a combinatorial invariant of X. In particular, N_X is independent of the choice of $\mathcal{X}_K$.

We need some combinatorial preliminaries. Fix an orientation (whose choice is irrelevant) on the dual graph $G = G_X$. Let $C_0(G, \mathbb{Z})$ and $C_1(G, \mathbb{Z})$ be the usual groups of i-chains, so that $C_0(G, \mathbb{Z})$ is the free abelian group on the vertex set V and $C_1(G, \mathbb{Z})$ is the free abelian group on the edge set E. Consider the boundary homomorphism $\partial : C_1(G, \mathbb{Z}) \to C_0(G, \mathbb{Z})$, mapping an edge e oriented from u to v to $u - v$. Let $\delta : C_0(G, \mathbb{Z}) \to C_1(G, \mathbb{Z})$ be the coboundary, mapping a vertex v to $\sum e_v^+ - \sum e_v^-$ where the first sum is over all edges originating from v and the second is over all edges ending at v. Now we can introduce the finite group

$$\Phi_G := \frac{\partial \delta C_0(G, \mathbb{Z})}{\partial C_1(G\mathbb{Z})}.$$

This group is well known in graph theory. By the Kirchhoff–Trent (or Kirchhoff matrix) theorem, its cardinality is equal to the number of spanning trees of G (i.e., the connected subgraphs of G having the same vertices as G and the first Betti number equal to 0). We have (see [26], [25], [4]) the following.

Proposition 3.3.2. *Let $\mathcal{X} \to \operatorname{Spec} R$ have a stable special fiber X and a nonsingular total space $\mathcal{X}$. Then*

$$N_X \cong \bigsqcup_{i \in \Phi_{G_X}} (J_X)_i. \tag{11}$$

Using compactifications of Néron models and Jacobians, we shall give, in (22), a concrete realization of the above isomorphism.

Example 3.3.3. By what we said before the statement, the number of irreducible components of N_X equals the number of spanning trees of G_X. In particular, if X is of compact type or is irreducible, we have $N_X \cong J_X$.

By contrast, if G is a graph consisting of two vertices joined by $\ell \geq 2$ edges, we have ℓ spanning trees, and one can prove that $\Phi_G \cong \mathbb{Z}/\ell\mathbb{Z}$.

It is not hard to find other graphs G with $\Phi_G \cong \mathbb{Z}/\ell\mathbb{Z}$, for example, a cycle with ℓ vertices and edges.

There is a clear relation between the Néron model and the relative Picard scheme. Indeed $N(J_{\mathcal{X}_K}) \to \operatorname{Spec} R$ is the *maximal separated quotient* of the Picard scheme $\operatorname{Pic}^0_{\mathcal{X}/R} \to \operatorname{Spec} R$. Moreover, there is an embedding of the relative Jacobian in the Néron model; see [26] and [7].

3.4. **Compactified Jacobians and Néron models.** A *compactified Jacobian* for a (stable) curve X is a projective variety $\overline{P}_X$ containing J_X not necessarily as a dense subset (which is why we use the notation $\overline{P}_X$ rather than $\overline{J_X}$) and satisfying the following requirement. For any family $\mathcal{X} \to \operatorname{Spec} R$ as in (8), there is a projective morphism

$$\overline{P}_R \longrightarrow \operatorname{Spec} R \tag{12}$$

having $J_{\mathcal{X}_K}$ as generic fiber and $\overline{P}_X$ as special fiber, and such that $\overline{P}_R$ has a moduli interpretation extending that of $J_{\mathcal{X}/R}$.

Going back to the issues illustrated in Example 3.2.1, we observe that since the morphism (12) is projective, any map $\mu_K : \operatorname{Spec} K \to J_{\mathcal{X}_K}$ (associated to a line bundle $\mathcal{L}_K$ on $\mathcal{X}_K$) admits a unique extension to a map $\mu_R : \operatorname{Spec} R \to \overline{P}_R$.

From now on we apply the notation introduced in Proposition 3.3.2 and, for a connected nodal curve X, we denote by N_X the special fiber of the Néron model $N(J_{\mathcal{X}_K}) \to \operatorname{Spec} R$ of the Jacobian associated to a family $\mathcal{X} \to \operatorname{Spec} R$, with $\mathcal{X}$ nonsingular.

From Proposition 3.3.2 we see that N_X is not complete unless J_X is complete. We thus introduce a terminology to distinguish compactified Jacobians which also compactify the Néron model. We say that a compactified Jacobian $\overline{P}_X$ is of *Néron type* (or a Néron compactified Jacobian) if it contains N_X as a dense open subset.

Compactified Jacobians of Néron type do exist, but we postpone to the next section the discussion on their existence. Assuming it, we describe how these Néron compactified Jacobians have a recursive structure in terms of Néron models.

We introduced in (4) the lattice $\mathcal{H}_G$ associated to the graph G. We now introduce a subposet of $\mathcal{H}_G$,

$$\mathcal{H}_G^1 := \{H \in \mathcal{H}_G : H \text{ is connected}\}.$$

The maximum of $\mathcal{H}_G^1$ is G, and the minimal elements are the graphs H such that $[H]$ is a spanning tree. Moreover, just as $\mathcal{H}_G$, the poset $\mathcal{H}_G^1$ is graded by the rank function $H \mapsto g(H)$.

If G is the dual graph of the curve X, then for every $H \in \mathcal{H}_G^1$ we have a set $S = S_H$ of nodes of X (and edges of G) such that $H = G - S_H^o$. We denote by X_S^ν the desingularization of X at S, so that X_S^ν is a connected nodal curve of genus $g(G - S)$ whose dual graph is $[H]$.

