BOOK REVIEWS

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Boundary value problems, Weyl functions, and differential operators, by J. Behrndt, S. Hassi, and H. de Snoo, Monographs in Mathematics, Vol. 108, Birkhäuser/-Springer, Cham, 2020, vii+772 pp., ISBN 978-3-030-36713-8

The book under review is a comprehensive treatment of boundary value problems from the unifying point of view of boundary triplets, offering a panoramic view ranging from abstract problems to concrete examples of ordinary and partial differential operators.

Briefly and from a purely abstract point of view, the notion of boundary triplets, in the simplest and rather special context of self-adjoint problems, centers around a densely defined, closed, symmetric operator S in a complex Hilbert space $\mathcal{H}$ (often called the *minimal operator* in concrete applications) with equal and nonzero deficiency indices and its self-adjoint extensions denoted by S_Θ , with Θ an appropriate operator (resp., relation) parameter indexing the self-adjoint extensions of S (see (3)). Thus, self-adjoint extensions of S can equivalently be viewed as self-adjoint restrictions of the adjoint S^* of S ; S^* is typically called the *maximal operator* in applications.

Given an auxiliary complex Hilbert space $\mathcal{G}$, one seeks surjective *boundary maps* $\Gamma_j: \text{dom}(S) \rightarrow \mathcal{G}$, $j = 0, 1$, such that an *abstract second Green identity* holds in the form

$$(S^*f, g)_{\mathcal{H}} - (f, S^*g)_{\mathcal{H}} = (\Gamma_1 f, \Gamma_0 g)_{\mathcal{G}} - (\Gamma_0 f, \Gamma_1 g)_{\mathcal{G}}, \quad f, g \in \text{dom}(S^*),$$

giving rise to an abstract *boundary triplet* $(\mathcal{G}, \Gamma_0, \Gamma_1)$ for S^* . (Here $(\cdot, \cdot)_{\mathcal{H}}$ denotes the scalar product in $\mathcal{H}$ and $\text{dom}(\cdot)$ abbreviates the domain of a linear operator or relation.)

To describe two more crucial ingredients in this abstract approach to self-adjoint extensions of S —the notion of an abstract (operator-valued) *Weyl–Titchmarsh function* M and *Mark Krein’s resolvent formula* for resolvent differences of self-adjoint extensions—one fixes a convenient self-adjoint extension of S denoted by S_{Θ_0} as a reference operator (e.g., if S is bounded from below, the Friedrichs extension S_F of S is a natural candidate), such that

$$\ker(\Gamma_0) = \text{dom}(S_{\Theta_0})$$

with $\ker(\cdot)$ denoting the nullspace and

$$\text{dom}(S^*) = \text{dom}(S_{\Theta_0}) \dot{+} \ker(S^* - z_0 I), \quad z_0 \in \rho(S_{\Theta_0})$$

with $\rho(\cdot)$ abbreviating the resolvent set and $\dot{+}$ denoting the direct sum of linear subspaces. Then, given the boundary triplet $(\mathcal{G}, \Gamma_0, \Gamma_1)$ for S^* , one introduces the associated γ -field as the bijection

$$\gamma(z) = \begin{cases} \mathcal{G} \rightarrow \ker(S^* - zI), \\ \varphi \mapsto [\Gamma_0|_{\ker(S^* - zI)}]^{-1}\varphi, \end{cases} \quad z \in \rho(S_{\Theta_0}),$$

implying

$$(1) \quad \gamma(z) = [I + (z - z_0)(S_{\Theta_0} - zI)^{-1}]\gamma(z_0), \quad z, z_0 \in \rho(S_{\Theta_0}).$$

