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SELECTED MATHEMATICAL REVIEWS

related to the paper in the previous section by
ALEX WRIGHT

MR0690844 (85e:32028) 32G15; 22E40, 30F20, 57M99, 57N10, 58F17
Wolpert, Scott
On the symplectic geometry of deformations of a hyperbolic surface.
Annals of Mathematics. Second Series **117** (1983), *no.* 2, 207–234.

The author describes a fresh and interesting aspect of the structure of Teichmuller space $T(R)$ for a compact Riemann surface R. By dint of a detailed and quite explicit analysis of the deformations known as Fenchel-Nielsen twists (along simple closed geodesics), the author is able to establish a deep connection between the variation of the hyperbolic geometry of a Riemann surface, as manifested by the changes in length of systems of simple geodesic loops, and the symplectic geometry of the space $T(R)$ deriving from the Kahler form defined by Weil using the Petersson inner product on quadratic automorphic forms.

To each simple loop α in R there is associated a 1-parameter family of diffeomorphisms of $T(R)$ obtained by cutting each marked surface along the unique geodesic in the homotopy class of α and sliding one side relative to the other a distance t in the hyperbolic metric to obtain a new marked surface. The infinitesimal generator of this is denoted t_α, the Fenchel-Nielsen vector field of α on $T(R)$. The author described this construction in terms of quasiconformal mappings in an earlier article [ame journal 115 (1982), no. 3, 501–528; MR0657237]. Here it is shown that, if l_α denotes the length function on $T(R)$ which measures the length of the geodesic on R_x in the homotopy class of α, then $\omega(t_\alpha, -) = -dl_\alpha$, where ω is the Kahler form; consequently the vector fields t_α are Hamiltonian for the symplectic form ω. There also follows a beautiful formula signifying the interdependence of the symplectic geometry of $T(R)$ and the hyperbolic geometry of the surfaces R, viz. $\omega(t_\alpha, t_\beta) = \sum \cos\theta_p$, the sum taken over intersections p (with angle θ_p) of α and β. Finally a structure of Lie algebra over $\mathbf{Z}$ is defined on the span of the normalised twist vector fields.

The techniques employed in deriving the analytic structure of the twist deformations involve identifying the differential of the cross-ratio χ-invariant of four points on $\mathbf{R} \cup \{\infty\}$ and interpreting this for the function l_α and the angle θ between two geodesics, both expressible in terms of χ's. These result in formulae for variation of l_α and $\cos\theta$ first derived by D. A. Hejhal [Bull. Amer. Math. Soc. 84 (1978), no. 3, 339–376; MR0492237] in a study of Poincare series and periods of holomorphic differentials, whose relationship to the twist derivative was exhibited in the previous article by the author [op. cit.]. The author now proves an important formula for the second twist derivative of the length function l_α, which implies a result of S. P. Kerckhoff that l_α is a convex function when viewed along integral curves in $T(S)$ of a twist vector field t_β (see the review of the paper by S. P. Kerckhoff which follows immediately after this review for further comments).

The paper combines clear exposition with explicit computations, which provides a very satisfying picture of the local real analytic structure of Teichmuller space.

William Harvey

From MathSciNet, April 2020

MR1677888 (2000b:57025) 57M50

Otal, Jean-Pierre

Thurston's hyperbolization of Haken manifolds.

Surveys in differential geometry, Vol. III (Cambridge, MA, 1996), 77–194, *Int. Press, Boston, MA*, 1998.

Thurston's hyperbolization theorem for Haken manifolds says that if M is a compact, oriented, irreducible, atoroidal 3-manifold, and if M is Haken, meaning that M contains a properly embedded incompressible surface, then M has a geometrically finite hyperbolic structure. Thurston's original proof was expounded through a combination of lectures, survey articles, and research articles, and it had a profound influence on low-dimensional topology and geometry as it filtered through the mathematical community. The proof has been well digested, and various steps have been re-proved using different techniques. After two decades, complete published proofs are now appearing.

The proof of the hyperbolization theorem for the case where M fibers over the circle is different from the non-fibered case, and has been given a complete treatment in the author's previous work [Astérisque No. 235 (1996), x+159 pp.; MR1402300]. The current article covers the non-fibered case, with a further restriction to the case where M has no torus boundary components (the general case is covered by informal notes of Otal and Paulin). With these restrictions, the article gives a complete proof, starting from basic knowledge of 3-dimensional topology, hyperbolic geometry, and quasiconformal maps. The proof combines elements of Thurston's original proof with methods of McMullen.

Thurston's proof is inductive, based on a Haken hierarchy which cuts M into successively simpler pieces. The last step of the induction is the final gluing theorem, which says that if N is a compact, oriented manifold with incompressible boundary, if N is not an interval bundle, if τ is an orientation-reversing involution of ∂N which exchanges the boundary components by pairs, if the quotient N/τ is atoroidal, and if N has a geometrically finite hyperbolic structure, then N/τ is hyperbolic. Even the induction step can be reduced to the final gluing theorem, using a mirroring trick of Thurston.

Thurston reduced the final gluing theorem to a fixed point problem on $\mathcal{T}(\partial N)$, the Teichmüller space of ∂N. To simplify the explanation we assume ∂N is connected. A geometrically finite hyperbolic structure on N corresponds, by a theorem of Ahlfors-Bers, to a conformal structure on ∂N, or more precisely, a point s of $\mathcal{T}(\partial N)$, the Teichmüller space of ∂N. The covering space of N corresponding to ∂N is $N \times [0,1]$, with ∂N lifting to $\partial N \times 1$ and with $\partial N \times 0$ corresponding to $\overline{\partial N}$, the orientation reversal of ∂N. Lifting the hyperbolic structure on N one obtains a geometrically finite hyperbolic structure on $N \times [0,1]$. Another application of the Ahlfors-Bers theorem returns the point $s \in \mathcal{T}(\partial N)$ as well as a point $s' \in \mathcal{T}(\overline{\partial N})$. The map $\sigma\colon \mathcal{T}(\partial N) \to \mathcal{T}(\overline{\partial N})$ defined by $\sigma(s) = s'$ is the so-called skinning map. Composing with the orientation reversal map $\tau^*\colon \mathcal{T}(\overline{\partial N}) \to \mathcal{T}(\partial N)$ one obtains a map $\tau^* \circ \sigma\colon \mathcal{T}(\partial N) \to \mathcal{T}(\partial N)$. Thurston's fixed point theorem says that if N/τ is atoroidal then $\tau^* \circ \sigma$ has a fixed point, and this, combined with Maskit's combination theorem, provides a proof of the final gluing theorem.

C. T. McMullen [Invent. Math. **99** (1990), no. 2, 425–454; MR1031909] gave a new proof of the fixed point theorem. As recounted in the current article, "This approach originated in an observation that J. Hubbard made shortly after Thurston enunciated the fixed point theorem. Hubbard noticed that the formula for the coderivative of the skinning map involved a well-known operator in Teichmüller theory, the Poincaré series operator" or Theta operator. Kra had earlier formulated a conjecture about the norm of the Theta operator, and McMullen proved this "Theta conjecture" and generalizations thereof in [C. T. McMullen, Invent. Math. **97** (1989), no. 1, 95–127; MR0999314]. The current article follows McMullen's method, tailored by an alternative approach to the Theta conjecture due to D. E. Barrett and J. Diller [Michigan Math. J. **43** (1996), no. 3, 519–538; MR1420590].

The article is complete and detailed, in particular containing full accounts of the mirroring trick and the reduction to the fixed point theorem, McMullen's proof of the fixed point theorem, and the results of Barrett and Diller.

Lee Mosher

From MathSciNet, April 2020

MR1354958 (97h:57028) 57M50; 57N10

Gabai, David

On the geometric and topological rigidity of hyperbolic 3-manifolds.

Journal of the American Mathematical Society **10** (1997), *no.* 1, 37–74.

In this important and impressive paper, the author considers rigidity problems for 3-manifolds. He considers a closed hyperbolic 3-manifold N which satisfies a certain condition which he calls the insulator condition. The precise statement of this condition is technical and will be discussed later, but very many hyperbolic 3-manifolds satisfy this condition. In particular, if the shortest closed geodesic in N is very short, or is very long, then N satisfies the insulator condition. Precise bounds for these lengths are given in the paper. The main theorem of this paper is as follows: Theorem: Let N be a closed hyperbolic 3-manifold satisfying the insulator condition. (1) If M is a closed irreducible 3-manifold and $f\colon M \to N$ is a homotopy equivalence, then f is homotopic to a homeomorphism. (2) If f and g are homotopic homeomorphisms of N, then they are isotopic. (3) The space of hyperbolic metrics on N is path connected. The author conjectures that all closed hyperbolic 3-manifolds satisfy the insulator condition. In March 1996, Gabai, Meyerhoff and N. Thurston announced that they had proved this conjecture.

Before the review itself, here is a brief discussion of how these results relate to previous results. For brevity, I will restrict attention to the situation where M and N are both closed, orientable and irreducible with infinite fundamental group. In [Ann. of Math. (2) **87** (1968), 56–88; MR0224099], F. Waldhausen showed that Parts (1) and (2) hold under the additional assumption that N is Haken. (Actually he used the term "sufficiently large" but this term has been replaced by "Haken" these days.) In [Ann. of Math. (2) **117** (1983), no. 1, 35–70; MR0683801; Topology **24** (1985), no. 3, 341–351; MR0815484], the reviewer showed that (1) and (2) hold under the additional assumption that N is a Seifert fibre space. In [Topology **31** (1992), no. 3, 493–517; MR1174254; Comment. Math. Helv. **68** (1993), no. 3, 341–364; MR1236759], the reviewer and J. Hass showed that (1) and (2) hold if N admits a π_1-injective map f of a closed orientable surface, not S^2, which satisfies some technical conditions which mean that f is not very singular. Also Mostow's rigidity theorem implies that (1) holds if both M and N are assumed to be hyperbolic. Finally, the author of the paper under review showed in a recent paper [J. Amer. Math. Soc. **7** (1994), no. 1, 193–198; MR1205445] that under the hypotheses of Part (1) of the above theorem, and without any insulator condition, there is a common finite cover of M and N. Both Waldhausen's result and this result are used in the proof of the paper under review. It should be pointed out that Part (3) does not follow from Mostow's rigidity theorem, as that result shows only that given two metrics on a closed hyperbolic 3-manifold N, there is an isometry between them which is homotopic to the identity, whereas the result of Part (3) asserts that the isometry can be chosen to be isotopic to the identity.