With our notation, $N_{X_S^\nu}$ is the special fiber of the Néron model of the Jacobian of a smooth curve specializing to X_S^ν.

The following statement (from [10]) describes the compactification of N_X provided by a Néron compactified Jacobian in terms of the Néron models of all the connected partial normalizations of X. This is another instance of a widespread recursive phenomenon for compactified moduli spaces. Namely, to compactify a space (e.g., N_X), one adds at the boundary the analogous spaces associated to simpler objects (e.g., $N_{X'}$ with X' a connected partial normalization of X).

Theorem 3.4.1. *Let X be a stable curve, and let G be its dual graph. Then there exists a Néron compactified Jacobian $\overline{P}_X$ such that*

$$\overline{P}_X = \bigsqcup_{H \in \mathcal{H}_G^1} N_H \tag{13}$$

with $N_H \cong N_{X_{S_H}^\nu}$ for every $H \in \mathcal{H}_G^1$. Moreover, (13) *is a graded stratification, i.e., the following hold.*

(1) *$N_H \cap \overline{N_{H'}} \neq \emptyset \Leftrightarrow N_{H'} \subset \overline{N_H} \Leftrightarrow H' \leq H$.*
(2) *N_H is locally closed of pure dimension $g(H)$.*
(3) *The map $\mathcal{H}_G^1 \to \mathbb{Z}$ mapping H to $\dim N_H$ is a rank on $\mathcal{H}_G^1$.*

Notice the similarities between this theorem and Theorem 2.3.2.

Remark 3.4.2. The strata of minimal dimension in (13) are Néron models of curves whose dual graph is a spanning tree of G_X, hence they are irreducible and projective. By Proposition 3.3.2, the number of such strata is equal to the number of irreducible components of $\overline{P}_X$.

The strata of (13) are, in general, not connected. Hence it is quite natural to ask whether the stratification can be refined so as to have connected strata. We will answer this question in the affirmative after Theorem 4.2.3.

4. Compactifying Jacobians of any degree

4.1. **Universal Jacobians of any degree.** We now concentrate on compactified Jacobians and, before continuing, we pause for a moment to recall that the Jacobian is the moduli space for line bundles of multidegree $(0, \dots, 0)$ on a curve X, and we ask, why not extend our consideration to all degrees and multidegrees?

We have mentioned that for any $\underline{d} \in \mathbb{Z}^V$, where $G_X = (V, E)$ as usual, there are isomorphisms $\mathrm{Pic}^{\underline{d}}(X) \cong J_X$. How are they defined? For every vertex v, pick a smooth point p_v of X lying on the component corresponding to v. Then the following is an isomorphism:

$$J_X \longrightarrow \mathrm{Pic}^{\underline{d}}(X), \qquad L \mapsto L(\sum_{v \in V} d_v p_v),$$

which is obviously not canonical, as it depends on the choice of the points p_v. For a smooth curve $\mathcal{X}_K$ over a discrete valuation field K, we have similar isomorphisms $\mathrm{Pic}^d(\mathcal{X}_K) \cong J_{\mathcal{X}_K}$ for every $d \in \mathbb{Z}$.

We can define compactified degree d Jacobians, written $\overline{P}^d_X$, as we did in section 3.4 for $d = 0$. So, for any family of curves $\mathcal{X} \to \mathrm{Spec}\, R$ as in (8), our $\overline{P}^d_X$ is the special fiber of a projective morphism

$$\overline{P}^d_R \longrightarrow \mathrm{Spec}\, R \tag{14}$$

having $\mathrm{Pic}^d(\mathcal{X}_K)$ as generic fiber.

In generalizing our analysis to all degrees, we lose the group structure, but we gain a better understanding on the geometric complexity of the situation.

We shall approach the problem of compactifying the Jacobians of any degree from the point of view of the moduli theory of stable curves, described earlier. First of all, over the moduli space of stable curves, we have a *universal curve*, denoted by $\mathcal{X}_g \longrightarrow \overline{M}_g$, whose fiber over an automorphism-free curve X is the curve itself. The requirement that X be free from automorphisms is a bit annoying, but it is needed if, as is done in this paper, we work with varieties and schemes rather than stacks. On the other hand, if $g \geq 3$, the curves in $\overline{M}_g$ admitting nontrivial automorphisms form a proper closed subset, i.e., the general stable curve has no nontrivial automorphisms. To simplify the forthcoming description, from now on we assume that curve X is general in this sense.

Now, as we did for curves over valuation rings, we can consider the relative degree d Jacobian associated to the universal curve over M_g. This is often called the *universal degree d Jacobian*, and it is given by a morphism $P^d_g \to M_g$, whose fiber over the point parametrizing the curve X is $\mathrm{Pic}^d(X)$. We point out that, as d varies, the varieties P^d_g are not always isomorphic.

Just as M_g is not complete, P^d_g is not complete, and we want to construct a compactification $\overline{P}^d_g$ of P^d_g parametrizing compactified degree d Jacobians for all stable curves of genus g. We refer to such a space as a compactified universal degree d Jacobian.

The spaces $\overline{P}_g^d$ can be constructed for all $d \in \mathbb{Z}$ by imitating the Geometric Invariant Theory (GIT) construction of $\overline{M}_g$. Indeed, $\overline{M}_g$ was constructed in [16] as the GIT quotient of the Hilbert scheme of n-canonically embedded curves ($n \gg 0$), i.e., curves in projective space embedded by the nth power of their dualizing line bundle. In view of the fact that the Hilbert scheme of n-canonically embedded curves has such a remarkable GIT quotient, one is led to ask about the Hilbert scheme of all curves of degree $d \gg 0$ and genus g in a fixed projective space. Can we construct its GIT quotient? And, if so, is this quotient a good compactification of the universal degree d Jacobian?