With γ in place this permits one to introduce the abstract *Weyl–Titchmarsh function* M by

$$(2) \quad M(z) = \Gamma_1\gamma(z) = \Gamma_1[\Gamma_0|_{\ker(S^* - zI)}]^{-1}, \quad z \in \rho(S_{\Theta_0}),$$

implying for $z, z_0 \in \rho(S_{\Theta_0})$ that

$$M(z) = M(z_0) + (z - z_0)\gamma(\bar{z})^*\gamma(z_0),$$

$$M(z)^* = M(\bar{z}), \quad \text{Im}(M(z)) = \text{Im}(z)\gamma(z)^*\gamma(z),$$

$$M(z) = \text{Re}(M(z_0)) + \gamma(z_0)^*[z - \text{Re}(z_0) + (z - z_0)(z - \bar{z}_0)(S_{\Theta_0} - zI)^{-1}]\gamma(z_0),$$

in particular, since $M(\cdot)$ is analytic in $\mathbb{C}_+$ (the open complex upper half-plane) and $\text{Im}(M(z)) \geq 0$ for $\text{Im}(z) > 0$, $M(\cdot)$ is an operator-valued *Nevanlinna–Herglotz function*.

Utilizing the basic fact that all self-adjoint extensions of S are in a one-to-one correspondence with all self-adjoint relations Θ in $\mathcal{G}$ via

$$\text{dom}(S_\Theta) = \{f \in \text{dom}(S^*) \mid \{\Gamma_0 f, \Gamma_1 f\} \in \Theta\}$$

(now clarifying the nature of the relation parameter Θ), we can describe the final pillar of the boundary triplet approach, viz., *Krein’s resolvent formula*, as

$$(3) \quad (S_\Theta - zI)^{-1} = (S_{\Theta_0} - zI)^{-1} + \gamma(z)[\Theta - M(z)]^{-1}\gamma(\bar{z})^*, \quad z \in \rho(S_\Theta) \cap \rho(S_{\Theta_0}).$$

Naturally, spectral theory for S_Θ is given in terms of boundary values on the real line of the resolvent of S_Θ , and hence, via (1), (2), and (3), by the boundary values $[\Theta - M(\lambda + i0)]^{-1}$, $\lambda \in \mathbb{R}$, underscoring the fundamental importance of $M(\cdot)$.

While we have clearly oversimplified matters (e.g., by ignoring the more general situation where self-adjoint extensions are replaced by closed extensions, etc.), the ideas outlined apply to even order ordinary and partial differential operators, as derived in great detail in the second order case in the book under review.

While the theory of boundary triplets is typically illustrated by choosing the by far simplest case of one-dimensional Schrödinger operators on a finite interval, we will now try a different route and employ the example of multidimensional Schrödinger operators on a bounded domain to describe the theory at hand.

Assuming $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, to be open and bounded, with a C^2 -boundary $\partial\Omega$, ν the unit normal vector pointing outwards of $\partial\Omega$, $V \in L^\infty(\Omega)$ real-valued a.e., we are interested in *self-adjoint realizations* in $L^2(\Omega)$ of the differential expression $\tau = -\Delta + V(x)$, $x \in \Omega$.

For notational simplicity we omit Lebesgue measure $d^n x$ in $L^2(\Omega)$, and in the same vein we will write $L^2(\partial\Omega)$, omitting the standard surface measure on $\partial\Omega$.

The map

$$C^\infty(\bar{\Omega}) \ni f \mapsto \{f|_{\partial\Omega}, \partial f / \partial \nu|_{\partial\Omega}\} \in H^{3/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)$$

extends by continuity to a bounded surjective operator

$$H^2(\Omega) \ni f \mapsto \{\tau_D f, \tau_N f\} \in H^{3/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)$$

with τ_D (resp., τ_N) denoting the *Dirichlet* (resp., *Neumann*) trace operator, implying

$$H_0^2(\Omega) = \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{H^2(\Omega)}} = \{f \in H^2(\Omega) \mid \tau_D f = 0 = \tau_N f\}.$$

Here $H^s(\Omega)$, $s \geq 1$, and $H^t(\partial\Omega)$, $t \geq 1/2$, denote the standard Sobolev spaces.

Minimal and maximal operators in $L^2(\Omega)$ associated with the differential expression τ are then given by

$$\begin{aligned} T_{\min} f &= \tau f, & f &\in \text{dom}(T_{\min}) = H_0^2(\Omega), \\ T_{\max} f &= \tau f, & f &\in \text{dom}(T_{\max}) = \{f \in L^2(\Omega) \mid \tau f \in L^2(\Omega)\}, \\ T_{\min}^* &= T_{\max}, & T_{\max}^* &= T_{\min}, \end{aligned}$$

in particular, $H^2(\Omega) \subsetneq \text{dom}(T_{\max})$.