Now we turn to the author's proof of his main theorem. This is a major piece of work. I will attempt to describe the ideas in simple terms, but several parts of the argument are very difficult to carry out. The initial idea for the proof is that one should choose a simple closed curve γ in M in such a way that f sends γ to a simple closed curve δ in N by a homeomorphism, sends a regular neighbourhood $N(\gamma)$ of γ to a regular neighbourhood $N(\delta)$ of δ by a homeomorphism, and in addition f maps $M - N(\gamma)$ into $N - N(\delta)$. If one can do this, then Waldhausen's theorem on Haken manifolds implies that the restricted map $f\colon M - N(\gamma) \to N - N(\delta)$ is homotopic to a homeomorphism by a homotopy fixing the boundary of $N(\gamma)$, and hence that f itself is homotopic to a homeomorphism. This is not a new idea, but the problem was that there seemed no way of choosing γ and δ appropriately. In [op. cit.], the author showed that these choices can be made if one is prepared to replace M and N by appropriate finite covers. In this paper, he shows that these choices can be made if one has the insulator condition.

As there is no natural way to choose any particular simple closed curve in M, one starts by considering the choice of δ in N. Here one has the slight advantage that there is a canonical choice for δ within any homotopy class. Namely one can choose δ to be the closed geodesic in that class. It is possible that this geodesic might be singular, but this can be avoided by choosing δ to be the shortest closed geodesic in N, as this geodesic is automatically simple. For the moment, let δ denote any simple closed geodesic in N and consider the problem of choosing γ correctly. The choice of δ determines the homotopy class of the curve γ in M, but there are infinitely many isotopy classes to choose from. If M and N are indeed homeomorphic there is exactly one "correct" isotopy class of γ, and we can think of all the other isotopy classes of homotopic curves as being knotted in some sense. Thus the problem can be thought of as the problem of choosing the unknotted isotopy class for γ. But as we do not yet know that M and N are homeomorphic, it is not at all clear that there is any isotopy class of γ which will have the required property.

Given a simple closed geodesic δ in the hyperbolic 3-manifold N, the author describes a canonical solid torus neighbourhood W of δ, which he calls the Dirichlet neighbourhood of δ, as its definition is analogous to that of the Dirichlet fundamental region of a discrete group acting on hyperbolic space. Consider the picture in the universal cover $\widetilde{N}$ of N. There will be an infinite family Δ of disjoint geodesics forming the pre-image of δ. Fix one of these geodesics δ_0. Given any other geodesic δ_i in this family, there is a shortest line segment d_i between δ_0 and δ_i, and d_i is itself a geodesic segment and is perpendicular to both δ_0 and δ_i. Let D_i denote the hyperbolic plane which meets d_i orthogonally at its midpoint. Thus D_i is the plane halfway between δ_0 and δ_i. This gives rise to an infinite family D_0 of planes in $\widetilde{N}$ which will be invariant under the stabiliser of the geodesic δ_0 and will cut $\widetilde{N}$ into regions. Let W_0 denote the region which contains δ_0. This region has closure homeomorphic to $\mathbf{R} \times D^2$ and its image in N has closure W which is a solid torus neighbourhood of the closed geodesic δ. Thus the canonical family D_0 of planes in $\widetilde{N}$ defines a canonical solid torus neighbourhood of δ in N. The key idea of the whole paper is that one should construct a family of planes with similar properties in the universal cover of M, but without having first chosen γ. If this can really be done, then there is a natural choice of γ, namely the core of the solid torus obtained in M. The construction of a suitable family of planes is described in the next few paragraphs.

At this point one uses the earlier result of the author which states that M and N must have a common finite cover. This makes it easy to identify the universal covers of M and N with each other and with hyperbolic 3-space H^3 in such a way that there is a lift of $f\colon M \to N$ to a map $\widetilde{f}\colon H^3 \to H^3$, which extends to the identity map on the sphere at infinity. The fundamental groups of M and N both act on H^3 as covering groups, and both actions extend to the same action on the sphere at infinity. (The actions are the same when one identifies the two groups via the isomorphism given by the map f.) There is no natural metric on $\widetilde{M}$, but if one chooses a metric on M, this will yield a metric on $\widetilde{M}$, and $\pi_1(M)$ will act by isometries. Of course, this metric will almost certainly not be hyperbolic.

Now consider the family Δ of geodesics in $\widetilde{N}$ and the family D_0 of planes midway between δ_0 and its translates. One wants to construct a similar picture in $\widetilde{M}$. As these two universal covers have already been identified, it is tempting to simply consider the images of these geodesics and planes under this identification. But this family need not be invariant under the action of $\pi_1(M)$. However, if we consider only the sphere at infinity of H^3, the endpoints of the geodesics forming Δ will be invariant under the action of $\pi_1(M)$, as the actions of $\pi_1(M)$ and $\pi_1(N)$ are the same. Similarly, the circles at infinity of the planes forming D_0 are also invariant under the action of $\pi_1(M)$. The idea now is to span these circles by planes invariant under the action of $\pi_1(M)$ which are area minimising using the metric in the universal cover of M. The first difficulty here is that it is not clear that there exists an area-minimising plane spanning each circle. The author needs to work very hard to get round this problem and shows instead that there is an area-minimising lamination spanning each circle. In what follows, I will ignore this problem and continue to refer to planes for simplicity of exposition.

Suppose now that we can span one of these circles by an area-minimising plane. By taking translates of this plane, we will obtain a family E_0 of planes invariant under the action of $\pi_1(M)$ which are area minimising using the metric in the universal cover $\widetilde{M}$ of M. Standard cut-and-paste arguments show that two of these planes are disjoint if they have disjoint boundary circles. The converse is trivial because any two of the circles being considered in the sphere at infinity must be disjoint or cross transversely. As two of the hyperbolic planes in D_0 are also disjoint if and only if their boundary circles are disjoint, it follows that two of the area-minimising planes in E_0 are disjoint if and only if the corresponding planes of the family D_0 are disjoint. Thus the way in which the new planes intersect each other is identical to that of the original hyperbolic planes if we only consider pairs of planes. However, there is no obvious information about how triples of planes should meet. The family of planes E_0 might divide $\widetilde{M}$ into regions in quite a different way from the way in which the hyperbolic planes in D_0 divided $\widetilde{N}$. We are interested in the component of the complement of the planes of E_0 which contains the two points at infinity a and b which form the endpoints of the geodesic δ_0. We would like this region to be homeomorphic to $\mathbf{R} \times D^2$ and to cover a solid torus in M, as for the family of hyperbolic planes D_0. The main problem is that there may be no such component. In other words, the two endpoints a and b of δ_0 might lie in different components of the complement of the family of planes E_0. This is where the insulator condition is used.

Now I will discuss what the insulator condition is about. First we consider a condition which the author calls "tri-linking". Recall that we are interested in a pair of points $\{a, b\}$ in the sphere, namely the endpoints of δ_0, and a family of circles in the sphere with the property that each circle separates the pair $\{a, b\}$ from some translate of $\{a, b\}$. Thus each individual circle does not separate a from b. This family of circles is said to exhibit tri-linking if there are three of them whose union separates a from b. (It is clearly impossible for the union of two of them to do this.) If tri-linking occurs, then it is possible to isotop the three planes so as to separate a from b in the 3-ball. Thus there is no reason why this should not occur within the family of planes E_0. However, the author proves that if no tri-linking occurs, then a and b must lie in one component V_0 of the complement of the planes in E_0, and that V_0 covers a solid torus V in M. A special case of the insulator condition occurs when there is some simple closed geodesic δ in N such that if one proceeds as above to choose a geodesic δ_0 and the family of hyperbolic planes D_0, then no tri-linking occurs. The most general form of the insulator condition asserts the existence of a family of circles on the sphere at infinity which again satisfy the no tri-linking condition but need not be the boundaries of hyperbolic planes, but I will say no more about that.

The conclusion so far is that if no tri-linking occurs in the family of planes D_0 in $\widetilde{N}$, then the corresponding family of planes E_0 in $\widetilde{M}$ yields a solid torus embedded in M, so that we can define γ to be the core of this solid torus. However, this is not the end of the argument by any means. First note that there is a worry here, as we know that there is only one correct isotopy class for γ, but the above construction seems as if it might yield different isotopy classes as one alters the choices of the planes E_0. And we still need to prove that f can be homotoped to send a regular neighbourhood $N(\gamma)$ of γ to a regular neighbourhood $N(\delta)$ of δ in N by a homeomorphism and in addition to send $M - N(\gamma)$ into $N - N(\delta)$. It is trivial to arrange that f is a homeomorphism between regular neighbourhoods of γ and δ. The difficulty is to arrange that f maps $M - N(\gamma)$ into $N - N(\delta)$. The author deals with these problems using some more impressive arguments. First he shows that the isotopy class of γ is independent of the choice of planes forming E_0 and is also independent of the choice of metric on M. Next he applies this result with the manifold M replaced by the common finite cover M_1 of M and N. This implies that the lifts of γ and δ to M_1 must be isotopic. It follows that the link Γ in H^3 formed by the pre-image of γ is equivalent to that formed by the pre-image Δ of δ, meaning that there is a homeomorphism of H^3 which sends one link to the other and is the identity on the sphere at infinity. In the last part of the argument, the author shows that this implies that f can be homotoped in the required way. Note that there are many homeomorphisms from $N(\gamma)$ to $N(\delta)$, but probably only one which extends to a homeomorphism from M to N. Thus there is a problem of choosing the correct homeomorphism from $N(\gamma)$ to $N(\delta)$. The details of this part of the argument are interesting but technical, and I will say nothing more about them.

G. Peter Scott

From MathSciNet, April 2020

MR1478844 (98i:32030) 32G15; 20H10, 30F60, 57N05

McMullen, Curtis T.

Complex earthquakes and Teichmüller theory.

Journal of the American Mathematical Society **11** (1998), *no.* 2, 283–320.

One can cut a hyperbolic surface S along a simple geodesic, twist through an angle t, and reglue, obtaining a marked surface representing a point in the Teichmüller space $T(S)$. An earthquake path $\mathbf{R} \to T(S)$ is defined by sending t to the resulting point. This construction extends easily to collections of disjoint geodesics individually weighted with twisting angles, and less easily, through work of Thurston, to measured laminations. Earthquakes have proven to be a useful and interesting structure; notably, they were used by Kerckhoff in the solution of the long-standing Nielsen realization problem.

In this paper, the author shows that any earthquake path extends to a proper holomorphic map $D \to T(S)$, where D is a simply-connected domain in $\mathbf{C}$ which contains the closed upper half-plane. This map is called a complex earthquake. The imaginary parameter s is related to a second construction called grafting. In the case of a simple geodesic, a grafting cuts S along the geodesic and inserts a flat cylinder of length s connecting the two copies. This also extends to collections of geodesics and to measured laminations. For $s > 0$, the complex earthquake carries $t + is$ to the point obtained by a real earthquake using the parameter t and a grafting using s. This extends to the rest of D by analytic continuation.