The answer to these questions is positive, and we have, for every $d \in \mathbb{Z}$, a projective morphism

$$\psi_d : \overline{P}_g^d \longrightarrow \overline{M}_g$$

whose fiber over the curve X is a compactified degree d Jacobian, written $\overline{P}_X^d$; see [9]. Now, $\overline{P}_X^d$ is a connected, projective variety, all of whose irreducible components have dimension g. It is singular in general, and its smooth locus, written P_X^d, parametrizes line bundles on X of suitable multidegree. More exactly, we have an identification between the smooth locus of $\overline{P}_X^d$ and a (disjoint) union of components isomorphic to J_X,

$$P_X^d = \bigsqcup_{\underline{d} \in \Delta_d} \mathrm{Pic}^{\underline{d}}(X), \tag{15}$$

where Δ_d is a well-defined finite set of multidegrees of total degree d. From Proposition 3.3.2 and the discussion in the earlier sections, we expect

$$|\Delta_d| \leq |\Phi_{G_X}|, \tag{16}$$

with equality for Néron type Jacobians. This expectation does hold, and we shall discuss some examples later,

The points of the boundary, $\overline{P}_X^d \smallsetminus P_X^d$, parametrize equivalence classes of certain line bundles on nodal curves having X as stable model (i.e., admitting a genus-preserving birational morphism onto X). As we shall see in the next sections, these line bundles are determined, recursively, by suitable line bundles on the partial normalizations of the given curve X.

Going back to the construction of $\overline{P}_g^d$ as a GIT quotient, there is a main difference with the GIT quotient defining $\overline{M}_g$. Namely, in the latter case the quotient is a so-called *geometric* quotient, which implies that every point in $\overline{M}_g$ corresponds to exactly one orbit in the Hilbert scheme. For $\overline{P}_g^d$ the quotient is geometric only for certain values of d. When $\overline{P}_g^d$ is not a geometric quotient, some of its points parametrize more than one orbit and its moduli description becomes more complex.

The degrees d for which $\overline{P}_g^d$ is a geometric quotient are precisely those satisfying the condition $(d - g + 1, 2g - 2) = 1$. In these cases, and only in these cases, all fibers of ψ_d are Néron compactified Jacobians.

Now, the above numerical condition holds if $d = g$, but it fails if $d = g - 1$. We shall concentrate on these two cases, which are interesting for different reasons, and give a combinatorial analysis of the compactified Jacobian, highlighting its recursive structure.

4.2. **Compactified Jacobians in degree g.** As we said, the compactified Jacobians $\overline{P}^g_X$ are of Néron type for every stable curve X. We shall now describe them closely by adapting to the present case a method introduced in [5] to handle the case $d = g - 1$.

Let $G = (V, E)$ be the dual graph of X. We know, by the general results mentioned in the previous section, that our compactification has finitely many irreducible components all of dimension g. Moreover, by (15), each component contains a dense subset parametrizing line bundles on X of a fixed multidegree $\underline{d} \in \mathbb{Z}^V$ such that $|\underline{d}| = g$; we wrote Δ_g for the set of these special multidegrees. As $\overline{P}^g_X$ is of Néron type, we have $|\Delta_g| = |\Phi_G|$, and the question is how to interpret the multidegrees lying in Δ_g in a geometrically meaningful way.

At a very basic level, this amounts to distributing the integer g among the vertices of G. We start from the identity

$$g = \sum_{v \in V} \big(g(v) - 1\big) + |E| + 1.$$

The first term on the right, the summation, contains the topological data of each component (or vertex) of X (or G), while the second part, $|E| + 1$, is not related to the vertices. It is thus natural to try and distribute it among the vertices in a combinatorially meaningful way. A natural approach in the graph-theoretic setting is to consider an *orientation* O on G and, for any vertex v, denote by $\underline{t}^O_v$ the number of edges having v as target. Now we define a multidegree $\underline{d}^O$ associated to O:

$$\underline{d}^O_v := (g(v) - 1) + \underline{t}^O_v \qquad \forall v \in V. \tag{17}$$

By definition of orientation, every edge has exactly one target, and hence $\sum_{v \in V} \underline{t}^O_v = |E|$, so that $|\underline{d}^O| = g - 1$, which is off by 1 from what we want (i.e., g). A way to fix this problem is to modify the orientation O by allowing one edge to be bi-oriented (i.e., having both ends as targets). So, we define a 1-*orientation* on a graph G as the datum of a bi-oriented edge e and of an orientation on $G - e$; see Figure 7 for some examples. Now, if O is any 1-orientation, we have $\sum_{v \in V} \underline{t}^O_v = |E| + 1$. Therefore, if we define the multidegree of O as in (17), we have $|\underline{d}^O| = g$ for any 1-orientation O.

In this way we have selected a special set of multidegrees of total degree g, and this set is finite as there are only finitely many 1-orientations on a graph. Now, it may happen that two 1-orientations, O and O', have the same multidegree. If that is the case, we say that O and O' are *equivalent*.

We now ask whether we got the correct set of multidegrees Δ_g. It is immediately clear that the answer is negative, as this set is still too big. Indeed, from our earlier discussion we expect $|\Delta_g| = |\Phi_G|$, but the set of equivalence classes of 1-orientations on G is bigger than $|\Phi_G|$, as Example 4.2.1 shows.