The self-adjoint Dirichlet extension T_D of $T_{\min}$ (resp., restriction of $T_{\min}^* = T_{\max}$) is then characterized by

$$\begin{aligned} \text{dom}(T_D) &= \{f \in H_0^1(\Omega) \mid \tau f \in L^2(\Omega)\} \\ &= \{f \in H^2(\Omega) \mid \tau_D f = 0\} \\ &= H^2(\Omega) \cap H_0^1(\Omega); \end{aligned}$$

moreover, $T_D = T_{\min, F}$, the *Friedrichs extension* of $T_{\min}$. Similarly,

$$\text{dom}(T_N) = \{f \in H^2(\Omega) \mid \tau_N f = 0\}$$

characterizes the self-adjoint *Neumann extension* of $T_{\min}$. Since

$$\ker(\tau_D) = \text{dom}(T_D), \quad \ker(\tau_N) = \text{dom}(T_N),$$

one verifies that

$$\begin{aligned} H^2(\Omega) &= \ker(\tau_D) \dot{+} \{f_z \in H^2(\Omega) \mid \tau f_z = z f_z\}, \quad z \in \rho(T_D), \\ H^2(\Omega) &= \ker(\tau_N) \dot{+} \{f_z \in H^2(\Omega) \mid \tau f_z = z f_z\}, \quad z \in \rho(T_N). \end{aligned}$$

The (z -dependent) *Dirichlet-to-Neumann map* $D(\cdot)$ then reads

$$D(z): \begin{cases} H^{3/2}(\partial\Omega) \rightarrow H^{1/2}(\Omega), & \tau f_z = z f_z, \, z \in \rho(T_D). \\ \tau_D f_z \mapsto \tau_N f_z, \end{cases}$$

If $z \in \rho(T_D) \cap \rho(T_N)$, $D(z): H^{3/2}(\partial\Omega) \rightarrow H^{1/2}(\Omega)$ is a bijection.

In addition, $\tau_D: H^2(\Omega) \rightarrow H^{3/2}(\Omega)$ and $\tau_N: H^2(\Omega) \rightarrow H^{1/2}(\Omega)$ permit unique, bounded, and surjective extensions

$$\tilde{\tau}_D: \text{dom}(T_{\max}) \rightarrow H^{-1/2}(\partial\Omega), \quad \tilde{\tau}_N: \text{dom}(T_{\max}) \rightarrow H^{-3/2}(\partial\Omega),$$

where $\text{dom}(T_{\max})$ is equipped with the graph norm. Moreover,

$$\ker(\tilde{\tau}_D) = \ker(\tau_D) = \text{dom}(T_D), \quad \ker(\tilde{\tau}_N) = \ker(\tau_N) = \text{dom}(T_N),$$

and the fundamental *second Green identity* (i.e., artful integration by parts) holds in the form

$$\begin{aligned} (T_{\max} f, g)_{L^2(\Omega)} - (f, T_{\max} g)_{L^2(\Omega)} \\ = \langle \tilde{\tau}_D f, \tau_N g \rangle_{H^{-1/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)} - \langle \tilde{\tau}_N f, \tau_D g \rangle_{H^{-3/2}(\partial\Omega) \times H^{3/2}(\partial\Omega)}, \\ f \in \text{dom}(T_{\max}), \, g \in H^2(\Omega). \end{aligned}$$

One can show that $D(\cdot)$ extends to a bounded operator

$$\tilde{D}(z): \begin{cases} H^{-1/2}(\partial\Omega) \rightarrow H^{-3/2}(\partial\Omega), \\ \tilde{\tau}_D f_z \mapsto \tilde{\tau}_N f_z, \end{cases} \quad f_z \in \ker(T_{\max} - zI), \quad z \in \rho(T_D).$$