The author gives numerous results which relate complex earthquakes to other constructions and structures. These include bending, projective structures on S, the Weil-Petersson metric on $T(S)$, extremal length of laminations, and rigidity of cone-manifolds.

Some of the most surprising results concern the space of quasi-Fuchsian representations $\mathrm{QF}(S)$. This is a subset of the representation variety $V(S)$, the irreducible representations of $\pi_1(S)$ into $\mathrm{PSL}(2, \mathbf{C})$ sending elements of $\pi_1(\partial S)$ to parabolic elements, modulo conjugacy in $\mathrm{PSL}(2, \mathbf{C})$. A representation lies in $\mathrm{QF}(S)$ if its image is a quasi-Fuchsian group. By a theorem of Bers, $\mathrm{QF}(S)$ is a manifold, and the author proves that it is disc-convex in $V(S)$. That is, if $f\colon \overline{\Delta} \to V(S)$ is a continuous map from the closure of the unit disc Δ, which is holomorphic on Δ and sends the boundary circle of $\overline{\Delta}$ to $\mathrm{QF}(S)$, then f must also send Δ into $\mathrm{QF}(S)$. This disc-convexity is needed in the proof that the domain D of a complex earthquake is a disc. In contrast, the author adapts an ingenious construction of Anderson and Canary to prove that the closure of $\mathrm{QF}(S)$ in $V(S)$ is not a topological manifold with boundary. A computer-generated picture of a slice of $\mathrm{QF}(S)$, where S is a punctured torus, illustrates this result.

The case when $T(S)$ has complex dimension 1 is studied extensively. The complex earthquake map is then biholomorphic, and grafting along a fixed measured lamination λ defines a bijection from $T(S)$ to $T(S)$. Also, complex earthquakes are used to produce bending coordinates on any Bers slice of $\mathrm{QF}(S)$; this generalizes work of Keen and Series.

The paper is beautifully written, with extensive references and attention to the historical development of the many ideas in play.

Darryl McCullough
From MathSciNet, April 2020

MR1730906 (2001d:32016) 32G15; 30-02, 30C62, 30F60

Gardiner, Frederick P.; Lakic, Nikola

Quasiconformal Teichmüller theory. (English)

Mathematical Surveys and Monographs, 76.
American Mathematical Society, *Providence, RI*, 2000, xx+372 pp., \$89.00,
ISBN 0-8218-1983-6

Gardiner and Lakic have produced a formidable treatise on the modern theories of quasiconformal mappings, Riemann surfaces and Teichmüller spaces. They have gathered, into a unified exposition, results which, for the most part, have not previously been found in book form or are otherwise scattered in the literature. Many of the approaches and results are new; others are more detailed than can be found elsewhere.

A quasiconformal mapping between two Riemann surfaces R_1 and R_2 is a homeomorphism $f\colon R_1 \to R_2$, satisfying some mild analytic assumptions, with the defining property that $\|\mu_f(z)\|_\infty \le k < 1$ where $\mu_f(z) = f_{\bar{z}}/f_z$. (Here subscripts denote partial derivatives in terms of any holomorphic local coordinates z and w on R_1 and R_2 respectively.) μ is called the complex dilatation of f.

From the point of view of this reviewer, there is a theme which underlies much of the book. It is the extension, when possible, of the Teichmüller theoretic concepts which are valid for compact Riemann surfaces (more precisely, those of finite conformal or topological type) to a wide range of more general Riemann surfaces. Attempts to find general theories of this type have met only limited success in the past. With the exception of the uniformization theorem and its immediate consequences or related arguments, few results are known about generic Riemann surfaces without strong side assumptions.

The first chapter introduces quasiconformal mappings, while the second focuses on Riemann surfaces. Both are notably terse—the Ahlfors-Bers form of the solution to the Beltrami equation is outlined in the exercises, whereas it previously has been most of the content of Ahlfors' 1966 monograph. Following earlier work of Grötzsch, Ahlfors, Lavrent′ev and Schiffer, Teichmüller discovered a deep relationship between the space of integrable quadratic differentials on a surface R and the space of quasiconformal deformations of R. The requisite structure theorem for quadratic differentials and some of the basic relationships to quasiconformal mappings are in Chapters 3 and 4. Deeper results of the same style are in Chapter 11. The Teichmüller space of a Riemann surface may be defined in several ways, each elucidating a different aspect of its structure. Henceforth we assume our surfaces are hyperbolic; the deformation theory of the few other surfaces has been long understood. Two Riemann surfaces R_1 and R_2 are quasiconformally equivalent if there is a quasiconformal mapping $f\colon R_1 \to R_2$. A point in the Teichmüller space $T(R)$ of R is an equivalence class of quasiconformal deformations of R. Two deformations $f_i\colon R \to R_i$ are equivalent if $f_j \circ f_i^{-1}$ is homotopic to a conformal mapping. A quasiconformal mapping $f\colon R_1 \to R_2$ is extremal (in its equivalence class) if it has minimal dilatation $K := (1 + \|\mu\|)/(1 - \|\mu\|)$ in its class. $\log K$ is the Teichmüller distance from R_1 to R_2. An integrable holomorphic quadratic differential on R is a symmetric 2-form $\omega := q\,dz^2$ with q holomorphic in the local coordinate z and satisfying $\int_R |q| dxdy < \infty$. For analytically finite Riemann surfaces, there is an integrable holomorphic quadratic differential ω, unique up to positive rescaling, with the following property: away from any poles (only at punctures) and zeroes of ω, f is affine in the holomorphic local coordinate $\zeta = \int_a \sqrt{q} dz$. Earlier authors have shown that $T(R)$ carries a complex structure. Bers' method is particularly nice in that it gives a holomorphic embedding of $T(R)$ into a bounded domain in a complex Banach space. The Ahlfors-Bers theory of the 1960's gave a method of simultaneously studying the Teichmüller spaces of all hyperbolic surfaces. The object is called the universal Teichmüller space and may be realized, as the quasi-symmetric maps, in the homeomorphism group of the circle or as a Bers embedding as above. Following earlier work by Gardiner, this book presents an elucidation of the partial topological group structure on the quasi-symmetric maps.

For any complex manifold, Kobayashi showed that there is a natural pseudometric. About 1970, using a different model of $T(R)$, Royden, followed by Earle and Kra, showed that for finite-dimensional Teichmüller spaces, the Kobayashi and Teichmüller metrics are equal. In 1984, Gardiner extended this result to more general surfaces.

The discussion of the Kobayashi metric and its geometry is given in Chapters 7 and 8. The automorphism group of $T(R)$ coincides with its group of orientation-preserving isometries. More is known for surfaces of finite genus. Royden's pioneering work was followed by various papers of Earle, Gardiner, Kra and Lakic. They showed that, for finite genius surfaces—with a few, well-understood exceptions—the isometry group of $T(R)$ is isomorphic to the mapping class group of R. In infinite genus, the question remains open.

The main new work reported in this monograph involves the asymptotic Teichmüller space $AT(R)$ studied by the authors together with Clifford Earle. The fundamental object (deformation) studied here is a quasiconformal mapping of R to R_1 which is almost conformal off any compact set. Thus the points in $AT(R)$ are the deformations of R which are conformal at the ideal boundary of R. Very limited flexibility is permitted arbitrarily close to the ideal boundary. $AT(R)$ has an impressive structure akin to its much simpler cousin $T(R)$ when R is analytically finite.

The roles of many basic tools and their consequences are clarified in the book—I cite only a few: the body of work due to Reich and Strebel and, independently, Jenkins on the trajectory structure of quadratic differentials (later generalized by Kerckhoff and Hubbard and Masur), the Douady-Earle extension, the Earle-McMullen isotopy and Slodkowski's extension theorem. Applications are given of Teichmüller theory and quasiconformal mappings to related areas of geometric function theory. A generalization of the classical slit mapping theorems is offered as a consequence of a constrained maximization problem. For general plane domains Ω, the authors study a notion of strength of a boundary point in terms of boundary dilatation of a

Teichmüller equivalence class of quasiconformal maps of Ω into the plane. Other applications of the methods are given, in particular, to earthquake theorems in the sense of Thurston.

The authors have informed Mathematical Reviews that they had intended to include a reference to the paper by V. Božin et al. [J. Anal. Math. **75** (1998), 299–338; MR1655836] in the bibliography. The presentation is marred by a number of typos and grammatical errors some of which might prove misleading to non-experts. The authors have recognized the difficulties this poses to both the newcomers to the field and researchers seeking to use their results. They have set up a webpage accessible at http://comet.lehman.cuny.edu/lakic/html/books.html. It should be consulted by all but the most casual readers. With the above caution, this monograph is now the standard reference on two-dimensional quasiconformal mappings and Teichmüller theory and is likely to remain so for many years.

William Abikoff

From MathSciNet, April 2020

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Eskin, Alex; Masur, Howard; Zorich, Anton

Moduli spaces of abelian differentials: the principal boundary, counting problems, and the Siegel-Veech constants. (English)

Publications Mathématiques. Institut de Hautes Études Scientifiques (2003), *no.* 97, 61–179.

This paper, along with a related paper [Invent. Math. **153** (2003), no. 3, 631–678 MR2000471] by M. Kontsevich and A. Zorich, concerns the geometry, topology and arithmetic of translation surfaces. We refer to the review of the latter paper for motivations and some basic definitions.

The principal interest in the paper under review is in the asymptotics of various counting functions associated with a translation surface. The cone singularities are traditionally called saddles, and a saddle connection is a geodesic segment whose end points are saddles and which does not contain saddles in the interior. A saddle connection may be closed. In polygonal billiards, saddle connections correspond to generalized diagonals, that is, billiard trajectories that start and end at the vertices of the polygon.

One fixes two saddles and a positive number L and asks how many saddle connections there are joining these points of length less than L. Likewise, one asks how many homotopy classes of primitive closed geodesics there are of length less than L. In the basic example of translation surface, the torus, this amounts to counting lattice points inside a circle of radius L in the plane. For distinct saddles, the first number has the asymptotics πL^2, and the second $(3/\pi^2)L^2$. Similar quadratic asymptotics, $c\pi L^2$, hold for generic translation surfaces, and the constant c depends only on the connected component of the stratum corresponding to the surface [see A. Eskin and H. A. Masur, Ergodic Theory Dynam. Systems **21** (2001), no. 2, 443–478; MR1827113]. One of the main goals of the present paper is to compute the constants c involved.

