MATHEMATICAL PERSPECTIVES

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 58, Number 1, January 2021, Pages 107–118
https://doi.org/10.1090/bull/1698
Article electronically published on July 6, 2020

HILBERT 13: ARE THERE ANY GENUINE CONTINUOUS MULTIVARIATE REAL-VALUED FUNCTIONS?

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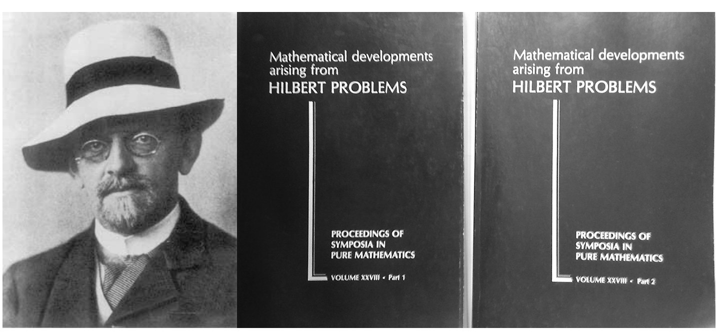
To my friend and coauthor Karl Heinrich Hofmann—
a mathematical grandchild of David Hilbert

Abstract. This article begins with a provocative question: Are there any genuine continuous multivariate real-valued functions? This may seem to be a silly question, but it is in essence what David Hilbert asked as one of the 23 problems he posed at the second International Congress of Mathematicians, held in Paris in 1900. These problems guided a large portion of the research in mathematics of the 20th century. Hilbert's 13th problem conjectured that there exists a continuous function $f : \mathbb{I}^3 \to \mathbb{R}$, where $\mathbb{I} = [0,1]$, which cannot be expressed in terms of composition and addition of continuous functions from $\mathbb{R}^2 \to \mathbb{R}$, that is, as composition and addition of continuous real-valued functions of two variables. It took over 50 years to prove that Hilbert's conjecture is false. This article discusses the solution.

1. Introduction

We begin by asking a provocative question: Are there any genuine continuous multivariate real-valued functions? This may seem to be a silly question, but it is in essence what David Hilbert (1862–1943) asked as one of the 23 problems he posed at the second International Congress of Mathematicians, held in Paris in 1900. (See [3,4].) These problems guided a large portion of the research in mathematics of the 20th century.



In his last mathematical paper [11] in 1927, David Hilbert reported on the progress on his 23 problems.[1] He devoted five pages to his 13th problem and only three pages to the remaining 22 problems. Our article is devoted to the mathematics surrounding Hilbert's 13th problem.

Remark 1.1. To set the stage, let us consider functions we often meet in a course on functions of several variables. As usual we denote the set of all real numbers with the euclidean topology by $\mathbb{R}$, the product $\mathbb{R} \times \mathbb{R}$ with the euclidean product topology by $\mathbb{R} \times \mathbb{R}$, the closed unit interval $[0, 1]$ with its subspace topology from $\mathbb{R}$ by $\mathbb{I}$, and the set $\mathbb{I}^n$, $n \in \mathbb{N}$, with the product topology by $\mathbb{I}^n$.

(i) The function $f_1 : \mathbb{I}^2 \to \mathbb{R}$ is defined by $f_1(x, y) = x + y$. We see that

$$f_1(x, y) = \phi(x) + \phi(y), \tag{1}$$

where $\phi : \mathbb{I} \to \mathbb{R}$ is given by $\phi(x) = x$. So f_1, which is a function of two variables, can be represented in terms of functions of one variable using addition.

(ii) The function $f_2 : \mathbb{I}^2 \to \mathbb{R}$ is defined by $f_2(x, y) = x.y$. Observe that

$$\begin{aligned} f_2(x, y) = x.y &= (x + 1)(y + 1) - (x + \frac{1}{2}) - (y + \frac{1}{2}) \\ &= e^{[\log_e(x+1)+\log_e(y+1)]} + (-x - \frac{1}{2}) + (-y - \frac{1}{2}). \end{aligned}$$

(We need to avoid $\log_e x$, when $x = 0$.)

So if we define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = \exp(x)$, $\phi_1 : \mathbb{I} \to \mathbb{R}$ by $\phi_1(x) = \log_e(x + 1)$, and $\phi_2 : \mathbb{I} \to \mathbb{R}$ by $\phi_2(x) = -x - \frac{1}{2}$, we see that

$$f_2(x, y) = g(\phi_1(x) + \phi_1(y)) + \phi_2(x) + \phi_2(y). \tag{2}$$

So this function f_2 of two variables can be represented in terms of functions of one variable using addition and composition.