Next, suppose $\mathcal{G}$ and $\mathcal{H}$ are Hilbert spaces such that $\mathcal{G}$ is densely and continuously embedded in $\mathcal{H}$, the dual space $\mathcal{H}'$ of $\mathcal{H}$ is identified with $\mathcal{H}$, but the dual space $\mathcal{G}'$ of continuous antilinear functionals is not identified with $\mathcal{G}$, implying

$$\mathcal{G} \xhookrightarrow{\iota} \mathcal{H} = \mathcal{H}' \xhookrightarrow{\iota'} \mathcal{G}'.$$

Here the embedding operator $\iota: \mathcal{G} \rightarrow \mathcal{H}$ is bounded, has dense range and trivial kernel, $\ker(\iota) = \{0\}$ (implying that ι' is injective with dense range in $\mathcal{G}'$). In this case $\{\mathcal{G}, \mathcal{H}, \mathcal{G}'\}$ is called a *Gelfand triple* (or a *rigged Hilbert space*). The antilinear dual pairing $g'(g) = \langle g', g \rangle_{\mathcal{G}' \times \mathcal{G}}$ is then compatible with the scalar product in $\mathcal{H}$, that is,

$$\langle \iota' h, g \rangle_{\mathcal{G}' \times \mathcal{G}} = (h, g)_{\mathcal{H}}, \quad g \in \mathcal{G}, h \in \mathcal{H}.$$

In this context one has that $\{H^{1/2}(\partial\Omega), L^2(\partial\Omega), H^{-1/2}(\partial\Omega)\}$ is a Gelfand triple and there exist isometric isomorphisms $\iota_{\pm}: H^{\pm 1/2}(\partial\Omega) \rightarrow L^2(\partial\Omega)$ such that

$$\langle \varphi, \psi \rangle_{H^{-1/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)} = (\iota_- \varphi, \iota_+ \psi)_{L^2(\partial\Omega)}, \quad \varphi \in H^{-1/2}(\partial\Omega), \quad \psi \in H^{1/2}(\partial\Omega).$$

Recalling the decomposition

$$(4) \quad \text{dom}(T_{\max}) = \text{dom}(T_D) \dot{+} \ker(T_{\max} - \eta I), \quad \eta \in \rho(T_D),$$

one introduces the boundary maps

$$(5) \quad \begin{aligned} \Gamma_0: & \begin{cases} \text{dom}(T_{\max}) \rightarrow L^2(\partial\Omega), \\ \Gamma_0 f = \iota_- \tilde{\tau}_D f, \end{cases} & \Gamma_1: & \begin{cases} \text{dom}(T_{\max}) \rightarrow L^2(\partial\Omega), \\ \Gamma_1 f = -\iota_+ \tilde{\tau}_N f_D, \end{cases} \\ & f = f_D + f_{\eta}, \quad f_D \in \text{dom}(T_D), \quad f_{\eta} \in \ker(T_{\max} - \eta I), \quad \eta \in \rho(T_D) \cap \mathbb{R}, \end{aligned}$$

and notes

$$\ker(\Gamma_0) = \text{dom}(T_D),$$

that Γ_0, Γ_1 are surjective, and that the abstract Green identity,

$$(T_{\max} f, g)_{L^2(\Omega)} - (f, T_{\max} g)_{L^2(\Omega)} = (\Gamma_1 f, \Gamma_0 g)_{L^2(\partial\Omega)} - (\Gamma_0 f, \Gamma_1 g)_{L^2(\partial\Omega)},$$

$$f, g \in \text{dom}(T_{\max}),$$

holds. Equivalently, $\{L^2(\partial\Omega), \Gamma_0, \Gamma_1\}$ defines a *boundary triplet* for $T_{\max}(= T_{\min}^*)$.