Now, let us observe that the definition of a 1-orientation depends on the choice of the bi-oriented edge, which is quite arbitrary from our point of view. To eliminate this problem, we ask whether this dependence disappears when passing to equivalence classes of 1-orientations. The answer is no in general, but it is yes for a special type of 1-orientations, called *rooted* orientations.

A 1-orientation O is rooted if, for every edge e, the class of O contains a representative having e as bi-oriented edge.

A more geometric definition is the following. A 1-orientation with bi-oriented edge e is rooted if for every vertex v there exists a directed path from e to v. In

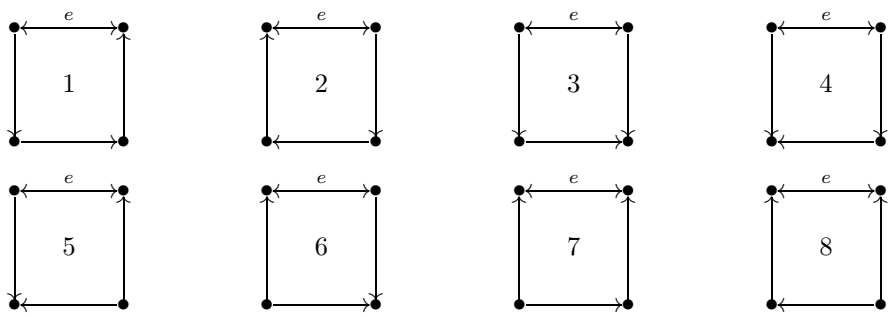


FIGURE 7. The eight 1-orientations on a 4-cycle

the next example the orientations from 1 to 4 are rooted, the ones from 5 to 8 are not.

Example 4.2.1. In Figure 7 we have all the 1-orientations on a 4-cycle with the same bi-oriented edge. One checks easily that they are not equivalent to one another. Since $|\Phi_G| = 4$, we see that the number of multidegrees corresponding to 1-orientations is greater than $|\Phi_G|$.

We denote by $\overline{O}^1(G)$ the set of equivalence classes of rooted orientations on G. We have two facts; see [14], [11].

Remark 4.2.2.

(a) $\overline{O}^1(G) = \Delta_g$.

(b) $\overline{O}^1(G)$ is not empty if and only if G is connected.

We can go back to our main problem, the description of $\overline{P}^g_X$. Recall that we denote by $\mathcal{H}^1_G$ the poset of connected spanning subgraphs of G having legs corresponding to the removed edges. In our case, since G has no legs, an element in $\mathcal{H}_G$ is a graph H of the form $G - S^o$ having $2|S|$ legs. We denoted by $[H]$ the graph obtained by removing all legs from H, hence $[H] = G - S$.

We have defined neither orientations nor rooted orientations on a graph with legs. We do it now in the simplest possible way, namely by simply disregarding the legs and using exactly the same terminology. In particular, for any $H \in \mathcal{H}^1_G$, we have

$$\overline{O}^1(H) = \overline{O}^1([H]).$$

Now consider the set of all rooted orientations on all (connected) spanning subgraphs of G:

$$\overline{\mathcal{OP}}^1_G := \bigsqcup_{H \in \mathcal{H}^1_G} \overline{O}^1(H). \tag{18}$$

The above set admits a natural partial order. To define it recall that for any $H \in \mathcal{H}^1_G$, we denote by $S_H \subset E$ the set of edges such that $H = G - S^o_H$. We write O_H for an orientation on H. Then for two classes of rooted orientations, $\overline{O}_{H_1}$ and $\overline{O}_{H_2}$, we set $\overline{O}_{H_1} \leq \overline{O}_{H_2}$ if $H_1 \subset H_2$ and if the restriction of O_{H_2} to H_1 is equivalent to O_{H_1}.

The forgetful map below is a (surjective) quotient of posets

$$\varphi : \overline{\mathcal{OP}}^1_G \longrightarrow \mathcal{H}^1_G, \qquad \overline{O}_H \mapsto H. \tag{19}$$

The following result of [14], using the terminology of Theorem 3.4.1, states that $\overline{P}^g_X$ admits a recursive graded stratification governed by rooted orientations.

Theorem 4.2.3. *Let X be a stable curve of genus g, and let G be its dual graph. Then $\overline{P}^g_X$ admits the graded stratification*

$$\overline{P}^g_X = \bigsqcup_{\overline{O}_H \in \overline{\mathcal{OP}}^1_G} P^{O_H}_X, \tag{20}$$

with a natural isomorphism $P^{O_H}_X \cong \mathrm{Pic}^{\underline{d}^{O_H}}(X^\nu_{S_H})$ for every $\overline{O}_H \in \overline{\mathcal{OP}}^1_G$.

4.3. **Néron compactified Jacobians.** Now let us recall that $\overline{P}^g_X$ is of Néron type. Hence by Theorem 3.4.1 we have the following stratification of $\overline{P}^g_X$ and the associated stratification map σ_N,

$$\sigma_N : \overline{P}^g_X = \bigsqcup_{H \in \mathcal{H}^1_G} N_H \longrightarrow \mathcal{H}^1_G \tag{21}$$

such that $\sigma_N^{-1}(H) = N_H$. We know that σ_N is surjective and its fibers are not always connected.