More precisely, one considers $\widetilde{S}$, the universal covering of the translation surface S with saddles removed. One has the developing isometric map $\widetilde{S} \to \mathbf{R}^2$, called the holonomy. To a saddle connection γ there corresponds a vector $\mathrm{hol}(\gamma) \in \mathbf{R}^2$. It may be that $\mathrm{hol}(\gamma) = \mathrm{hol}(\beta)$ for different saddle connections. For a generic surface, this implies that γ is homologous to β, and in particular that their end points coincide. One considers configurations $\mathcal{C}$ of p saddle connections joining two saddles with fixed cone angles. One is interested in counting the number of such configurations whose holonomy vector has length at most L. By a theorem of Eskin and Masur, for almost every translation surface, this number has the asymptotics, as $L \to \infty$, of $c(\alpha, \mathcal{C})\pi L^2$ where the constant $c(\alpha, \mathcal{C})$ depends only on the connected component of the stratum $\mathcal{H}(\alpha)$ that contains the surface and the configuration $\mathcal{C}$.

This has a surprising consequence: configurations of pairs of parallel saddle connections of the same length and direction can be found on almost every flat surface, and their number has quadratic asymptotics.

The strata $\mathcal{H}(\alpha)$ have a natural measure. Given a translation surface S_0 with cone points $P_1, \dots, P_k$, choose a basis in $H_1(S_0, \{P_1, \dots, S_k\}; \mathbf{Z})$ represented by saddle connections. For every surface S near S_0, the holonomy vectors associated to these saddle connections serve as local coordinates on the stratum. One gets a measure on $\mathcal{H}(\alpha)$. Likewise, one considers $\mathcal{H}_1(\alpha)$, the set of translation surfaces of unit area. This hypersurface in $\mathcal{H}(\alpha)$ also has a natural measure.

The strategy of evaluating the numbers $c(\alpha, \mathcal{C})$ is as follows. Given a configuration $\mathcal{C}$ and a surface S, let $V_{\mathcal{C}}(S)$ be the discrete set in the plane consisting of the holonomy vectors of the respective saddle connections. One constructs an operator $f \mapsto \widehat{f}$ from the space of integrable functions with compact support on $\mathbf{R}^2$ to functions on $\mathcal{H}(\alpha)$ given by the formula

$$\widehat{f}(S) = \sum_{v \in V_{\mathcal{C}}(S)} f(v).$$

One has the Siegel-Veech formula

$$\frac{1}{\operatorname{Vol}(\mathcal{H}_1(\alpha))} \int_{\mathcal{H}_1(\alpha)} \widehat{f}(S) d \operatorname{vol}(S) = c(\alpha, \mathcal{C}) \int_{\mathbf{R}^2} f$$

where the constant on the right-hand side is called the Siegel-Veech constant; this constant is the same as in the quadratic asymptotics above. To evaluate the constant, one chooses a convenient function f_ε, the characteristic function of the ε-disc. Then the left-hand side is the average number of configurations with holonomy vector shorter than ε.

Let $\mathcal{H}_1^\varepsilon(\alpha, \mathcal{C})$ be the support of f_ε, the set of surfaces with a saddle connection in the configuration $\mathcal{C}$ shorter than ε. One partitions $\mathcal{H}_1^\varepsilon(\alpha, \mathcal{C})$ into "thick" and "thin" parts; the former consists of surfaces with exactly one holonomy vector in the ε-disc, and $f_\varepsilon = 1$ on $\mathcal{H}_1^{\varepsilon,\mathrm{thick}}(\alpha, \mathcal{C})$. The contribution of the thin part to the integral is negligible, and one has

$$c(\alpha, \mathcal{C}) = \lim_{\varepsilon \to 0} \frac{1}{\pi \varepsilon^2} \frac{\operatorname{Vol}(\mathcal{H}_1^\varepsilon(\alpha, \mathcal{C}))}{\operatorname{Vol}(\mathcal{H}_1(\alpha))}.$$

As $\varepsilon \to 0$, the saddle connections from a configuration $\mathcal{C}$ collapse to a point, and one obtains a surface from a more degenerate stratum. The resulting surface is said to belong to the principal boundary of the original stratum. A substantial part of the paper is devoted to the study of this principal boundary.

The picture is as follows: $\mathcal{H}_1^{\varepsilon,\mathrm{thick}}(\alpha, \mathcal{C})$ is a ramified covering over $\mathcal{H}_1(\alpha') \times B(\varepsilon)$ where α' describes the stratum obtained after degeneration of the configuration $\mathcal{C}$ and $B(\varepsilon)$ is the ε-disc. The measure on the thick part is the product of the two measures. Denoting the degree of the covering by M, one has the answer

$$c(\alpha, \mathcal{C}) = M \frac{\mathrm{Vol}(\mathcal{H}_1(\alpha'))}{\mathrm{Vol}(\mathcal{H}_1(\alpha))}.$$

Similarly one treats closed saddle connections.

As a consequence of the computation of the volumes of the strata by Eskin and A. Okounkov [Invent. Math. **145** (2001), no. 1, 59–103; MR1839286], the Siegel-Veech constants corresponding to configurations of closed saddle connections are always equal to a rational number divided by π^2, and the ones corresponding to saddle connections joining distinct points are rational.

The paper is written very well; the 15-page introduction provides an excellent panorama of the paper and related works. Numerous examples are worked out in detail; in particular, all possible strata and configurations in genera 2 and 3 are described and the respective computations are presented (the two appendices cover the case $g = 4$).

Serge L. Tabachnikov
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Kontsevich, Maxim; Zorich, Anton

Connected components of the moduli spaces of Abelian differentials with prescribed singularities. (English)

Inventiones Mathematicae **153** (2003), *no.* 3, 631–678.

The paper under review, and a related paper [Publ. Math. Inst. Hautes Études Sci. No. 97 (2003), 61–179 MR2010740] by A. Eskin, H. Masur and A. Zorich, concern the geometry, topology and arithmetic of flat surfaces. This is an active research area, and the authors of these papers are among the main players in the field. Let me describe some motivations (see [H. A. Masur and S. L. Tabachnikov, in *Handbook of dynamical systems, Vol. 1A*, 1015–1089, North-Holland, Amsterdam, 2002; MR1928530] for a detailed discussion).

One is the billiard inside a plane polygon. The billiard system describes the motion of a free point-mass inside a domain; the point reflects off the boundary elastically so that the angle of incidence equals the angle of reflection. If a point hits a corner, its motion terminates. Billiards in polygons are poorly understood (for example, it is not known whether every triangle has a periodic billiard trajectory). A notable exception is rational polygons whose angles are rational multiples of π.

In a rational polygon, the billiard has a kind of integral of motion: a billiard trajectory may have only a finite number of directions. The simplest example is the unit square. Reflect the square in its sides to obtain a 2×2 square and paste its parallel sides pairwise to obtain a torus. The billiard flow in a fixed direction becomes a constant flow on the torus. In general, the billiard flow in rational polygons decomposes into directional flows on closed surfaces obtained by pairwise pasting the sides of a number of copies of the original polygon.

The resulting surface of genus g (depending on the angles of the billiard polygon) has a flat metric, coming from the billiard polygon, with isolated cone singularities and cone angles multiples of 2π. A flat surface with trivial linear holonomy is called a translation surface. Equivalently, a translation surface can be described as a smooth compact complex curve of genus g with a holomorphic 1-form, i.e., an abelian differential. The 1-form has zeroes at the cone points. A translation surface can be described in terms of coordinate charts and atlases; the group $\mathrm{GL}(2/\mathbf{R})_+$ acts on the space of translation surfaces by its natural action on the atlases.

Given a partition α of $2g-2$, let $\mathcal{H}(\alpha)$ be the moduli space of pairs (M,ω) where M is a closed Riemann surface of genus g and ω is an abelian differential whose orders of zeros are given by α (the zeros may be labelled or unlabelled; this gives slightly different settings, but this does not affect the number of connected components of the moduli spaces). $\mathcal{H}(\alpha)$ is called a stratum of the moduli space of abelian differentials; it has complex dimension $2g+n-1$ where n is the number of parts in α.

A breakthrough in the study of rational polygonal billiards was made by S. P. Kerckhoff, Masur and J. Smillie in [Ann. of Math. (2) **124** (1986), no. 2, 293–311; MR0855297], in which methods of Teichmüller theory were applied to the study of the billiard flow on invariant surfaces. One of the results was that this flow was uniquely ergodic for almost all directions. Another result in this area, due to Masur, is that, in a rational polygon, there is a dense set of directions each with a periodic trajectory.

Another motivation for the study of moduli spaces of translation surfaces comes from the dynamics of interval exchange maps. An interval exchange map $T\colon [0,1]\to[0,1]$ is determined by a partition of the unit interval into n subintervals $I_1,\dots,I_n$ and a permutation σ of n symbols. The restriction of T on each I_j is a parallel translation, and the new intervals $T(I_1),\dots,T(I_n)$ follow from left to right in the order $\sigma(1),\dots,\sigma(n)$. Given a translation surface with its vertical foliation, consider a transversal interval I. Then the first return map to I is an interval exchange map, and every interval exchange map can be obtained in this way. Although the abelian differential is not uniquely determined by the interval exchange map, the multiplicities of its zeros α and the connected component of the stratum $\mathcal{H}(\alpha)$ depend only on the permutation σ. It follows from previous work by Masur and W. Veech that the dynamical properties of a generic interval exchange map depend only on the respective connected component of $\mathcal{H}(\alpha)$.

The main result of the paper is a classification of connected components of the strata $\mathcal{H}(\alpha)$. To formulate this result, one needs to consider also moduli spaces of meromorphic quadratic differentials with simple poles. Let ϕ be such a quadratic differential on a Riemann surface C, not equal to the square of an abelian differential. One can canonically construct another connected Riemann surface C' with an abelian differential ω on it: C' is the double covering of C, possibly ramified at singularities of ϕ, such that the pull-back of ϕ is ω^2. The following connected components are called hyperelliptic:

$$\mathcal{H}^{\mathrm{hyp}}(2g-2)\subset\mathcal{H}(2g-2),$$

consisting of abelian differentials on curves of genus g corresponding to meromorphic quadratic differentials with $2g+1$ simple poles and a zero of multiplicity $2g-3$, and

$$\mathcal{H}^{\mathrm{hyp}}(g-1,g-1)\subset\mathcal{H}(g-1,g-1),$$

corresponding to meromorphic quadratic differentials with $2g+2$ simple poles and a zero of multiplicity $2g-2$.

Another invariant which distinguishes connected components of strata is called the parity of a Spin structure. One of the definitions is as follows. Let α be a partition with all even parts. Assign to a smooth closed curve a on a translation surface the non-negative integer $\mathrm{ind}(a)$, the rotation number of its tangent vector. Choose a symplectic basis in the first homology of the surface represented by closed curves $a_1, \ldots, a_g, b_1, \ldots, b_g$ where the intersections are given by $a_i \cap a_j = b_i \cap b_j = 0$ and $a_i \cap b_j = \delta_{ij}$. The parity of a Spin structure is

$$\sum_{i=1}^{g} (\mathrm{ind}(a_i) + 1)\ (\mathrm{ind}(b_i) + 1) \bmod 2.$$

The parity of a Spin structure is constant on a connected component of a stratum; one uses the notation $\mathcal{H}^{\mathrm{ev}}(\alpha)$ or $\mathcal{H}^{\mathrm{odd}}(\alpha)$ for even and odd spin structures.