Recalling

$$\Gamma_0 f = \iota_- \tilde{\tau}_D f = \iota_- \tilde{\tau}_D f_{\eta}, \quad f = f_D + f_{\eta},$$

with respect to the decomposition (4), since $\ker(\tilde{\tau}_D) = \ker(\tau_D) = \text{dom}(T_D)$ and since $\Gamma_0: \ker(T_{\max} - \eta I) \rightarrow L^2(\partial\Omega)$ is a bijection, one can introduce the γ -field $\gamma(z)$, $z \in \rho(T_D)$, corresponding to $\{L^2(\partial\Omega), \Gamma_0, \Gamma_1\}$ via

$$\gamma(z)\varphi = [I + (z - \eta)(T_D - zI)^{-1}]f_{\eta}(\varphi), \quad \varphi \in L^2(\partial\Omega),$$

where $f_{\eta}(\varphi) \in \ker(T_{\max} - \eta I)$ is the unique element satisfying $\Gamma_0 f_{\eta}(\varphi) = \varphi$. In this case $f_z(\varphi) = \gamma(z)\varphi \in \ker(T_{\max} - zI)$ is the unique element such that $\Gamma_0 f_z(\varphi) = \varphi$. Then

$$\gamma(z)^* = -\iota_+ \tau_N (T_D - \bar{z})^{-1}, \quad z \in \rho(T_D),$$

and the (operator-valued) Weyl–Titchmarsh function $M(\cdot)$, one of the pillars of the boundary triplet approach, associated to the boundary triplet $\{L^2(\partial\Omega), \Gamma_0, \Gamma_1\}$ is defined by

$$\begin{aligned} M(z)\varphi &= (\eta - z)\iota_+\tau_N(T_D - zI)^{-1}f_\eta(\varphi) \\ &= (\eta - z)\iota_+\tau_N(T_D - zI)^{-1}\gamma(\eta)\varphi \\ &= (z - \eta)\gamma(\bar{z})^*\gamma(\eta)\varphi, \quad \varphi \in L^2(\partial\Omega) \quad z \in \rho(T_D). \end{aligned}$$

Returning to the decomposition (4), (5), $f = f_D + f_\eta \in \text{dom}(T_{\max})$, one has

$$\tau_N f_D = \tilde{\tau}_N f_D = \tilde{\tau}_N f - \tilde{\tau}_N f_\eta,$$

and employing $\tilde{D}(\eta)\tilde{\tau}_D f_\eta = \tilde{\tau}_N f_\eta$ and $\ker(\tilde{\tau}_D) = \text{dom}(T_D)$, one obtains

$$\tau_N f_D = \tilde{\tau}_N f - \tilde{D}(\eta)\tilde{\tau}_D f_\eta = \tilde{\tau}_N f - \tilde{D}(\eta)\tilde{\tau}_D f.$$

The regularization property $\text{ran}(\tilde{D}(\eta) - \tilde{D}(z)) \subseteq H^{1/2}(\partial\Omega)$ then implies the intimate connection between $M(\cdot)$ and $\tilde{D}(\cdot)$ in the form

$$\begin{aligned} M(z)\varphi &= \iota_+[\tilde{D}(\eta) - \tilde{D}(z)]\iota_-^{-1}\varphi, \quad \varphi \in L^2(\partial\Omega), \quad z, \eta \in \rho(T_D), \\ M(z)\varphi &= \iota_+[D(\eta) - D(z)]\iota_-^{-1}\varphi, \quad \varphi \in H^2(\partial\Omega), \quad z, \eta \in \rho(T_D). \end{aligned}$$

To demonstrate the fundamental importance of M in connection with spectral properties of self-adjoint extensions T_Θ of $T_{\min}$ (equivalently, self-adjoint restrictions of $T_{\max} = T_{\min}^*$), we recall that T_Θ are in a one-to-one correspondence to all self-adjoint relations (cf. [1, Sects. 1.3–1.5]) Θ in $L^2(\partial\Omega)$ via

$$\begin{aligned} \text{dom}(T_\Theta) &= \{f \in \text{dom}(T_{\max}) \mid \{\Gamma_0 f, \Gamma_1 f\} \in \Theta\} \\ &= \{f \in \text{dom}(T_{\max}) \mid \{\iota_- \tilde{\tau}_D f, -\iota_+ \tau_N f_D\} \in \Theta\}. \end{aligned}$$

If Θ is a linear operator in $L^2(\partial\Omega)$, this reduces to

$$\text{dom}(T_\Theta) = \{f \in \text{dom}(T_{\max}) \mid \Theta \iota_- \tilde{\tau}_D f = -\iota_+ \tau_N f_D\}.$$