Similarly, Theorem 4.2.3 gives us another stratification map

$$\sigma_P : \overline{P}^g_X \longrightarrow \overline{\mathcal{OP}}^1_G,$$

whose fiber over the class of O_H is the stratum $P^{O_H}_X$. In this case the stata are connected, so that (20) is a refinement of (21) with connected strata. Combining with the map in (19), we have the commutative diagram

$$\begin{tikzcd} \overline{P}^g_X \arrow[r, two heads, "\sigma_P"] \arrow[dr, two heads, "\sigma_N"'] & \overline{\mathcal{OP}}^1_G \arrow[d, "\varphi"] \\ & \mathcal{H}^1_G . \end{tikzcd}$$

In other words, for any connected spanning subgraph H of G, we have

$$N_H = \bigsqcup_{O_H \in \overline{O}^1(H)} P^{O_H}_X$$

which, by Theorems 3.4.1 and 4.2.3, implies that we have an isomorphism

$$N_X \cong \bigsqcup_{O \in \overline{O}^1(G)} \mathrm{Pic}^{\underline{d}^O}(X). \tag{22}$$

This is an explicit description of the isomorphism (11), within a specific moduli problem.

We now show how the stratification of $\overline{M}_{g,n}$ of Theorem 2.3.2 fits together with the stratifications of Theorems 3.4.1 and 4.2.3. We defined above two stratification maps, σ_N and σ_P. We can define the analogous stratification map for $\overline{M}_g$, using Theorem 2.3.2:

$$\sigma_M : \overline{M}_g \longrightarrow \mathcal{G}_g$$

whose fiber over the graph G is the stratum M_G.

Now, consider the union of all posets $\mathcal{H}^1_G$ for all stable graphs G,

$$\mathcal{H}^1_g := \bigsqcup_{G\in\mathcal{G}_g} \mathcal{H}^1_G.$$

We have an obvious map

$$\gamma_g : \mathcal{H}^1_g \longrightarrow \mathcal{G}_g$$

mapping $\mathcal{H}^1_G$ to G. Now, $\mathcal{H}^1_g$ has a natural poset structure which restricts to the poset structure of $\mathcal{H}^1_G$ for every G and is compatible with the poset structure of $\mathcal{G}_g$. More precisely, the map γ_g is a quotient of posets.

We can argue in a similar way for the posets of rooted orientations up to equivalence. Namely, we define

$$\overline{\mathcal{OP}}^1_g := \bigsqcup_{G\in\mathcal{G}_g} \overline{\mathcal{OP}}^1_G$$

with a natural map

$$\varphi_g : \overline{\mathcal{OP}}^1_g \longrightarrow \mathcal{H}^1_g$$

mapping an orientation class $\overline{O}_H \in \overline{O}^1(H) \subset \overline{\mathcal{OP}}^1_G$ to the graph H on which the orientation is defined.

It turns out that $\overline{\mathcal{OP}}^1_g$ also has a poset structure extending that of the $\overline{\mathcal{OP}}^1_G$ and is compatible with that of $\mathcal{H}^1_g$, so that the above map φ_g is a quotient of posets; see [11].

Now we need a bit of extra care to describe the global picture over $\overline{M}_g$. In fact there are, as we mentioned earlier, technical problems for curves having nontrivial automorphisms. To avoid dealing with them, we here restrict to the locus in $\overline{M}_g$ of curves free from automorphisms, denoted by $\widetilde{M}_g$; we refer to [11] for the general case. Recall that if $g \geq 3$, then $\widetilde{M}_g$ is open and dense in $\overline{M}_g$. We write $\widetilde{P}^g_g$ for the restriction of $\overline{P}^g_g$ over $\widetilde{M}_g$.

In conclusion, we have the following commutative diagram involving the maps we described earlier.

(23)
$$\begin{tikzcd}
\widetilde{P}^g_g \arrow[r, "\sigma_P"] \arrow[dr, "\sigma_N"] \arrow[dd, "\psi_g"'] & \overline{\mathcal{OP}}^1_g \arrow[d, "\varphi_g"] \\
& \mathcal{H}^1_g \arrow[d, "\gamma_g"] \\
\widetilde{M}_g \arrow[r, "\sigma_M"] & \mathcal{G}_g
\end{tikzcd}$$

4.4. **Compactified Jacobians in degree $g-1$.** The case $d = g-1$ has been the object of much attention because of its connections with the Theta divisor, the Torelli and the Schottky problems, and the Prym varieties. On the other hand, the compactified Jacobian $\overline{P}^{g-1}_X$ is never of Néron type, and its moduli properties are not as good.

Let us analyze $\overline{P}^{g-1}_X$ by the same pattern used for $\overline{P}^g_X$. We need to describe the special set Δ_{g-1} of multidegrees of total degree $g-1$ which correspond to the irreducible components of $\overline{P}^{g-1}_X$. The basic identity to look at is, again, $g-1 = \sum_{v\in V}\big(h(v)-1\big)+|E|$.

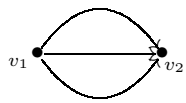
How can we partition $|E|$ among the components of X? As before, if we consider an orientation O on G and denote by $\underline{t}_v^O$ the number of edges ending at the vertex v, we have $\sum_{v\in V} \underline{t}_v^O = |E|$. We define the multidegree $\underline{d}^O$ of O exactly as we did in (17), by setting $\underline{d}_v^O = h(v) - 1 + \underline{t}_v^O$ for every $v \in V$. Then, as we already noticed, we have $|\underline{d}^O| = g - 1$, which now is what we need.

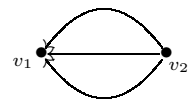
As for 1-orientations, two orientations with the same multidegree are defined to be equivalent.

How many equivalence classes of orientations do we have on a graph G? One easily sees, as is Example 4.4.2, that there are more than $|\Phi_G|$, which is against our expectation (16).

Therefore, to compactify the degree $(g-1)$ Jacobian, we must select a special type of orientation, or we must exclude some of them. To explain which orientations to exclude, we connect to Example 3.2.1 and argue as in the next example.