(iii) The function $f_3 : \mathbb{I}^2 \to \mathbb{R}$ is defined by $f_3(x, y) = \sin(x + \cos y)$. If we define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = \sin x$, $\phi_1 : \mathbb{I} \to \mathbb{R}$ by $\phi_1(x) = x$, and $\phi_2 : \mathbb{I} \to \mathbb{R}$ by $\phi_2(x) = \cos x$, then

$$f_3(x, y) = g(\phi_1(x) + \phi_2(y)). \tag{3}$$

So the function f_3 of two variables can be represented in terms of functions of one variable using addition and composition.

We notice that all three functions of two variables, f_1, f_2, and f_3, can be expressed in terms of functions of one variable using addition and composition. This observation and representations (1), (2), and (3) are a hint about what is to come. □

Hilbert was aware that for $n > 4$ the Tschirnhausen transformation [29] can be used to reduce the general nth degree equation $x^n + a_1x^{n-1} + \cdots + a_n = 0$ to one of the form $y^n + b_{n-4}y^{n-4} + \cdots + b_1y + 1 = 0$. Thus Hilbert formulated his 13th problem as follows:

Prove that the equation of the seventh degree

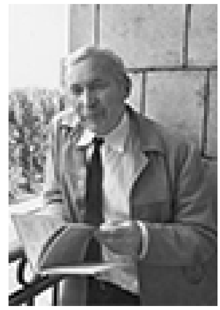
$$x^7 + ax^3 + bx^2 + cx + 1 = 0$$

is not solvable with the help of any continuous functions of only two variables.

[1] Around 1925 Hilbert apparently developed pernicious anemia, a then-untreatable vitamin B_{12} deficiency whose primary symptom is exhaustion, and he was not himself after this time. (See [28].)

This, in particular, conjectures that there are continuous functions of three variables (for example, $x(a, b, c)$) which are not representable as continuous functions of two variables (such as $\{a, b\}$, $\{a, c\}$, and $\{b, c\}$).

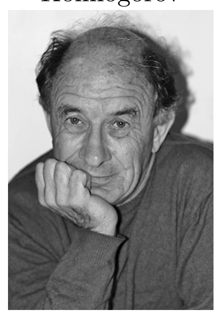
Hilbert's 13th problem conjectured that there are continuous functions of several variables which *cannot* be expressed as composition and addition of continuous functions of two variables.



Kolmogorov

To be more precise, Hilbert conjectured that there exists a continuous function $f : \mathbb{I}^3 \to \mathbb{R}$, where $\mathbb{I} = [0, 1]$, which cannot be expressed in terms of composition and addition of continuous functions from $\mathbb{R}^2 \to \mathbb{R}$, i.e., as composition and addition of continuous functions of two variables.

It took over 50 years to prove that Hilbert's conjecture is *false*. In 1957, 14 years after Hilbert's death, the solution was provided by Vladimir Igorevich Arnol′d (1937–2010). His solution built on the work of his PhD advisor, Andrej Nikolajewitsch Kolmogorov (1903–1987). Vitushkin [31] says, "No one before Kolmogorov dared to doubt the validity of Hilbert's conjecture."

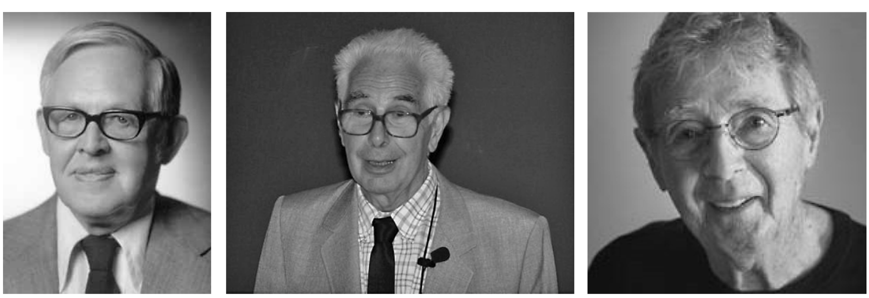


Arnol′d

In 1956 Kolmogorov proved the surprising and remarkable result that every continuous function of *any* finite number of variables can be expressed in terms of composition and addition of continuous functions of three (or fewer) variables. In 1957 Arnol′d, at the age of 19, showed that 3 can be replaced by 2, thereby proving that Hilbert's conjecture is false. Soon thereafter, Kolmogorov showed that 2 can be replaced by 1. (See [1, 13, 14].)