Returning to the case where Θ is a self-adjoint relation in $L^2(\partial\Omega)$, it is known that Θ can be represented in terms of bounded operators $\mathcal{A}, \mathcal{B}$ in $L^2(\partial\Omega)$ satisfying

$$\mathcal{A}^* \mathcal{B} = \mathcal{B}^* \mathcal{A}, \quad \mathcal{A} \mathcal{B}^* = \mathcal{B} \mathcal{A}^*, \quad \mathcal{A}^* \mathcal{A} + \mathcal{B}^* \mathcal{B} = I = \mathcal{A} \mathcal{A}^* + \mathcal{B} \mathcal{B}^*$$

as

$$\Theta = \{\{\mathcal{A}\varphi, \mathcal{B}\varphi\} \mid \varphi \in L^2(\partial\Omega)\} = \{\{\psi, \psi'\} \mid \mathcal{A}^* \psi' = \mathcal{B}^* \psi\}.$$

In this case the self-adjoint extension T_Θ of $T_{\min}$ is the restriction of $T_{\max}$ to the domain

$$\text{dom}(T_\Theta) = \{f \in \text{dom}(T_{\max}) \mid \mathcal{B}^* \iota_- \tilde{\tau}_D f = -\mathcal{A}^* \iota_+ \tau_N f_D\},$$

and the celebrated Krein resolvent formula, another pillar of the boundary triplet approach, becomes

$$\begin{aligned} (T_\Theta - zI)^{-1} &= (T_D - zI)^{-1} + \gamma(z)[\Theta - M(z)]^{-1}\gamma(\bar{z})^* \\ (6) \quad &= (T_D - zI)^{-1} + \gamma(z)\mathcal{A}[\mathcal{B} - M(z)\mathcal{A}]^{-1}\gamma(\bar{z})^*, \\ &\quad z \in \rho(T_\Theta) \cap \rho(T_D). \end{aligned}$$

As an illustrative special case we mention local and nonlocal Robin-type self-adjoint extensions T_{Θ_B} of $T_{\min}$. In this case

$$\Theta \equiv \Theta_B = \iota_+[D(\eta) - B]^{-1}\iota_-^{-1}, \quad \eta \in \rho(T_D) \cap \rho(T_N) \cap \mathbb{R},$$

where $B: H^{3/2}(\partial\Omega) \rightarrow H^{1/2}(\partial\Omega)$ is compact and B on $\text{dom}(B) = H^{3/2}(\partial\Omega)$ is symmetric in $L^2(\Omega)$, and T_{Θ_B} are restrictions of $T_{\max}$ given by

$$\text{dom}(T_{\Theta_B}) = \{f \in H^2(\Omega) \mid \tau_N f = B\tau_D f\},$$

and Krein's resolvent formula becomes

$$(T_{\Theta_B} - zI)^{-1} = (T_D - zI)^{-1} + \gamma(z)\iota_-[D(z) - B]^{-1}\iota_+^{-1}\gamma(\bar{z})^*, \quad z \in \rho(T_{\Theta_B}) \cap \rho(T_D).$$

The Neumann extension T_N of $T_{\min}$ is then given by the choice $B = 0$.

Krein's resolvent formula (6) illustrates that spectral theory for T_Θ is intimately related to the boundary values $[\Theta - M(\lambda + i\varepsilon)]^{-1}$, $\lambda \in \mathbb{R}$, as $\varepsilon \downarrow 0$. In particular, since $\sigma(T_D)$ is purely discrete (as $\Omega \subset \mathbb{R}^n$ is bounded), one infers that

$$\text{if } \lambda \in \rho(T_D), \text{ then } \lambda \in \sigma_p(T_\Theta) \text{ if and only if } \ker(\Theta - M(\lambda)) \supsetneq \{0\},$$

$$\text{or, equivalently, } \ker(\mathcal{B} - M(\lambda))\mathcal{A} \supsetneq \{0\},$$

and

$$\ker(T_\Theta - \lambda I) = \gamma(\lambda) \ker(\Theta - M(\lambda)) = \gamma(\lambda) \ker(\mathcal{B} - M(\lambda))\mathcal{A}.$$