Example 4.4.1. Consider the two following orientations, O_1 and O_2, on the same graph.

$O_1 =$ 

$O_2 =$ 

Suppose $g(v_1) = g(v_2) = 1$. Then $\underline{d}^{O_1} = (0,3)$ and $\underline{d}^{O_2} = (3,0)$; hence, O_1 and O_2 are not equivalent.

Now let X be a curve having the above graph as a dual graph, and view it as the special fiber of a family $\mathcal{X} \to \operatorname{Spec} R$ with $\mathcal{X}$ nonsingular, as in Example 3.2.1. So, X has three nodes and two irreducible components, C_1 and C_2. Now let $\mathcal{L}$ be a line bundle on $\mathcal{X}$ such that its restriction to X has the multidegree

$$\underline{\deg}\mathcal{L}_{|X} = (3,0) = \underline{d}^{O_2}$$

(it is not hard to prove such an $\mathcal{L}$ exists). Now, since $\mathcal{X}$ is nonsingular, C_1 is a Cartier divisor on $\mathcal{X}$ of multidegree $(-3,3)$. Hence we can consider the line bundle $\mathcal{L}' := \mathcal{L}(C_1)$, whose multidegree on X satisfies

$$\underline{\deg}\mathcal{L}'_{|X} = \underline{\deg}\mathcal{L}_{|X} + (-3,3) = (3,0) + (-3,3) = (0,3) = \underline{d}^{O_1}.$$

As the restrictions of $\mathcal{L}$ and $\mathcal{L}'$ to $\mathcal{X}_K$ coincide, we conclude that $\underline{d}^{O_1}$ and $\underline{d}^{O_2}$ are the multidegrees of two different limits of the same line bundle.

Now, compactified Jacobians are projective and, hence, separated. Therefore the moduli map of $\mathcal{L}$ and $\mathcal{L}'$ from $\operatorname{Spec} R$ to P_R^{g-1} must coincide. In fact, the image of the special point of $\operatorname{Spec} R$ under this moduli map turns out to lie in the boundary of $\overline{P}_X^{g-1}$ and to parametrizes both $\mathcal{L}_{|X}$ and $\mathcal{L}'_{|X}$.

The conclusion we want to draw is that the two multidegrees $\underline{d}^{O_1}$ and $\underline{d}^{O_2}$ cannot lie in Δ_{g-1}; hence, we need to exclude the orientations O_1 and O_2. A close look shows that the reason why O_1 and O_2 must be excluded is that they have a vertex with no incoming edge.

With this example in mind, we define an orientation to be *totally cyclic* if every set of vertices Z admits at least one incoming edge, i.e., an edge with target in Z and source not in Z. In analogy with rooted orientation, we mention that totally cyclic orientations are characterized by the property that any two vertices in the

same connected component lie in a directed cycle. The set of equivalence classes of totally cyclic orientations on G is denoted by $\overline{O}^0(G)$.

Example 4.4.2. In Figure 8 we have the four orientations on a 2-cycle. The first two are totally cyclic and equivalent. The last two are not totally cyclic and not equivalent. In this case $|\overline{O}^0(G)| < |\Phi_G| = 2$.

FIGURE 8. The four orientations on a 2-cycle

Now, as we did for $d = g$, we want to consider totally cyclic orientations on all subgraphs of G. For this reason we need to extend our consideration to disconnected graphs.

Remark 4.4.3. Let H be a (possibly disconnected) graph.

(a) If H is connected, then $\overline{O}^0(H) = \Delta_{g-1}$, and if $|V| \geq 2$, $|\overline{O}^0(H)| < |\Phi_H|$.
(b) $\overline{O}^0(H)$ is empty if and only if H contains some bridge (an edge whose removal disconnects the connected component in which it lies).

Since totally cyclic orientations exist only on bridgeless graphs, we introduce a new sublattice of $\mathcal{H}_G$:

$$\mathcal{H}_G^0 := \{H \in \mathcal{H}_G : H \text{ has no bridge}\}.$$

The maximum of $\mathcal{H}_G^0$ is the graph obtained from G by removing all bridges, the minimum is $G - E^o$. The poset $\mathcal{H}_G^0$ is graded by $H \mapsto g(H)$. We write

$$\overline{\mathcal{OP}}_G^0 := \bigsqcup_{H \subset \mathcal{H}_G^0} \overline{O}^0(H)$$

for the set of all classes of totally cyclic orientations on all spanning subgraphs of G. We have a partial order on $\overline{\mathcal{OP}}_G^0$ exactly as for $\overline{\mathcal{OP}}_G^1$. Now, rephrasing results from [12], we are ready to exhibit a graded stratification of $\overline{P}_X^{g-1}$ governed by totally cyclic orientations, and we use the same terminology as in Theorem 4.2.3.

Theorem 4.4.4. *Let X be a stable curve of genus g, and let G be its dual graph. Then we have a graded stratification*

$$\overline{P}_X^{g-1} = \bigsqcup_{\overline{O}_H \in \overline{\mathcal{OP}}_G^0} P_X^{O_H} \tag{24}$$

and a natural isomorphism for every $\overline{O}_H \in \overline{\mathcal{OP}}_G^0$,

$$P_X^{O_H} \cong \mathrm{Pic}^{\underline{d}^{O_H}}(X_{S_H}^\nu). \tag{25}$$

The isomorphism (25) exhibits a recursive behavior which we have already encountered in our earlier statements. Indeed, $\mathrm{Pic}^{\underline{d}^{O_H}}(X_{S_H}^\nu) \cong J_{X_{S_H}^\nu}$ and $\underline{d}^{O_H}$ is the multidegree associated to a totally cyclic orientation on $G - H$. Hence the boundary of the compactified degree $(g-1)$ Jacobian of X is stratified by Jacobians of degree $(g(X') - 1)$ of partial normalizations X' of X.