Kolmogorov's generalization is often referred to in the literature as the Kolmogorov superposition theorem. Over the 60 years since then, the Kolmogorov superposition theorem has been generalized substantially, its statement and proof simplified, and myriads of applications found in approximation theory, image processing, neural networks, and topological groups. It may even be of importance in the modern world of big data. (See [2, 5–8, 10, 12, 15–20, 25–27].)

In this paper we state one of the generalizations of the Kolmogorov superposition theorem and outline its proof by the Swedish mathematician Torbjörn Hedberg, which uses the work of George Gunter Lorentz (1910–2006), Jean-Pierre Kahane (1926–2017) and David A. Sprecher.



Lorentz, Kahane, and Sprecher

2. The superposition theorem: Preliminaries

Theorem 2.1 (Kolmogorov, Arnol′d, Kahane, Lorentz, and Sprecher)**.** *For any $n \in \mathbb{N}$, $n \geq 2$, there exist real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ and continuous functions $\phi_k : \mathbb{I} \to \mathbb{R}$, for $k = 1, \ldots, 2n+1$, with the property that for every continuous function $f : \mathbb{I}^n \to \mathbb{R}$ there exists a continuous function $g : \mathbb{R} \to \mathbb{R}$ such that for each $(x_1, x_2, \ldots, x_n) \in \mathbb{I}^n$,*

$$f(x_1, \ldots, x_n) = \sum_{k=1}^{2n+1} g\left(\lambda_1 \phi_k(x_1) + \cdots + \lambda_n \phi_k(x_n)\right).$$

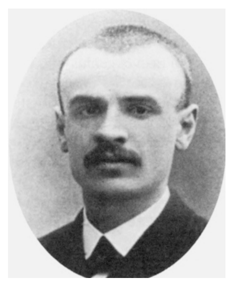
To see how beautiful and simple this theorem is, look at the case $n = 2$.

Theorem 2.2. *There exists a real number λ and continuous functions $\phi_k : \mathbb{I} \to \mathbb{R}$, for $k = 1, \ldots, 5$, with the property that for every continuous function $f : \mathbb{I}^2 \to \mathbb{R}$ there exists a continuous function $g : \mathbb{R} \to \mathbb{R}$ such that for each $(x_1, x_2) \in \mathbb{I}^2$,*

$$f(x_1, x_2) = \sum_{k=1}^{5} g(\phi_k(x_1) + \lambda \phi_k(x_2)). \tag{4}$$

Remark 2.3. We emphasize that the λ_i *and ϕ_i do not depend on the function f.* In fact, the proof will show that $\lambda_1, \ldots, \lambda_n$ can be chosen such that all $(2n+1)$-tuples $(\phi_1, \ldots, \phi_{2n+1})$ except for a "small set" (namely, one of first category in the metric space $C(\mathbb{I}^{2n+1})$) have the stated property. □

Key to Kahane's and Hedberg's (1970–1971) proof of the Kolmogorov–Arnol′d–Kahane–Lorentz–Sprecher Theorem 2.1 is the Baire category theorem, proved by the French mathematician René-Louis Baire (1874–1932) in his doctoral dissertation in 1899. T. W. Körner in his book *Linear Analysis* says, "The Baire Category is a profound triviality which condenses the folk wisdom of a generation of ingenious mathematicians into a single statement."



Baire

A proof of the Baire category theorem can be found in most topology textbooks, including in the online text *Topology Without Tears* [21].

Theorem 2.4 (Baire category theorem)**.** *Let (X, d) be a complete metric space. If $X_1, X_2, \ldots, X_n, \ldots$ is a sequence of open dense subsets of X, then the set $\bigcap_{n=1}^{\infty} X_n$ is also dense in X.*

We recall that the set $C(\mathbb{I})$ of all continuous functions from $\mathbb{I}$ into $\mathbb{R}$ is a metric space if given the metric d defined by

$$d(f, g) = \sup_{x \in [0,1]} |f(x) - g(x)|.$$

In fact, $C(\mathbb{I})$ is a Banach space with the norm $||f|| = \sup_{x \in [0,1]} |f(x)|$, for $f \in C(\mathbb{I})$. Similarly, we can think of $C(\mathbb{I}^n)$ as a metric space and as a Banach space.