The fact that T_D has purely discrete spectrum, equivalently, empty essential spectrum, by no means implies the same for T_Θ in general. In fact, one can show that

$$\begin{aligned} &\text{if } \lambda \in \rho(T_D), \text{ then } \lambda \in \sigma_c(T_\Theta) \text{ (resp., } \lambda \in \sigma_{\text{ess}}(T_\Theta), \text{ or } \lambda \in \sigma_d(T_\Theta) \text{ if and only if} \\ &0 \in \sigma_c(\Theta - M(\lambda)) \text{ (resp., } 0 \in \sigma_{\text{ess}}(\Theta - M(\lambda)), \text{ or } 0 \in \sigma_d(\Theta - M(\lambda))). \end{aligned}$$

Here $\sigma(\cdot)$, $\sigma_p(\cdot)$, $\sigma_c(\cdot)$, $\sigma_{\text{ess}}(\cdot)$, and $\sigma_d(\cdot)$, abbreviate the spectrum, point spectrum (i.e., the set of eigenvalues), continuous spectrum, essential spectrum, and discrete spectrum, respectively. We also mention a characterization of the absolutely continuous spectrum $\sigma_{\text{ac}}(T_\Theta)$ of T_Θ in the form

$$\sigma_{\text{ac}}(T_\Theta) = \overline{\bigcup_{\varphi \in L^2(\partial\Omega)} \{\lambda \in \mathbb{R} \mid 0 < \text{Im}(M_\Theta(\lambda + i0)\varphi, \varphi)_{L^2(\partial\Omega)} < \infty\}}^{\text{ess}},$$

where $\overline{A}^{\text{ess}}$ denotes the essential closure of a Borel set $A \subset \mathbb{R}$.

Again, this just scratches the surface of the theory, especially, in the case of Schrödinger operators bounded from below, which permit an elegant treatment of boundary triplets invoking quadratic forms bounded from below.

The book under review develops the abstract part of the theory in chapters 1–5, and treats concrete applications to ordinary and partial differential equations in chapters 6–8. The notion of linear relations in Hilbert spaces (including adjoint, symmetric, and self-adjoint), maximal dissipative relations, and von Neumann's formulas are presented in detail in chapter 1, as are notions of convergence, resolvent operators with respect to bounded operators, and Nevanlinna–Herglotz families and their representations. Chapter 2, the centerpiece of this monograph, presents the fundamentals of boundary triplets, γ -fields, and Weyl–Titchmarsh functions M in connection with closed symmetric relations S and their adjoints. This chapter culminates in Krein's formula for resolvent differences of intermediate extensions of S including Krein's formula for exit-space extensions in the spirit of Krein, Naimark, and Straus. The connection between boundary values to the real axis of Weyl–Titchmarsh functions and spectra of self-adjoint relations, a discussion of eigenvalues and eigenspaces, and the characterization of the singular behavior of the Weyl–Titchmarsh function at an eigenvalue (such as a simple pole behavior in the case of an isolated eigenvalue) is the content of chapter 3 (complementing chapter 2).

It has repeatedly been mentioned that Weyl–Titchmarsh functions are (operator-valued) Nevanlinna–Herglotz functions. More precisely, they are uniformly strict Nevanlinna–Herglotz functions (i.e., their imaginary parts are boundedly invertible on $\mathbb{C}_+$). Employing a model space approach that invokes reproducing kernel Hilbert spaces, chapter 4 is devoted to proving the converse; that is, every uniformly strict Nevanlinna–Herglotz function arises as a Weyl–Titchmarsh function. The special case of relations bounded from below, a situation of fundamental importance, is dealt with in great depth in chapter 5, naturally employing the theory of forms bounded from below. In particular, the celebrated cases of the Friedrichs extension and the Krein-type extension (and its special case, the Krein–von Neumann extension) are discussed. (The Krein–von Neumann extension was originally found by von Neumann in his quest to establish what we now call the Friedrichs extension; Mark Krein and, subsequently, Birman and Vishik, extended and completed the theory of semibounded self-adjoint extensions of closed symmetric operators bounded from below.) The extremal nature of the Friedrichs and Krein-type extensions with respect to semibounded self-adjoint extensions with a given lower bound are explored, and the notion of a boundary pair for relations bounded from below, its connection with an abstract first Green identity, and the Birman–Krein–Vishik theory is discussed. This completes the abstract and first half of this monumental treatise.