Now, having diagram (23) in mind, we ask whether the stratifications of Theorem 4.4.4 glue together over $\overline{M}_g$ consistently with the stratification of $\overline{M}_g$ of Theorem 2.3.2. The answer is positive, and, reasoning as we did for $d = g$ with some obvious modification, we have the commutative diagram below, where the posets $\overline{\mathcal{OP}}_g^0$ and $\mathcal{H}_g^0$ are defined exactly as $\overline{\mathcal{OP}}_g^1$ and $\mathcal{H}_g^1$ (in the previous section). We refer to [11] for details.

$$\begin{tikzcd}
\widetilde{P}_g^{g-1} \arrow[r, "\sigma_P"] \arrow[d, "\psi_{g-1}"'] & \overline{\mathcal{OP}}_g^0 \arrow[dr, "\varphi_{g-1}"] & \\
& & \mathcal{H}_g^0 \arrow[dl, "\gamma_{g-1}"] \\
\widetilde{M}_g \arrow[r, "\sigma_M"] & \mathcal{G}_g &
\end{tikzcd} \tag{26}$$

From Remark 4.4.3(a), it is clear that $\overline{P}_X^{g-1}$ is not of Néron type unless X is irreducible or of compact type. Nevertheless $\overline{P}_X^{g-1}$ as been used in various applications.

An example of these applications concerns the Theta divisors and the generalized Torelli map. Recall that on a smooth curve X a general line bundle of degree $g-1$ has no nontrivial sections, whereas as soon as the degree is at least g, Riemann–Roch theory predicts the existence of nontrivial sections for every line bundle of that degree. The subset in $\mathrm{Pic}^{g-1}(X)$ of all line bundles admitting some nontrivial global section is a proper closed subset of codimension 1, called the Theta divisor, denoted by Θ_X,

$$\Theta_X := \{L \in \mathrm{Pic}^{g-1}(X) : \ h^0(L) \neq 0\}. \tag{27}$$

It is well known that Θ_X is a prime divisor giving a principal polarization on $\mathrm{Pic}^{g-1}(X)$. By using the isomorphisms $\mathrm{Pic}^{g-1}(X) \cong J_X$, we obtain Theta divisors on the Jacobian (or on any other $\mathrm{Pic}^d(X)$), but since these isomorphisms are not natural, the Theta divisor of $\mathrm{Pic}^d(X)$ is canonically given only in the case where $d = g-1$ (in the other cases, its class is canonically given).

A famous theorem involving the Theta divisor is the Torelli theorem, stating that a smooth curve is uniquely determined, up to isomorphism, by the pair given by its Jacobian and its Theta divisor.

The definition of Θ_X given in (27), with small modifications, makes sense also for our singular curve X, and enables us to define the Theta divisor $\overline{\Theta}_X$ in $\overline{P}_X^{g-1}$ as the closure of the locus of line bundles on X parametrized by $\overline{P}_X^{g-1}$ admitting nontrivial global sections. Moreover, $\overline{\Theta}_X$ is a Cartier divisor and a principal polarization, and the pair $(\overline{P}_X^{g-1}, \overline{\Theta}_X)$ is endowed with a natural group action of J_X. Using the language of moduli theory for abelian varieties, $(\overline{P}_X^{g-1}, \overline{\Theta}_X)$ is a so-called *principally polarized stable semi-abelic pair*. Such pairs appear as the boundary points in the compactification $\overline{A}_g^{\mathrm{mod}}$ of the moduli space of principally polarized abelian varieties constructed in [2]. Moreover, they form the image of the compactified Torelli morphism

$$\overline{\tau} : \overline{M}_g \longrightarrow \overline{A}_g^{\mathrm{mod}}$$

mapping a curve X to the pair $(\overline{P}_X^{g-1}, \overline{\Theta}_X)$. By the Torelli theorem, the restriction of $\overline{\tau}$ to M_g is injective, but $\overline{\tau}$ is easily seen not to be injective. The combinatorial

structure of $\overline{P}_X^{g-1}$ described in Theorem 4.4.4 is a key tool in [12] to describe the fibers of $\overline{\tau}$ in detail.

We conclude by observing that, by the moduli properties of $\overline{A}_g^{\mathrm{mod}}$, the compactified Torelli map $\overline{\tau}$ can be viewed as the moduli map associated to the family $\psi_{g-1} : \overline{P}_g^{g-1} \to \overline{M}_g$.

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References

[1] Dan Abramovich, Lucia Caporaso, and Sam Payne, *The tropicalization of the moduli space of curves* (English, with English and French summaries), Ann. Sci. Éc. Norm. Supér. (4) **48** (2015), no. 4, 765–809, DOI 10.24033/asens.2258. MR3377065

[2] Valery Alexeev, *Complete moduli in the presence of semiabelian group action*, Ann. of Math. (2) **155** (2002), no. 3, 611–708, DOI 10.2307/3062130. MR1923963

[3] Enrico Arbarello, Maurizio Cornalba, and Pillip A. Griffiths, *Geometry of algebraic curves. Volume II*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 268, Springer, Heidelberg, 2011. With a contribution by Joseph Daniel Harris. MR2807457

[4] M. Artin, *Néron models*, Arithmetic geometry (Storrs, Conn., 1984), Springer, New York, 1986, pp. 213–230. MR861977