The complete metric space $C(\mathbb{I}^n)$ satisfies the Baire category theorem, Theorem 2.4, for every $n \in \mathbb{N}$. This is what makes the Kahane–Hedberg proof of the Kolmogorov–Arnol′d–Kahane–Lorentz–Sprecher Theorem 2.1 work.

We shall give a slightly modified version of Hedberg's proof in [10] (which is technical rather than particularly hard) of the Kolmogorov–Arnol′d–Kahane–Lorentz–Sprecher theorem for the special case that $n = 2$; that is, we prove Theorem 2.2. For this case we put $\lambda_1 = 1$ and $\lambda_2 = \lambda$. The proof for general $n \in \mathbb{N}$ is a straightforward generalization of what is presented here.

Next we introduce a little notation for the proof of Theorem 2.2.

$$\text{Put} \quad \Phi_k(x_1, x_2) = \phi_k(x_1) + \lambda\phi_k(x_2). \tag{5}$$

So equation (4) becomes

$$f(x_1, x_2) = \sum_{k=1}^{5} g(\Phi_k(x_1, x_2)); \quad \text{i.e., } f = \sum_{k=1}^{5} g \circ \Phi_k.$$

Now we clarify the existence/choice of the λ.

Lemma 2.5. *There exists a real number λ such that for any $x_1, x_2, y_1, y_2 \in \mathbb{Q}$,*

$$x_1 + \lambda y_1 = x_2 + \lambda y_2 \implies x_1 = x_2 \text{ and } y_1 = y_2.$$

Proof. Choose for λ any irrational number. Then $x_1 - x_2 = \lambda(y_2 - y_1)$.

Suppose $x_1 \neq x_2$ and $y_1 \neq y_2$. The left-hand side is a rational number and the right-hand side is an irrational number, which is impossible. So our superposition is false and $x_1 = x_2$ or $y_1 = y_2$. It is easily seen that this implies both $x_1 = x_2$ and $y_1 = y_2$. □

In the case of general n, rather than $n = 2$, we use the following.

Lemma 2.6. *For each $n \in \mathbb{N}$ there exist real numbers $\lambda_1, \ldots, \lambda_n$ with the property that for any rational numbers $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$,*

$$\lambda_1 x_1 + \cdots + \lambda_n x_n = \lambda_1 y_1 + \cdots + \lambda_n y_n \implies \text{each } x_i = y_i.$$

Proof. Choose λ_i which are independent over the rational number field $\mathbb{Q}$; for example, each $\lambda_i = \pi^{i-1}$. The lemma then follows easily. □

We see then that the choice of the λ is not only not unique, but "almost all" λ will work. (The set of λ which do not "work" has measure zero in $\mathbb{R}$.)

3. Using the Baire category theorem

Next we present a technical lemma which sets the stage for the use of the Baire category theorem, Theorem 2.4.

Lemma 3.1. *Fix λ satisfying Lemma 2.5. Let $f \in C(\mathbb{I}^2)$ with $||f|| = 1$. Let U_f be the subset of $[C(\mathbb{I})]^5$ described as $(\phi_1, \ldots, \phi_5) \in U_f$ if and only if there exists a $g \in C(\mathbb{R})$ such that*