A detailed application to the case of Sturm–Liouville differential expressions $\tau = r(x)^{-1}[-(d/dx)p(x)(d/dx) + q(x)]$, $x \in (a, b) \subseteq \mathbb{R}$, and their realizations in the weighted space $L^2((a, b); r(x)dx)$ is the topic of chapter 6. At this point one meets up with Herman Weyl, the creator of much of this subject, his dichotomy of limit-point versus limit-circle cases, the notion of regular and singular endpoints, and above all, the M -function named after him, are front and center in this chapter. (Incidentally, it should be noted that while Weyl introduced the M -function, it was Titchmarsh who thoroughly exploited its complex analytic properties, hence it seems fair to the reviewer to call it the Weyl–Titchmarsh function.) In the regular and limit-circle cases at an interval endpoint, associated boundary maps Γ_0, Γ_1 are formulated in terms of quasi-derivatives and appropriate limits of Wronskians, respectively. The spectral measure of self-adjoint realizations of τ in $L^2((a, b); r(x)dx)$ is explicitly connected to the measure in the Nevanlinna–Herglotz representation of $M(\cdot)$ in the case of one endpoint being a limit point, and the interface conditions in the case of both endpoints being limit points are presented. Also, the connection between the nature of the spectrum of self-adjoint realizations of τ in $L^2((a, b); r(x)dx)$, the boundary value behavior of $M(\cdot)$ on the real axis, and the notion of subordinated solutions is explored. The special but important case of minimal Sturm–Liouville operators bounded from below, associated quadratic forms, and principal and nonprincipal solutions are the subject of the second half of chapter 6.

The case of canonical 2×2 systems of differential equations in terms of boundary triplets and their M -functions is the subject of chapter 7. As in the previous chapter on Sturm–Liouville operators, the nature of the spectrum of self-adjoint realizations, the boundary value behavior of $M(\cdot)$ on the real axis, and the notion of subordinated solutions is developed. It should be emphasized that (with the exception of certain Dirac-type systems) in the context of canonical systems the use of linear relations (as opposed to the use of linear operators in the Sturm–Liouville chapter 6) is now of utmost importance.

Finally, the case of multidimensional Schrödinger operators associated with $L^2(\Omega)$ -realizations of the differential expression $\tau = -\Delta + V(x)$, $x \in \Omega$, with $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, open and bounded, with a C^2 -boundary $\partial\Omega$ (resp., a Lipschitz domain Ω) is developed in depth in chapter 8. We already outlined much of the content of this chapter above, following the discussion of the abstract part, but we did not elaborate on the special case of Schrödinger operators bounded from below that employ quadratic forms or on the possibility of coupling the bounded interior Ω_{int} and the exterior $\Omega_{\text{ext}} = \mathbb{R}^n \setminus \overline{\Omega_{\text{int}}}$ domain with the common C^2 -boundary $\partial\Omega_{\text{int}} = \partial\Omega_{\text{ext}}$ such that $L^2(\mathbb{R}^n) = L^2(\Omega_{\text{int}}) \oplus L^2(\Omega_{\text{ext}})$ which, together with a discussion of a bounded Lipschitz domain Ω , completes chapter 8.

The book under review represents a monumental effort, a mathematical *tour-de-force*, of describing boundary value problems, their Hilbert space realizations, and the underlying spectral theory. As such it offers an introduction to the subject and leads the reader all the way to current research in this area. The book cites a staggering 784 references, making it impossible to choose a representative (let alone, fair) selection for this review. Hence we decided to reference only a few monographs (see [2]–[6]) that deal with some aspects of the theory and direct the reader to the encyclopedic masterpiece [1] for additional hints to the literature.

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