Now we can formulate the classification theorem:

Let $g \geq 4$. The stratum $\mathcal{H}(2g-2)$ has three connected components: $\mathcal{H}^{\mathrm{hyp}}(2g-2)$, $\mathcal{H}^{\mathrm{ev}}(2g-2)$ and $\mathcal{H}^{\mathrm{odd}}(2g-2)$. The stratum $\mathcal{H}(2l, 2l)$, $l \geq 2$, has three connected components: $\mathcal{H}^{\mathrm{hyp}}(2l, 2l)$, $\mathcal{H}^{\mathrm{ev}}(2l, 2l)$ and $\mathcal{H}^{\mathrm{odd}}(2l, 2l)$. All other strata $\mathcal{H}(\alpha)$, where α has only even parts, have two connected components: $\mathcal{H}^{\mathrm{ev}}(\alpha)$ and $\mathcal{H}^{\mathrm{odd}}(\alpha)$. The strata $\mathcal{H}(2l-1, 2l-1)$, $l \geq 2$, have two connected components: one is hyperelliptic and the other is not. All the other strata are nonempty and connected.

In genus $g = 2$, there are two strata $\mathcal{H}(1,1)$ and $\mathcal{H}(2)$: each is connected and hyperelliptic. In genus $g = 3$, each of the strata $\mathcal{H}(2,2)$ and $\mathcal{H}(4)$ has two components: the hyperelliptic one and one having odd Spin structure. The other strata in genus 3 are connected.

The strategy of proof is first to study the stratum $\mathcal{H}(2g-2)$, called minimal. One can focus on abelian differentials whose horizontal foliations have all closed leaves; such translation surfaces are dense in every stratum. Abelian differentials of this type are represented combinatorially by diagrams, and diagrams corresponding to hyperelliptic abelian differentials are described. A surgery is introduced making it possible to go from the minimal stratum in genus g to that in genus $g+1$. Then one proves the classification result for the minimal stratum by induction in g.

The proof for other strata is based on the study of the adjacency of strata; one proves that the number of connected components of every stratum adjacent to the minimal one is not greater than the number of components of the latter. An abelian differential not in the minimal stratum can be degenerated so that the number of its zeros decreases. This makes it possible to proceed inductively.

This is, of course, just a brief sketch. The paper is written clearly, with attention to details, and is well illustrated by diagrams. There are two appendices, one of which concerns combinatorics of Rauzy classes of interval exchanges and its relations with translation surfaces and the other computes the parity of the Spin structure in the hyperelliptic case.

Serge L. Tabachnikov
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Liu, Kefeng; Sun, Xiaofeng; Yau, Shing-Tung

Canonical metrics on the moduli space of Riemann surfaces. I.

Journal of Differential Geometry **68** (2004), *no.* 3, 571–637.

For an integer $g \geq 2$, we denote by $\mathcal{M}_g$ the moduli space of Riemann surfaces of genus g, which is a complex orbifold of dimension $3g-3$ (quasi-projective) obtained as a quotient of the Teichmüller space $\mathcal{T}_g$ by the modular group. It is well known that these two objects are absolutely fundamental not only in topology, differential and algebraic geometry, but also for physicists via string theory (computations of path integrals are indeed reduced to integrals of Chern classes or currents on such moduli spaces). During the last thirty years, some natural (Kähler and Finsler) metrics on $\mathcal{M}_g$ have been extensively studied. A natural question raised by Yau [in *Proceedings of the International Congress of Mathematicians (Helsinki, 1978)*, 237–250, Acad. Sci. Fennica, Helsinki, 1980; MR0562611] is to what extent are these metrics equivalent? By equivalent, we mean that they are quasi-isometric to each other; i.e., we say that $\omega_1 \sim \omega_2$ if for some positive constant $c > 0$,

$$\frac{1}{c}\omega_1 \leq \omega_2 \leq c\omega_1.$$

Let's recall now some basic facts about these metrics.

(1) The Weil-Petersson metric ω_{WP}. It was introduced by Weil and is known to be Kähler, to have non-positive curvature operator and negative Ricci curvature, and to be geodesically convex. From the work of S. A. Wolpert, it is known that both its holomorphic sectional curvature and Ricci curvature have negative (genus dependent) upper bounds (but no lower bounds do exist). One key point is that it is unfortunately not complete. There are several ways to define this metric. For instance, Fenchel and Nielsen presented "twist-length" coordinates for $\mathcal{T}_g$ as the parameters $\{(\tau_j, l_j)\}$ for assembling pairs of pants (i.e. the set of lengths of all geodesics used in the decomposition and the set of the twisting parameters used to glue the pieces) to form hyperbolic surfaces. The Weil-Petersson form has a simple expression in terms of these coordinates

$$\omega_{WP} = \sum_{i=1}^{3g-3} dl_i \wedge d\tau_i.$$

Another way (and this is what the authors use) is to note that the cotangent space $T_X^*\mathcal{M}_g$ is canonically identified with the space $H^0(X, K_X^2)$ of holomorphic quadratic differentials $\phi(z)dz^2$ on $X \in \mathcal{M}_g$ and now the Weil-Petersson metric corresponds to the natural L^2 metric on $H^0(X, K_X^2)$, i.e.

$$\|\phi\|_{WP}^2 = \int_X |\phi(z)|^2 \frac{dz d\overline{z}}{\lambda(z)^2}$$

where $\lambda(z)dz \wedge d\overline{z}$ is the hyperbolic (Kähler-Einstein) metric on X.

(2) The Teichmüller metric ω_T. With previous notations, this can be defined as the L^1-norm in the cotangent space, i.e.

$$\|\phi\|_T^2 = \int_X |\phi(z)| dz d\overline{z}.$$

By a result of Royden, it is known that it is also the usual Kobayashi metric on $\mathcal{M}_g$ that we present now. For $\Delta_r \in \mathbb{C}$ the disk with radius r endowed with the Poincaré metric and $\mathrm{Hol}(\mathcal{M}_g, \Delta_r)$ the space of holomorphic maps from $\mathcal{M}_g$ to Δ_r, then for any vector $v \in T_p\mathcal{M}_g$, the Kobayashi norm of v is

$$\|v\|_K = \inf_{\{f \in \mathrm{Hol}(\Delta_r, X), f(0)=p, f'(0)=v\}} \frac{2}{r}.$$

It is a complete Finsler metric (not Riemannian); i.e., its norm is given pointwisely by a positive complex homogeneous function. Roughly speaking, since there is always a quasiconformal map from a Riemann surface to another isotopic to the identity, the Teichmüller metric measures the defect of that map from being conformal.

(3) The Carathéodory metric ω_C. For any vector $v \in T_p\mathcal{M}_g$, one has

$$\|v\|_{h_C} = \sup_{\{f\in \mathrm{Hol}(\mathcal{M}_g,\Delta_r), f(p)=0, |df(v)|=1\}} \frac{2}{r}.$$

It is also a nondegenerate complete Finsler metric.

(4) The Bergman metric ω_B. Let $K_{\mathcal{T}_g}$ be the canonical bundle of $\mathcal{T}_g$. The Bergman kernel form is defined by

$$B_{\mathcal{T}_g}(z) = \sum_{j=1}^{\infty} (\sqrt{-1})^{n^2} S_j(z) \wedge \overline{S_j(z)}$$

for any basis (S_i) of L^2-orthonormal (with respect to the natural inner product) sections of $H^0(\mathcal{T}_g, K_{\mathcal{T}_g})$. Then one can define the Bergman metric by

$$\omega_B = \sqrt{-1}\partial\overline{\partial} \log B_{\mathcal{T}_g}(z),$$

which induces on $\mathcal{M}_g$ a Kähler complete metric that we call the (induced) Bergman metric.

(5) The McMullen metric ω_M. It is complete and Kähler hyperbolic in the sense of Gromov, has positive first eigenvalue for its Laplace operator and has bounded geometry. Nevertheless, it has the disadvantage that the sign of its curvatures cannot be controlled. From the work of C. T. McMullen [Ann. of Math. (2) **151** (2000), no. 1, 327–357; MR1745010], it is known to be equivalent to the Teichmüller metric. It is defined for suitable choices of constants ϵ, δ, by

$$\omega_M = \omega_{WP} + \sqrt{-1}\delta \sum_{\{\gamma:\, l_\gamma<\epsilon\}} \frac{\partial l_\gamma}{l_\gamma} \wedge \overline{\frac{\partial l_\gamma}{l_\gamma}},$$

where the sum is taken over all primitive closed geodesics γ of length l_γ less than ϵ.

(6) The Kähler-Einstein metric ω_{KE}. Its Ricci curvature is -1. It has been studied by Cheng and Yau, who proved its completeness.

Moreover, we shall explain now what it means for a metric to have Poincaré asymptotic behaviour. Let's define the (asymptotic) Poincaré metric ω_P for a compact Kähler metric $(\overline{M}^n, \omega)$ and $Y \subset \overline{M}$ a divisor with normal crossings. Let's cover $\overline{M}$ by coordinate charts U_i such that $\left(\overline{U_{p+1}} \cap \cdots \cap \overline{U_q}\right) \cup Y = \emptyset$ and there exists n_i $(i \le p)$ with $U_i \backslash Y = (D_*)^{n_i} \times (D_*)^{n-n_i}$ (with D_* a punctured disk of radius $1/2$) and on U_i, Y is given by $Y = \{z_1^i \cdots z_{n_i}^i = 0\}$. Then one can define for C large enough the form

$$\omega_P = \omega + \sum_i \sqrt{-1}\partial\overline{\partial}\left(\eta_i \sum_{j=1}^{n_i} \log\log \frac{1}{|z_1^i \cdots z_{n_i}^i|}\right)$$

with η_i a partition of unity subordinate to the cover (U_i). We will say that a metric has Poincaré type if it's equivalent to the asymptotic Poincaré metric.

The main idea of the authors is to introduce new metrics on $\mathcal{M}_g$ with nice geometric properties and to study their relationship with the metrics ω_{KE}, ω_B, ω_C, ω_T, ω_{WP}. Let's define precisely these new metrics.