$$|g(t)| \leq \frac{1}{7}, \text{ for } t \in \mathbb{R}, \tag{6}$$

and

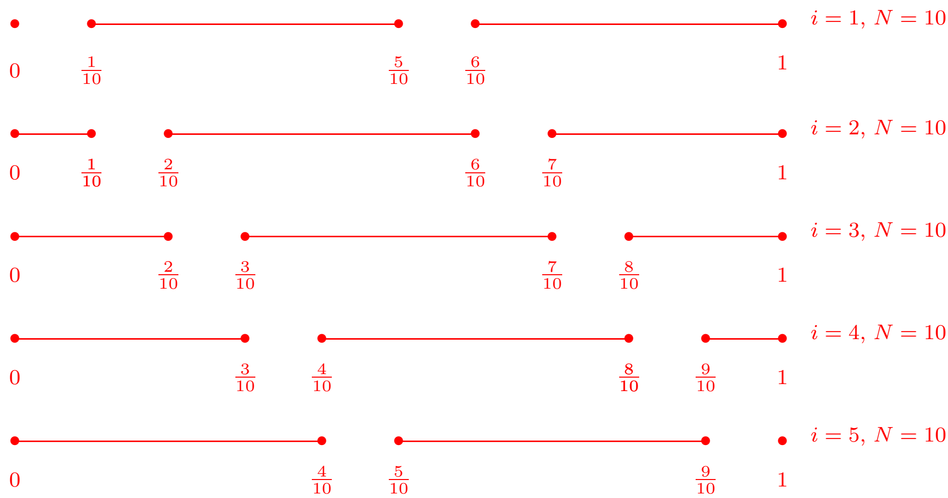
$$\Big| f(x, y) - \sum_{i=1}^{5} g(\phi_i(x) + \lambda\phi_i(y)) \Big| < \frac{7}{8}, \text{ for } (x, y) \in \mathbb{I}^2. \tag{7}$$

Then U_f is an open dense subset *of* $[\mathbb{C}(\mathbb{I})]^5$.

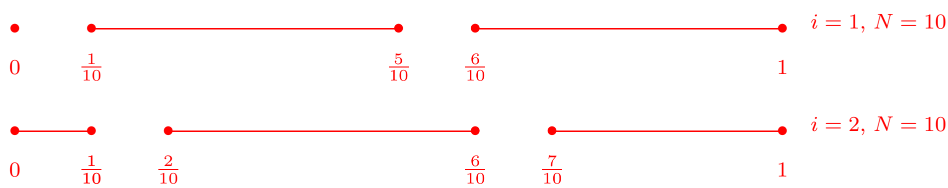
Proof. The set U_f is open, since if g "works" for some 5-tuple $(\phi_1, \ldots, \phi_5)$ in $[C(\mathbb{I})]^5$, it works also for all sufficiently close 5-tuples in the metric space $[C(\mathbb{I})]^5$.

To prove U_f is dense, let $(\phi_1^0, \ldots, \phi_5^0) \in [C(\mathbb{I})]^5$, and show that given $\varepsilon > 0$, there exists $(\phi_1, \ldots, \phi_5) \in [C(\mathbb{I})]^5$ and $g \in C(\mathbb{R})$ satisfying (6) and (7) such that each $||\phi_i - \phi_i^0|| < \varepsilon$.

Let N be a fixed positive integer which we shall specify later. If $i \in \{1, \ldots, 5\}$, consider the set of subintervals of $\mathbb{I}$ which remain when all of the intervals $[\frac{s}{N}, \frac{s+1}{N}]$ with $0 \leq s < N$, $s \equiv i - 1 \pmod 5$ are deleted. These remaining intervals, with endpoints adjoined so they are closed, shall be designated **red intervals of rank *i***.



Each red interval of rank i has length $\frac{4}{N}$, with two possible exceptions where the length is less:

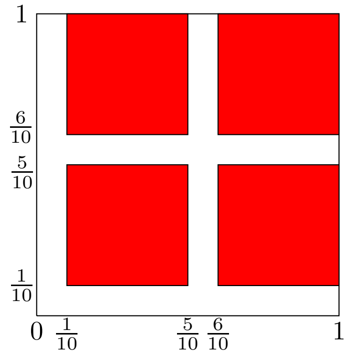


If N is large enough, clearly there exists $\phi_i \in C(\mathbb{I})$ with

(a) ϕ_i constant and equal to a rational number on each red interval of rank i;
(b) $\phi_i(x) \neq \phi_i(y)$ for x and y in distinct red intervals of rank i;
(c) $\phi_i(x) \neq \phi_j(z)$ for x in any red interval of rank i and z in any red interval of rank j; and
(d) $||\phi_i - \phi_i^0|| < \varepsilon$, $i = 1, \ldots, 5$.

A rectangle lying in $\mathbb{I}^2$, which is the cartesian product of two red intervals of rank i (one lying in $\{0 \leq x \leq 1\}$ and one lying in $\{0 \leq y \leq 1\}$), will be called a **red rectangle of rank *i*** (nearly all are squares of side $4/N$). The red rectangles of rank i will be denoted $R_{i,1}, R_{i,2}, \ldots$.

For $i = 1$ and $N = 10$:



We observe that the distance between any two points (x, y) and (x', y') in any one red rectangle is $\sqrt{(x-x')^2 + (y-y')^2} \leq \sqrt{(\frac{4}{N})^2 + (\frac{4}{N})^2} = \sqrt{\frac{32}{N^2}}$.