[5] Arnaud Beauville, *Prym varieties and the Schottky problem*, Invent. Math. **41** (1977), no. 2, 149–196, DOI 10.1007/BF01418373. MR0572974

[6] Vladimir G. Berkovich, *Spectral theory and analytic geometry over non-Archimedean fields*, Mathematical Surveys and Monographs, vol. 33, American Mathematical Society, Providence, RI, 1990. MR1070709

[7] Siegfried Bosch, Werner Lütkebohmert, and Michel Raynaud, *Néron models*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 21, Springer-Verlag, Berlin, 1990. MR1045822

[8] Silvia Brannetti, Margarida Melo, and Filippo Viviani, *On the tropical Torelli map*, Adv. Math. **226** (2011), no. 3, 2546–2586, DOI 10.1016/j.aim.2010.09.011. MR2739784

[9] Lucia Caporaso, *A compactification of the universal Picard variety over the moduli space of stable curves*, J. Amer. Math. Soc. **7** (1994), no. 3, 589–660, DOI 10.2307/2152786. MR1254134

[10] Lucia Caporaso, *Néron models and compactified Picard schemes over the moduli stack of stable curves*, Amer. J. Math. **130** (2008), no. 1, 1–47, DOI 10.1353/ajm.2008.0000. MR2382140

[11] Lucia Caporaso and Karl Christ, *Combinatorics of compactified universal Jacobians*. arXiv:1801.04098 (2018).

[12] Lucia Caporaso and Filippo Viviani, *Torelli theorem for stable curves*, J. Eur. Math. Soc. (JEMS) **13** (2011), no. 5, 1289–1329, DOI 10.4171/JEMS/281. MR2825165

[13] Renzo Cavalieri, Hannah Markwig, and Dhruv Ranganathan, *Tropicalizing the space of admissible covers*, Math. Ann. **364** (2016), no. 3-4, 1275–1313, DOI 10.1007/s00208-015-1250-8. MR3466867

[14] Karl Christ, *Orientations, break Divisors and compactified Jacobians*, PhD thesis in Mathematics, Roma Tre University (2017).

[15] P. Deligne and D. Mumford, *The irreducibility of the space of curves of given genus*, Inst. Hautes Études Sci. Publ. Math. **36** (1969), 75–109. MR0262240

[16] D. Gieseker, *Lectures on moduli of curves*, Tata Institute of Fundamental Research Lectures on Mathematics and Physics, vol. 69, Published for the Tata Institute of Fundamental Research, Bombay; Springer-Verlag, Berlin-New York, 1982. MR691308

[17] Alexander Grothendieck, *Technique de descente et théorèmes d'existence en géométrie algébrique. V. Les schémas de Picard: théorèmes d'existence* (French), Séminaire Bourbaki, Vol. 7, Soc. Math. France, Paris, 1995, pp. Exp. No. 232, 143–161. MR1611170

[18] Alexander Grothendieck, *Techniques de construction et théorèmes d'existence en géométrie algébrique. IV. Les schémas de Hilbert* (French), Séminaire Bourbaki, Vol. 6, Soc. Math. France, Paris, 1995, pp. Exp. No. 221, 249–276. MR1611822

[19] Joe Harris and Ian Morrison, *Moduli of curves*, Graduate Texts in Mathematics, vol. 187, Springer-Verlag, New York, 1998. MR1631825

[20] Joe Harris and David Mumford, *On the Kodaira dimension of the moduli space of curves*, Invent. Math. **67** (1982), no. 1, 23–88, DOI 10.1007/BF01393371. With an appendix by William Fulton. MR664324

[21] Finn F. Knudsen, *The projectivity of the moduli space of stable curves. II. The stacks $M_{g,n}$*, Math. Scand. **52** (1983), no. 2, 161–199, DOI 10.7146/math.scand.a-12001. MR702953

[22] Grigory Mikhalkin, *Moduli spaces of rational tropical curves*, Proceedings of Gökova Geometry-Topology Conference 2006, Gökova Geometry/Topology Conference (GGT), Gökova, 2007, pp. 39–51. MR2404949

[23] D. Mumford, J. Fogarty, and F. Kirwan, *Geometric invariant theory*, 3rd ed., Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)], vol. 34, Springer-Verlag, Berlin, 1994. MR1304906

[24] André Néron, *Modèles minimaux des variétés abéliennes sur les corps locaux et globaux* (French), Inst. Hautes Études Sci. Publ.Math. No. **21** (1964), 128. MR0179172

[25] Tadao Oda and C. S. Seshadri, *Compactifications of the generalized Jacobian variety*, Trans. Amer. Math. Soc. **253** (1979), 1–90, DOI 10.2307/1998186. MR536936

[26] M. Raynaud, *Spécialisation du foncteur de Picard* (French), Inst. Hautes Études Sci. Publ. Math. **38** (1970), 27–76. MR0282993

[27] Amaury Thuillier, *Géométrie toroïdale et géométrie analytique non archimédienne. Application au type d'homotopie de certains schémas formels* (French, with English summary), Manuscripta Math. **123** (2007), no. 4, 381–451, DOI 10.1007/s00229-007-0094-2. MR2320738

[28] Eckart Viehweg, *Quasi-projective moduli for polarized manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 30, Springer-Verlag, Berlin, 1995. MR1368632

[29] Filippo Viviani, *Tropicalizing vs. compactifying the Torelli morphism*, Tropical and non-Archimedean geometry, Contemp. Math., vol. 605, Amer. Math. Soc., Providence, RI, 2013, pp. 181–210, DOI 10.1090/conm/605/12116. MR3204272

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