(i) The (negative) Ricci curvature of the Weil-Petersson metric $\omega_\tau = -\mathrm{Ric}(\omega_{WP})$. This is a complete Kähler metric with bounded geometry. The authors manage to estimate the asymptotics of the curvature tensor of ω_τ near the boundary points. Indeed, one can give a technical estimate of it using an integral formula where the functions involved make derivatives appear only in the fiber direction. This relies essentially on a straightforward computation of the full Riemannian curvature tensor of ω_{WP} done by Siu, Schumacher and Wolpert. The (extremely) technical part of the paper consists in describing the asymptotics of the previous quantities by estimating the harmonic Beltrami differentials following the work of H. Masur [Duke Math. J. **43** (1976), no. 3, 623–635; MR0417456] on the construction of the holomorphic quadratic differentials. Note that each point of the boundary $\overline{\mathcal{M}_g}\backslash\mathcal{M}_g$ (here $\overline{\mathcal{M}_g}$ denotes the Deligne-Mumford compactification) corresponds to a stable nodal complex curve for which Wolpert has introduced some natural coordinates [J. Differential Geom. **31** (1990), no. 2, 417–472; MR1037410]. Using these tools, the authors are able to prove that the Ricci metric has Poincaré type boundary behaviour for $Y = \overline{\mathcal{M}_g}\backslash\mathcal{M}_g$, which is a divisor with normal crossings. Moreover, they show that the holomorphic sectional curvature of the Ricci metric is asymptotically negative in the degeneration directions and is bounded in the non-degeneration ones. Finally, on $\mathcal{M}_g$, the holomorphic bisectional curvature (and hence the sectional curvature) and the Ricci curvature of ω_τ are bounded from above and below.

(ii) The perturbed Ricci metric $\omega_{\widetilde{\tau}} = \omega_\tau + C\omega_{WP}$. For a certain choice of the constant $C > 0$, this is a Kähler metric which is complete with Poincaré type behaviour and whose holomorphic sectional curvatures and Ricci curvature are bounded from above and below by negative constants (see Part II [K. Liu, X. Sun and S.-T. Yau, J. Differential Geom. **69** (2005), no. 1, 163–216; MR2169586] for details on that important point). Its bisectional curvature is bounded from above and below and it also has bounded geometry.

Finally, the main result of the paper is the equivalence on $\mathcal{M}_g$ of the Teichmüller metric (or the Kobayashi metric), the Carathéodory metric, the Kähler-Einstein metric, the induced Bergman metric, the McMullen metric, the Ricci metric and the perturbed Ricci metric, i.e.,

$$\omega_T \sim \omega_C \sim \omega_{KE} \sim \omega_B \sim \omega_M \sim \omega_\tau \sim \omega_{\widetilde{\tau}} \sim \omega_P.$$

To compare two complete metrics on a noncompact manifold, either one can try to estimate their asymptotic behavior and compare it near infinity (which is not completely possible here), or one can try to apply Yau's Schwarz lemma [Amer. J. Math. **100** (1978), no. 1, 197–203; MR0486659] with the trivial map. It is here that the perturbed Ricci metric (in particular the precise understanding of its curvature and the asymptotic behavior of the geodesic length functions) plays a central role as a bridge among all the other metrics, which leads, using McMullen's result, to $\omega_T \sim \omega_{KE} \sim \omega_M \sim \omega_\tau \sim \omega_{\widetilde{\tau}} \sim \omega_P$. On the other hand, the authors compare ω_C with ω_B and the Kobayashi metric by using Bers' imbedding theorem (they obtain in fact a general result for the comparison of these metrics over certain complex manifolds, which has an independent interest).

All these results can actually be extended to the moduli of Riemann surfaces with n marked points. See also Part II [K. Liu, X. Sun and S.-T. Yau, op. cit.] for a continuation of this paper leading to some interesting geometric consequences, and for the detailed proofs of the geometric boundedness of the Ricci and perturbed Ricci metrics. Due to the length and the technicalities, and despite the natural concepts that are involved, the paper is hard to follow. The reviewer advises that one read in advance as an introduction [K. Liu, X. Sun and S.-T. Yau, Sci. China

Ser. A **48** (2005), suppl., 97–122; MR2156494]. See also [S.-K. Yeung, Int. Math. Res. Not. **2005**, no. 4, 239–255; MR2128436], in which some similar results were found.

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Julien Keller

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Fock, Vladimir; Goncharov, Alexander

Moduli spaces of local systems and higher Teichmüller theory. (English)

Publications Mathématiques. Institut de Hautes Études Scientifiques (2006), *no.* 103, 1–211.

This ambitious paper develops the theory of higher Teichmüller spaces over a compact connected oriented surface S with possibly nonempty boundary and punctures. These spaces generalize the classical Fricke-Teichmüller spaces whose points parametrize isometry classes of complete hyperbolic geometry structures on S, possibly with geodesic boundary.

Higher Teichmüller spaces originate with the moduli spaces $\mathcal{L}_{G,S}$ of G-local systems over S, where G is a connected $\mathbb{R}$-split semisimple algebraic Lie group. Such a local system is equivalent to a flat principal G-bundle over S. This, in turn, is equivalent to a conjugacy class of a representation $\pi_1(S) \xrightarrow{\rho} G$. The set $\mathrm{Hom}(\pi_1(S), G)$ has a natural structure of an affine algebraic set over $\mathbb{R}$, and $\mathcal{L}_{G,S}$ is its quotient (in the classical topology) by inner automorphisms of G.

The higher Teichmüller spaces involve some extra structure over the boundary, and correspond to a remarkable class of representations. Let B denote a Borel subgroup (minimal parabolic subgroup) of G and U its unipotent radical. A framing of a G-local system is a parallel section of the associated flat G/B-bundle over ∂S. A decoration of a G-local system is a parallel section of the associated flat G/U-bundle over ∂S, provided that the holonomy around each component of the boundary is unipotent. Equivalently a framing (respectively a decoration) corresponds to, for each component $\gamma \subset \partial S$, an element of G/B (respectively G/U) invariant under $\rho(\gamma)$. The authors define moduli spaces $\mathfrak{X}_{G,S}$ of framed local systems and $\mathcal{A}_{G,S}$ of decorated local systems, each of which maps to $\mathcal{L}_{G,S}$. They conjecture that these two moduli spaces are dual when the group G is replaced by its Langlands dual. Specifically, this means that there is a basis of functions on the framed moduli space $\mathfrak{X}_{G,S}$ parametrized by the points of the tropicalization of the decorated moduli space $\mathcal{A}_{G,S}$, and vice versa.

The paper develops a structure on these moduli spaces similar to that of a toric variety, whereby the space admits a dense open subset which looks like an affine torus $(\mathbb{G}_m)^N$ with a natural "symplectic geometry". When $\partial S = \emptyset$, this is the symplectic geometry given by the general construction in [W. M. Goldman, Adv. in Math. **54** (1984), no. 2, 200–225; MR0762512], which was based on [M. F. Atiyah and R. H. Bott, Philos. Trans. Roy. Soc. London Ser. A **308** (1983), no. 1505, 523–615; MR0702806] for the analogous case that G is compact. In the simplest case ($G = \mathrm{PSL}(2, \mathbb{R})$), all three moduli spaces are the Fricke-Teichmüller space $\mathfrak{T}(S)$ of marked hyperbolic structures on S, the moduli space in question, and the symplectic structure arises from the Kähler form of the Weil-Petersson metric on Teichmüller space.

When $\partial S \neq \emptyset$, the moduli space $\mathcal{A}_{G,S}$ of decorated local systems carries a natural degenerate closed exterior 2-form. Its "dual" moduli space $\mathfrak{X}_{G,S}$ of framed local systems carries a natural Poisson structure on $\mathcal{L}_{G,S}$. In the analogous case when G is compact, these structures relate to the Poisson structures on moduli spaces considered by K. Guruprasad et al. [Duke Math. J. **89** (1997), no. 2, 377–412; MR1460627] and the quasi-Hamiltonian moment maps considered by A. Yu. Alekseev, A. Z. Malkin and E. Meinrenken [J. Differential Geom. **48** (1998), no. 3, 445–495; MR1638045]. Closely related are the constructions, using quantum groups, of V. V. Fok and A. A. Roslyĭ [in *Moscow Seminar in Mathematical Physics*, 67–86, Amer. Math. Soc. Transl. Ser. 2, 191, Amer. Math. Soc., Providence, RI, 1999; MR1730456] and Alekseev and Malkin [Comm. Math. Phys. **169** (1995), no. 1, 99–119; MR1328263]

The locally toric structure on these moduli spaces has a remarkable property: one component of this space has a positive structure. From the viewpoint of the paper under review, the disconnectedness of the space $\mathrm{Hom}(\pi_1(S), G)$ (corresponding to the moduli space $\mathcal{L}_{G,S}$ of local systems) arises from the disconnectedness of the Lie group $\mathbb{R}^*$.

In the simplest case ($G = \mathrm{SL}(2,\mathbb{R})$), the Fricke-Teichmüller space $\mathfrak{T}(S)$ of marked complete hyperbolic structures on S embeds as a connected component in the moduli space, defined by real inequalities (conditions like $|\mathrm{tr}(\rho(\gamma))| \geq 2$, for example). The dense open affine torus $(\mathbb{G}_m)^N$ then corresponds to the shearing coordinates developed by W. Thurston and by R. C. Penner [Comm. Math. Phys. **113** (1987), no. 2, 299–339; MR0919235]. Since this plays a fundamental role in this theory, we briefly review the construction.

The starting point for this theory is an ideal triangulation of S and the shearing coordinates on $\mathfrak{T}(S)$ first studied in this context by Thurston and Penner. The surface S, with a convex hyperbolic structure, is decomposed into ideal polygons. (The first occurrence of this idea seems to be in the 1983 doctoral thesis of Lee Mosher ["Pseudo-Anosovs on punctured surfaces", Princeton Univ., Princeton, NJ, 1983] written under the supervision of Thurston.) When S has cusps, then the sides of the polygons may be simple geodesics which limit to the cusps. When $\partial\overline{S} \neq \emptyset$, then $\partial\overline{S}$ is assumed to be a union of closed geodesics, and the sides of the polygon may spiral around these closed geodesics, or other closed geodesics in the interior of S. The surface is reconstructed from this finite set of polygons by identifying sides (which appear as shears), and these gluing instructions furnish a convenient and computable set of coordinates for $\mathfrak{T}(S)$. As first observed by Penner [J. Differential Geom. **35** (1992), no. 3, 559–608; MR1163449], the Weil-Petersson symplectic form has a remarkably simple expression in these coordinates. Penner's construction is based in turn on S. A. Wolpert's theorem that Fenchel-Nielsen coordinates on $\mathfrak{T}(S)$ are canonical (Darboux) coordinates for the Weil-Petersson Kähler form [S. A. Wolpert, Amer. J. Math. **107** (1985), no. 4, 969–997; MR0796909].

The shearing coordinates provide instructions to assemble a hyperbolic surface out of ideal 2-simplices. The condition that the shear coordinates are positive implies that the union of ideal 2-simplices fit together to form a nonsingular hyperbolic surface. Otherwise the union is a hyperbolic surface folded along the geodesic 1-simplices. These correspond to representations in other components of

$\mathrm{Hom}(\pi_1(S), G)$, and have been investigated by R. M. Kashaev [Math. Res. Lett. **12** (2005), no. 1, 23–36; MR2122727].