We define

$$\Phi_i(x, y) = \phi_i(x) + \lambda\phi_i(y), \;\; i = 1, \ldots, 5.$$

Each Φ_i is obviously constant on each red rectangle of rank $\boldsymbol{i}$, and by our choice of λ in Lemma 2.5; the (constant) value, denoted by $\Phi(R_{i,r})$ which Φ_i takes on $R_{i,r}$, cannot equal the (constant) value which Φ_j takes on $R_{j,s}$ except trivially if $i = j$ and $r = s$.

Using (uniform) continuity of f on $\mathbb{I}^2$, choose N so that

$$|f(x,y) - f(x',y')| < \frac{1}{56} \text{ for } (x-x')^2 + (y-y')^2 \leq \frac{32}{N^2}. \tag{8}$$

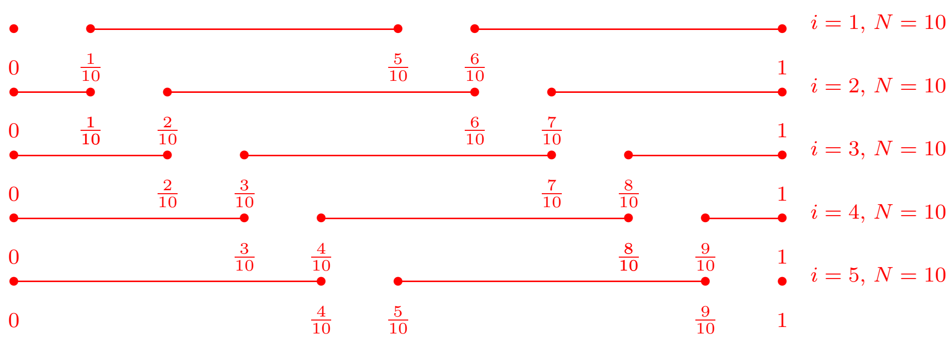
The final step in the proof is to define $g : \mathbb{I} \to \mathbb{R}$:

$$\text{If } f(x,y) > 0 \text{ throughout } R_{i,r}, \text{ define } g(\Phi(R_{i,r})) = \frac{1}{7}. \tag{9}$$

$$\text{If } f(x,y) < 0 \text{ throughout } R_{i,r}, \text{ define } g(\Phi(R_{i,r})) = -\frac{1}{7}. \tag{10}$$

Because the numbers $\Phi(R_{i,r})$ corresponding to distinct pairs (i, r) are distinct, this definition is valid. Now extend g to all of $\mathbb{R}$ in a piecewise-linear fashion so that $|g(t)| \leq \frac{1}{7}$, for all $t \in \mathbb{R}$. So $g \in C(\mathbb{R})$, and (6) is satisfied.

To complete the proof of the lemma, we need to verify inequality (7). Let (x', y') be a given point in $\mathbb{I}^2$. Since x' lies in red intervals of rank i (except perhaps for one value of i, and the same is true of y') (x', y') is contained in red rectangles of at least three different ranks.



There are three cases to consider.

The first case is $f(x', y') > \frac{1}{7}$. Then, by (8), $f(x, y) > 0$ throughout each red rectangle containing (x', y'), and we saw there are at least three such red rectangles. So using (9) and (6), we obtain

$$f(x,y) - \sum_{i=1}^{5} g(\Phi_i(x,y)) \leq 1 - \frac{3}{7} + \frac{2}{7} < \frac{7}{8}.$$

Also using (8) and (6), we see that

$$f(x,y) - \sum_{i=1}^{5} g(\Phi_i(x,y)) > (\frac{1}{7} - \frac{1}{56}) - \frac{5}{7} = -\frac{33}{56} > -\frac{7}{8}.$$

So (7) is true for this case.

The proof of the second case $f(x', y') < -\frac{1}{7}$ is analogous.

The third case is where $|f(x', y')| \leq \frac{1}{7}$. Using (6) and (8), we see that

$$\left|f(x,y) - \sum_{i=1}^{5} g(\Phi_i(x,y))\right| < \left|\frac{1}{7} + \frac{1}{56} + \frac{5}{7}\right| = \frac{7}{8},$$

as required. $\square$

Lemma 3.2. *Let λ be as in Lemma 3.1. There exist $\phi_1, \dots, \phi_5$ in $C(\mathbb{I})$, with the property that, given $f \in C(\mathbb{I}^2)$, there is a $g \in C(\mathbb{R})$ satisfying*

$$|g(t)| \leq \frac{1}{7} \, ||f||, \ t \in \mathbb{R}, \tag{11}$$

and

$$\left|\left|f - \sum_{k=1}^{5} g \circ \Phi_k\right|\right| < \frac{8}{9} \, ||f||, \tag{12}$$

where each $\Phi_k(x, y) = \phi_k(x) + \lambda\phi_k(y)$.

Proof. Without loss of generality we can assume $||f|| = 1$. Let $h_1, h_2, \dots, h_n, \dots$ be a sequence of functions in $C(\mathbb{I}^2)$ each having norm 1, such that $\{h_j : j \in \mathbb{N}\}$ is dense in the unit sphere of $C(\mathbb{I}^2)$. By Lemma 3.1, each h_j determines a set $U_j = U_{h_j} \subseteq (C(\mathbb{I}))^5$. As each such U_j is a dense open subset of the complete metric space $(C(\mathbb{I}))^5$, their intersection V is nonempty by the Baire category theorem, Theorem 2.4. Choose $(\phi_1, \dots, \phi_5) \in V$.

As $\{h_j : j \in \mathbb{N}\}$ is dense in the unit sphere of $C(\mathbb{I}^2)$, there is an $m \in \mathbb{N}$ such that $||f - h_m|| \leq \frac{1}{72}$. Since $(\phi_1, \dots, \phi_5) \in V \subseteq U_m$, there exists a $g \in C(\mathbb{R})$, $||g|| \leq \frac{1}{7}$ such that $||h_m - \sum_{k=1}^{5} g \circ \Phi_k|| < \frac{7}{8}$. Hence,

$$||f - \sum_{k=1}^{5} g \circ \Phi_k|| \leq ||f - h_m|| + ||h_m - \sum_{k=1}^{5} g \circ \Phi_k|| < \frac{1}{72} + \frac{7}{8} = \frac{8}{9}. \quad \square$$

4. Completing the proof of the superposition theorem

We now complete the proof of Theorem 2.2, the $n = 2$ case of the Kolmogorov–Arnol′d–Kahane–Lorentz–Sprecher Theorem 2.1.

Theorem 4.1. *There exists a real number λ and continuous functions $\phi_k : \mathbb{I} \to \mathbb{R}$ for $k = 1, \dots, 5$ with the property that for every continuous function $f : \mathbb{I}^2 \to \mathbb{R}$ there exists a continuous function $g : \mathbb{R} \to \mathbb{R}$ such that for each $(x_1, x_2) \in \mathbb{I}^2$,*

$$f(x_1, x_2) = \sum_{k=1}^{5} g(\phi_k(x_1) + \lambda\phi_k(x_2)). \tag{4}$$

Proof. From Lemma 3.2, there exists $\lambda \in \mathbb{R}$, $\phi_1, \dots, \phi_5$ in $C(\mathbb{I})$ and $g \in C(\mathbb{R})$, such that for each f in $C(I)$, with

$$|g(t)| \leq \frac{1}{7}\,||f||,\ t \in \mathbb{R}, \tag{11}$$

and

$$\Big|\Big|f - \sum_{k=1}^{5} g \circ \Phi_k\Big|\Big| < \frac{8}{9}\,||f||, \tag{12}$$

where each $\Phi_k(x, y) = \phi_k(x) + \lambda\phi_k(y)$.

Put $f_0 = f$ and $g_0 = g$. Recursively define $f_j \in C(\mathbb{I}^2)$ and $g_j \in C(\mathbb{R})$, for $j = 1, \dots, n, \dots,$ as $f_{j+1} = f_j - \sum_{k=1}^{5} g_j \circ \Phi_k$, and there exists a g_{j+1} satisfying (11) and (12) with g_{j+1} replacing g and f_{j+1} replacing f.