Such ideal triangulations are related by sequences of mutations, whereby one edge is removed and replaced by a geodesic with an alternate pair of endpoints, such as replacing one diagonal in a quadrilateral by the other diagonal. The coordinates transform birationally, preserving the symplectic geometry and the positivity. They define a groupoid, whose objects are ideal triangulations, and the morphisms are generated by the elementary moves. Because $\mathrm{Mod}(S)$ has only finitely many orbits on the set of ideal triangulations, this group is a finite extension of $\mathrm{Mod}(S)$.

Fok and Goncharov show that this theory extends to all split real forms G. In their generalized shearing coordinates, the elementary transformations are represented by rational functions whose numerators and denominators are polynomials whose coefficients are positive integers. Therefore inside the coordinate ring of the moduli space is a preserved subset of positive functions. Moreover this positive structure on the moduli space determines a preferred subset of positive points, which comprises a connected component in the classical topology of the set of $\mathbb{R}$-points. From its description this component is homeomorphic to a cell $\mathbb{R}^N$. This positivity was due to G. Lusztig in his theory of canonical bases [in *Lie theory and geometry*, 531–568, Progr. Math., 123, Birkhäuser Boston, Boston, MA, 1994; MR1327548; in *Algebraic groups and Lie groups*, 281–295, Cambridge Univ. Press, Cambridge, 1997; MR1635687], and independently to A. Zelevinsky.

The authors describe this in the general algebraic framework they call an orbi-cluster ensemble. The ideal triangulations correspond to the seeds and the mutations which closely relate to the cluster algebras developed by S. Fomin and Zelevinsky [J. Amer. Math. Soc. **15** (2002), no. 2, 497–529 (electronic); MR1887642; Invent. Math. **154** (2003), no. 1, 63–121; MR2004457; Adv. in Appl. Math. **28** (2002), no. 2, 119–144; MR1888840]. As noted in the paper the relationship between cluster algebras and Penner's Weil-Petersson symplectic geometry on decorated Teichmüller spaces was independently discussed by M. I. Gekhtman, M. Z. Shapiro and A. Vainshtein [Duke Math. J. **127** (2005), no. 2, 291–311; MR2130414; correction, Duke Math. J. **139** (2007), no. 2, 407–409; MR2352136]. While the mutations for higher groups appear the same as for SL(2) the expression of flips becomes increasingly complicated—for example for SL(3) flips require four mutations.

Furthermore the action of the mapping class group $\mathrm{Mod}(S)$ on these spaces preserves all this structure. Starting from the Poisson structure, the authors then develop a quantization of this space, from which new actions and extensions of the mapping class group derive. This generalizes earlier work in this direction by Fok and L. O. Chekhov [Teoret. Mat. Fiz. **120** (1999), no. 3, 511–528; MR1737362]. Extending the mapping class group of a surface to a groupoid generated by flips appears in earlier work of Penner [Adv. Math. **98** (1993), no. 2, 143–215; MR1213724; in *Geometric Galois actions, 2*, 293–312, Cambridge Univ. Press, Cambridge, 1997; MR1653016]. These quantum representations of the mapping class were an important motivation for this study, on which the authors have recently made progress [Invent. Math. **175** (2009), no. 2, 223–286; MR2470108].

The symplectic form is also described in terms of the algebraic K-theory of the moduli space $\mathcal{A}_{G,S}$ of decorated local systems. The (possibly degenerate) closed 2-form defines an element of K_2 of the function field of $\mathcal{A}_{G,S}$. Its explicit description [see A. B. Goncharov, in *I. M. Gel′fand Seminar*, 169–210, Amer. Math. Soc., Providence, RI, 1993; MR1237830] displayed the canonical coordinate systems, which initiated this investigation. As Fok has pointed out to the reviewer, pursuing this approach relates K_3 of this field with volumes of simplices in the symmetric space.

The positive structure allows one to tropicalize this variety. In the simplest case, the tropical points identify with measured geodesic laminations, whose projectivizations comprise Thurston's boundary for $\mathfrak{T}(S)$. The relation between Thurston's symplectic form on the measured lamination space and the Weil-Petersson Kähler form is due to A. Papadopoulos and Penner [Trans. Amer. Math. Soc. **335** (1993), no. 2, 891–904; MR1089420; C. R. Acad. Sci. Paris Sér. I Math. **312** (1991), no. 11, 871–874; MR1108510]. That Thurston's spaces tropicalize the real character variety is implicitly due to J. W. Morgan and P. B. Shalen [Ann. of Math. (2) **120** (1984), no. 3, 401–476; MR0769158] and is related to George Bergman's logarithmic limit set of an affine variety [Trans. Amer. Math. Soc. **157** (1971), 459–469; MR0280489].

For other G, this defines a new structure, which deserves further study. In particular the extension of Thurston's theory of measured laminations on hyperbolic surfaces (such as train track coordinates, earthquake deformations, bending, cataclysms) to higher Teichmüller theory raises many fascinating questions. The paper under review treats the case of $\mathrm{SL}(n)$, but for the other split real forms, the reader should consult the authors' sequel [in *Algebraic geometry and number theory*, 27–68, Progr. Math., 253, Birkhäuser Boston, Boston, MA, 2006; MR2263192], but the cluster theory for general G is not given here.

When $G = \mathrm{SL}(3,\mathbb{R})$ this is the deformation space of convex $\mathbb{R}\mathrm{P}^2$-structures on S, which was discussed in the authors' shorter paper [Adv. Math. **208** (2007), no. 1, 249–273; MR2304317]. For compact surfaces this deformation space was shown to be a cell when S is a compact surface with boundary by the reviewer [J. Differential Geom. **31** (1990), no. 3, 791–845; MR1053346].

In general, the higher Teichmüller space coincides with the Teichmüller component (now called the Hitchin component) of the space $\mathrm{Hom}(\pi, G)/G$ discovered by Nigel Hitchin [Topology **31** (1992), no. 3, 449–473; MR1174252]. Using gauge-theoretic techniques and a complex structure J on S, Hitchin identified a connected component of $\mathrm{Hom}(\pi, G)/G$ with the complex vector space of sections of a holomorphic vector bundle over the Riemann surface (S, J). F. Labourie [Invent. Math. **165** (2006), no. 1, 51–114; MR2221137] discovered strong dynamical properties of the representations in Hitchin's component, and proved that such representations quasi-isometrically embed $\pi_1(S)$ in G, and in particular define isomorphisms of $\pi_1(S)$ with discrete subgroups of G. O. Guichard [J. Differential Geom. **80** (2008), no. 3, 391–431; MR2472478] completed Labourie's characterization of these representations. Specifically the curve $S^1 \xrightarrow{f} \mathbb{P}^n$ is hyperconvex if for every collection $x_0, \ldots, x_n \in S^1$ consisting of distinct points, the lines in $\mathbb{R}^{n+1}$ corresponding to $f(x_0), \ldots, f(x_n)$ span $\mathbb{R}^{n+1}$. Crucial to this point of view is that the limit set of these groups is positive in the above sense; Labourie established that these positive curves are Hölder regular, which is an important feature in this theory.

Among the many intriguing questions raised in this paper is whether a representation $\pi_1(S) \longrightarrow G_{\mathbb{C}}$ (where $G_{\mathbb{C}}$ is the group of $\mathbb{C}$-points) which is close to a Hitchin representation in G determines a pair of Hitchin representations into G, that is, a pair of points in the "higher Teichmüller space". The evidence for this conjecture is the classical case when $G = \mathrm{SL}(2,\mathbb{R})$, in which L. Bers's simultaneous uniformization for quasi-Fuchsian deformations of Fuchsian representations parametrizes quasi-Fuchsian surface groups [L. Bers, Bull. Amer. Math. Soc. **66**

(1960), 94–97; MR0111834]. Bers's proof uses heavily the theory of quasiconformal mappings in dimension two, a tool which seems very difficult to extend to this more general setting where complicated integrability conditions are present.

Another provocative question arising from this theory is to what extent what the authors call "Weil-Petersson" is a mapping class group invariant Kähler geometry on the higher Teichmüller spaces.

Despite the length of the paper (211 pages), it is clearly written. The 30-page introduction is particularly helpful for an overview of the theory. Although parts of the paper are somewhat speculative, this paper contains a wealth of interesting new ideas and inter-relationships between several areas of mathematics. Undoubtedly this work will strongly impact and inspire future research.

William Goldman
From MathSciNet, April 2020

MR3418528 58D27; 22F10, 32G15, 37C85, 37D40, 60B15

Eskin, Alex; Mirzakhani, Maryam; Mohammadi, Amir

Isolation, equidistribution, and orbit closures for the $\mathrm{SL}(2,\mathbb{R})$ action on moduli space.

Annals of Mathematics. Second Series **182** (2015), *no.* 2, 673–721.

The results in this paper are analogous to the theory of unipotent flows and concern orbit closures and equidistribution for the $\mathrm{SL}(2,\mathbb{R})$-action on the moduli space of compact Riemann surfaces. Their number is such that a review can only give an impressionistic sampling. The proofs rely on the measure-classification theorem from [A. Eskin and M. Mirzakhani, "Invariant and stationary measures for the $SL(2,\mathbb{R})$ action on moduli space", preprint, arXiv:1302.3320], which is a partial analogue of M. Ratner's measure-classification theorem in the theory of unipotent flows [Ann. of Math. (2) **134** (1991), no. 3, 545–607; MR1135878], which was in turn motivated by the Raghunathan Conjecture [S. G. Dani, Invent. Math. **64** (1981), no. 2, 357–385; MR0629475; G. A. Margulis, in *Number theory, trace formulas and discrete groups (Oslo, 1987)*, 377–398, Academic Press, Boston, MA, 1989; MR0993328]. The second major ingredient is the main technical result of this paper (Proposition 2.13), an isolation property of closed $\mathrm{SL}(2,\mathbb{R})$-invariant manifolds, whose proof takes up of Sections 4–10.

The proofs of the principal results are in Section 3; these are actually simpler than the proofs of the analogous results in the theory of unipotent flows, due in no small part to the fact (Proposition 2.16, a consequence of the isolation property) that there are at most countably many affine invariant submanifolds in each stratum, while unipotent flows may have continuous families of invariant manifolds (which involve the centralizer and normalizer of the acting group). The proof of the isolation property in turn is based on the recurrence properties of the $\mathrm{SL}(2,\mathbb{R})$-action from [J. S. Athreya, Geom. Dedicata **119** (2006), 121–140; MR2247652] and on the uniform hyperbolicity in compact sets of the Teichmüller geodesic flow [G. Forni, Ann. of Math. (2) **155** (2002), no. 1, 1–103 (Corollary 2.1); MR1888794].