So $||g_j|| \leq \frac{1}{7}||f_j||$ and $||f_{j+1}|| = ||f_j - \sum_{k=1}^{5} g_j \circ \Phi_k|| < \frac{8}{9}||f_j||$. Thus

$$||f_{j+1}|| < \frac{8}{9}||f_j|| < (\frac{8}{9})^2||f_{j-1}|| < \cdots < (\frac{8}{9})^{j+1}||f_0|| = (\frac{8}{9})^{j+1}||f||$$

and

$$||g_{j+1}|| < \frac{1}{7}(\frac{8}{9})^{j+1}||f||.$$

Hence the series $\sum_{j=0}^{\infty} g_j$ converges in norm to a $g \in C(\mathbb{R})$, and we have

$$f = \sum_{j=0}^{\infty}(f_j - f_{j+1}) = \sum_{j=0}^{\infty}\sum_{k=1}^{5} g_j \circ \Phi_k = \sum_{k=1}^{5} g \circ \Phi_k.$$

This completes the proof of the theorem. □

With a small modification to the proof (see [10, pp. 272–273]), we can choose each ϕ_k to be a strictly increasing function, which yields the following.

Theorem 4.2. *For any $n \in \mathbb{N}$, $n \geq 2$, there exist real numbers $\lambda_1, \lambda_2, \dots, \lambda_n$ and strictly increasing continuous functions $\phi_k : \mathbb{I} \to \mathbb{R}$, for $k = 1, \dots, 2n+1$, with the property that for every continuous function $f : \mathbb{I}^n \to \mathbb{R}$ there is a continuous function $g : \mathbb{R} \to \mathbb{R}$ such that for each $(x_1, x_2, \dots, x_n) \in \mathbb{I}^n$,*

$$f(x_1, \dots, x_n) = \sum_{k=1}^{2n+1} g\left(\lambda_1\phi_k(x_1) + \cdots + \lambda_n\phi_k(x_n)\right).$$

5. Postscript

It is natural to ask if the Kolmogorov–Arnol′d–Kahane–Lorentz–Sprecher Theorem 2.1 would be true if we replaced "continuous" by a stronger property. In [30] in 1954, Anatoliy Georgievich Vitushkin (1931–2004), whose PhD advisor was Kolmogorov, proved that this theorem would be false if we insisted that f and g and all ϕ_k are continuously differentiable. However, the Kolmogorov–Arnol′d–Kahane–Lorentz–Sprecher Theorem 2.1 remains true [7] if we insist that each ϕ_k is Lip(1); that is, for all $x_1, x_2 \in \mathbb{I}$, $|f(x_1) - f(x_2)| \leq |x_1 - x_2|$, a condition weaker (for functions: $\mathbb{I} \to \mathbb{R}$) than differentiable but stronger than continuous.

Kolmogorov's proof, and the proof presented here, of the Kolmogorov–Arnol′d–Kahane–Lorentz–Sprecher Theorem 2.1, are not constructive. A *constructive* proof was given in 2007 in [2] by Jürgen Braun and Michael Griebel, clarifying and building on previous work of David A. Sprecher [23, 24], and Mario Köppen [15].

Our final theorem of Yasunao Hattori in 1993 [9] generalizes work of Phillip A. Ostrand (1936–1985) in 1965 [22], Yaki Sternfeld (1944–2001) in 1985 [26, 27], and Michael Levin in 1990 [18]. In particular, we see that the mysterious $2n+1$, which appeared in the Kolmogorov–Arnol′d–Kahane–Lorentz–Sprecher Theorem 2.1, is best possible.

Theorem 5.1. *Let $n \in \mathbb{N}$, $n \geq 2$. Further, let X be any compact metric space or, more generally, a locally compact separable metric space, of dimension n. Then there exist continuous functions $\theta_k : X \to \mathbb{R}$, for $k = 1, \ldots, 2n+1$, with the property that for every bounded continuous function $f : X \to \mathbb{R}$ there exists a continuous function $g : \mathbb{R} \to \mathbb{R}$ such that for each $x \in X$,*

$$f(x) = \sum_{k=1}^{2n+1} g(\theta_k(x)).$$

Further, $2n+1$ is "best possible", i.e., $2n+1$ is the smallest positive integer such that this is true for all f.

Noting that there are n-dimensional spaces that cannot be embedded in $\mathbb{R}^{2n}$, Levin [18] gives a relatively short proof that $2n+1$ is indeed best possible in Theorem 2.1.

Acknowledgments

This article is based on colloquia given at Federation University Australia and La Trobe University on the 85th birthday of Karl Heinrich Hofmann and also on a Plenary Lecture at the International Eurasian Conference on Mathematical Sciences and Applications (IECMSA) in Azerbaijan in 2019.

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