Some terminology to give formal statements: $H(\alpha)$ denotes a stratum of Abelian differentials, i.e., the space of pairs (M,ω) where M is a Riemann surface and ω is a holomorphic 1-form on M whose zeros have multiplicities $\alpha_1 \cdots \alpha_n$ with $\sum \alpha_i = \chi(M) \geq 0$. The form ω defines a canonical flat metric on M with cone points at the zeros of ω, i.e., a *flat surface* or *translation surface* [A. Zorich, in *Frontiers in number* *theory, physics, and geometry. I*, 437–583, Springer, Berlin, 2006; MR2261104]. The space $H(\alpha)$ admits an action of $\mathrm{SL}(2,\mathbb{R})$ which generalizes the action of $\mathrm{SL}(2,\mathbb{R})$ on the space $\mathrm{GL}(2,\mathbb{R})/\mathrm{SL}(2,\mathbb{Z})$ of flat tori. A "unit hyperboloid" $H_1(\alpha)$ is defined as a subset of translation surfaces in $H(\alpha)$ of area one: $\frac{i}{2}\int_M \omega \wedge \overline{\omega} = 1$.

The aforementioned measure-classification result of [A. Eskin and M. Mirzakhani, op. cit.] is: A probability measure on $H_1(\alpha)$ that is invariant under $P := \{\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)\} \subset \mathrm{SL}(2,\mathbb{R})$ is $\mathrm{SL}(2,\mathbb{R})$-invariant and affine (i.e., supported on an immersed submanifold and compatible with Lebesgue measure in a particular way; the submanifold is then also said to be affine).

With $a_t := \mathrm{diag}(e^t, e^{-t})$ and $r_\theta := \left(\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix}\right)$ we can now state the main isolation property of this paper.

If $\varnothing \subsetneqq M \subset H_1(\alpha)$ is an affine invariant submanifold, then there is an SO(2)-invariant $f\colon H_1(\alpha) \to [1,\infty]$ such that:

- $M = f^{-1}(\infty)$.
- f is bounded on compact subsets of $H_1(\alpha) \smallsetminus M$.
- $\overline{f^{-1}([1,\ell])}$ is compact for all ℓ.
- $\exists b$ (depending only on the "complexity" of M) $\forall c \in (0,1)$ $\exists T > 0$
$$(A_t f)(x) := \frac{1}{2\pi}\int_0^{2\pi} f(a_t r_\theta x)\, d\theta \le c f(x) + b$$
whenever $x \in H_1(\alpha) \smallsetminus M$ and $t > T$.
- There is $\sigma > 1$ such that $\sigma^{-1} f(x) \le f(gx) \le \sigma f(x)$ for all $x \in H_1(\alpha)$ and $g \in \mathrm{SL}(2,\mathbb{R})$ near the identity.

Here is an overview of the many consequences derived here.

Orbit closures in $H_1(\alpha)$ are affine invariant submanifolds (the unipotent counterpart is in [M. Ratner, Duke Math. J. **63** (1991), no. 1, 235–280; MR1106945]) and any closed P-invariant subset of $H_1(\alpha)$ is a finite union of affine invariant manifolds.

The space of ergodic P-invariant probability measures on $H_1(\alpha)$ is weak*-compact (the unipotent counterpart is in [S. Mozes and N. A. Shah, Ergodic Theory Dynam. Systems **15** (1995), no. 1, 149–159; MR1314973]).

Equidistribution for sectors, random walks and Følner sets (the first of which implies that for any $x \in H_1(\alpha)$ there is a unique affine invariant manifold of minimal dimension that contains x); uniform versions of the equidistribution results (Theorems 2.7, 2.9) are analogous to [S. G. Dani and G. A. Margulis, in *I. M. Gel′fand Seminar*, 91–137, Adv. Soviet Math., 16, Part 1, Amer. Math. Soc., Providence, RI, 1993 (Theorem 3); MR1237827], which plays a key role in applications of the theory.

Orbit counting in rational billiards: Let $N(Q,T)$ denote the number of cylinders of periodic trajectories of length at most T for the billiard flow on a rational polygon Q. It is known that this grows quadratically [H. A. Masur, Ergodic Theory Dynam. Systems **10** (1990), no. 1, 151–176; MR1053805; in *Holomorphic functions and moduli, Vol. I (Berkeley, CA, 1986)*, 215–228, Math. Sci. Res. Inst. Publ., 10, Springer, New York, 1988; MR0955824]: $N(Q,e^t)e^{-2t}$ is bounded above and away from 0 for $t > 1$. The uniform equidistribution result for sectors implies that
$$\frac{1}{t}\int_0^t N(Q,e^s)e^{-2s}\, ds \underset{t\to\infty}{\to} \to c,$$
where c is the Siegel-Veech constant [W. A. Veech, Ann. of Math. (2) **148** (1998), no. 3, 895–944; MR1670061; A. Eskin, H. A. Masur and A. Zorich, Publ. Math. Inst.

Hautes Études Sci. No. 97 (2003), 61–179; MR2010740] associated to the affine invariant submanifold $M = \mathrm{SL}(2,\mathbb{R})S$ with S the flat surface obtained by unfolding Q. The authors find it natural to conjecture that in fact $N(Q,e^t)e^{-2t}\,ds \underset{t\to\infty}{\to} \to c$, but this seems beyond current methods (yet is known in special cases [A. Eskin, H. A. Masur and M. Schmoll, Duke Math. J. **118** (2003), no. 3, 427–463; MR1983037; A. Eskin, J. Marklof and D. W. Morris, Ergodic Theory Dynam. Systems **26** (2006), no. 1, 129–162; MR2201941; K. Calta and K. Wortman, Ergodic Theory Dynam. Systems **30** (2010), no. 2, 379–398; MR2599885; M. Bainbridge, Geom. Funct. Anal. **20** (2010), no. 2, 299–356; MR2671280]).

Boris Hasselblatt

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Mirzakhani, Maryam; Vondrák, Jan

Sperner's colorings and optimal partitioning of the simplex. (English)

A journey through discrete mathematics, 615–631, *Springer, Cham*, 2017.

In this paper the authors study a number of variations of Sperner's lemma, a classical result in combinatorial topology. To state some of their results, let $\mathbf{e}_1,\dots,\mathbf{e}_k$ denote the standard basis for $\mathbb{R}^k$. Given $q \in \mathbb{N}$, let $\Delta_{k,q}$ denote the $(k-1)$-dimensional simplex

$$\Delta_{k,q} := \mathrm{conv}(q\mathbf{e}_1,\dots,q\mathbf{e}_k) = \Big\{\mathbf{x}\in\mathbb{R}^k : \sum_{i=1}^{k} x_i = q \text{ and } \mathbf{x}\geq \mathbf{0}\Big\}.$$

Also let $H_{k,q}$ denote the set of simplices $S(\mathbf{b}) := \mathrm{conv}(\mathbf{b}+\mathbf{e}_1,\dots,\mathbf{b}+\mathbf{e}_k)$ for $\mathbf{b}\in\Delta_{k,q-1}\cap\mathbb{N}^k$.

These simplices of $H_{k,q}$ are all contained in $\Delta_{k,q}$ and can be extended to a triangulation $\mathbf{T}$ of $\Delta_{k,q}$. Sperner's lemma shows that any Sperner-admissible colouring of the vertices of $\mathbf{T}$ contains a $(k-1)$-dimensional simplex whose vertices receive k distinct colours. On the other hand, these multicoloured simplices need not appear in $H_{k,q}$. The authors prove a tight lower bound on the number of non-monochromatic simplices of $H_{k,q}$ that must appear under a Sperner-admissible colouring. They also construct a Sperner-admissible colouring in which these simplices receive at most four colours, provided $k\geq 4$ and $q\geq k^2$.

The second focus of the paper is a variant of the Knaster-Kuratowski-Mazurkiewicz lemma, a geometric version of Sperner's lemma. This result says that given closed sets $A_1,\dots,A_k$ with $\Delta_{k,1}=\bigcup_{i\in[k]}A_i$ and $A_i\subset\Delta_{k,1}\cap\{\mathbf{x}: x_i>0\}$ for all $i\in[k]$, we have $\bigcap_{i\in[k]}A_i\neq\emptyset$. The authors prove an optimal lower bound on the measure of $\bigcup_{i\neq j}(A_i\cap A_j)$, showing that it is minimised when $A_1,\dots,A_k$ are Voronoi cells in $\Delta_{k,1}$ induced from the points $\{\mathbf{e}_1,\dots,\mathbf{e}_k\}$.

The paper concludes with a large number of open problems.

Eoin Long

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MR3814652 37D40; 22E50, 37C85

Eskin, Alex; Mirzakhani, Maryam

Invariant and stationary measures for the $\mathrm{SL}(2,\mathbb{R})$ **action on moduli space. (English)**

Publications Mathématiques. Institut de Hautes Études Scientifiques **127** (2018), 95–324.

This monumental work has a deceptively simple objective. There is a natural action of $\mathrm{SL}_2(\mathbb{R})$ on the space $\mathrm{GL}_2(\mathbb{R})/\mathrm{SL}_2(\mathbb{Z})$; its ergodic and dynamical properties are well understood, and there is an extensive arsenal of tools from entropy theory, conditional measure techniques, measure rigidity, and Ratner theory available to study it. Here this action is thought of as the natural action of $\mathrm{SL}_2(\mathbb{R})$ on the space of flat tori, and this action is generalized to an action of $\mathrm{SL}_2(\mathbb{R})$ on the space $\mathcal{H}(\alpha)$ of translation surfaces, parameterized by a partition $\alpha = (\alpha_1, \ldots, \alpha_n)$ of $2g - 2$ for a fixed genus $g \geqslant 1$. The main emphasis is on finding analogous rigidity and stationarity results in this setting, subsuming and generalizing much earlier work. While some of the results are inspired by the Ratner theory of unipotent flows on homogeneous spaces, much is different in this setting. In particular, the dynamical properties of the unipotent (upper triangular) flow are not understood well enough to be used, so the fundamental 'polynomial divergence' technique from unipotent flows on homogeneous spaces is not available. Instead, and in a setting where there is little control over the Lyapunov spectrum of the geodesic (diagonal) flow, new ideas are brought in to allow the 'exponential drift' technique of Y. Benoist and J.-F. Quint [Ann. of Math. (2) **174** (2011), no. 2, 1111–1162; MR2831114] to be used. This enormously understates the complexity of the work, which in fact makes use of many of the most significant results in the ergodic and rigidity theory of homogeneous dynamics. The authors have gone to great lengths to explain the overall view of the proofs, and take pains to explain where and why the main technical problems arise.

Thomas Ward

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