A tour of representation theory, by Martin Lorenz, Graduate Studies in Mathematics, Vol. 193, American Mathematical Society, Providence, RI, 2018, xvii+654 pp., ISBN 978-1-4704-3680-3

One central theme in the study of representation theory is the symmetries of the linear actions of various algebraic objects on vector spaces. The symmetries studied via advanced machinery of linear algebra as well as other sophisticated means in turn help uncover many interesting structures and properties of these algebraic objects. This subject dates back to the early 1800s when group theory was originated. By analyzing permutations of the roots, the theory of Évariste Galois (1811–1832) determines when a polynomial equation can be solved by radicals, and it shows that it is generally impossible for equations of degree no less than 5 (due to the fact that S_n , $n \geq 5$) are nonsolvable. Much more recently, the Langlands program, first proposed in the famous letter [La67], seeks a vast web of connections between automorphic representations and Galois representations. It is nowadays regarded by many people as a grand unified theory bridging different continents of mathematics.

The representation theory of the symmetric groups S_n plays a prominent role. Over an algebraically closed field $\mathbb{k}$ of characteristic 0, this has been well understood since the early work of Frobenius, Schur, and Young. The set $\text{Irr}_{\mathbb{k}} S_n$ of isomorphism classes of irreducible representations of S_n over $\mathbb{k}$ is parametrized by partitions $\mathcal{P}_n$ of n . Each partition $\lambda \in \mathcal{P}_n$ can be represented by a decreasing sequence of non-negative integers $\lambda_1 \geq \lambda_2 \geq \cdots$ such that $\sum_i \lambda_i = n$. For example, the trivial and sign representations of S_3 correspond to the partitions (3) and (1, 1, 1), respectively, and the unique two-dimensional irreducible representation of S_3 corresponds to the partition (2, 1). The length $\ell(\lambda)$ of λ is the number of nonzero λ_i 's. The partitions are also commonly visualized as Young diagrams, e.g., $\ell(\lambda)$ is then the number of rows in the Young diagram of λ .

Rather than the traditional approach of more than a century ago, the recent novel method of Okounkov and Vershik [OV96] makes use of the observation that each $V \in \text{Irr}_{\mathbb{k}} S_n$ restricts to a multiplicity-free direct sum of irreducible representations of S_{n-1} . This gives that the branching graph $\mathbb{B}$, with vertices $\bigcup_{n \geq 1} \text{Irr}_{\mathbb{k}} S_n$ and edges $W \rightarrow V$ if $W \in \text{Irr}_{\mathbb{k}} S_{n-1}$, occurs in the restriction of $V \in \text{Irr}_{\mathbb{k}} S_n$. The Young graph $\mathbb{Y}$ on the other hand, consists of vertices $\mathcal{P} := \bigcup_{n \geq 1} \mathcal{P}_n$ and edges $\mu \rightarrow \lambda$ if the Young diagram of $\mu \in \mathcal{P}_{n-1}$ is obtained from removing one “southeast” corner box from the Young diagram of $\lambda \in \mathcal{P}_n$. Then the graphs $\mathbb{B}$ and $\mathbb{Y}$ are isomorphic, and are stated as the graph isomorphism theorem in [L18, §4.3.2]. Imitating the representation theory of semisimple Lie algebras, the Okounkov–Vershik approach uses the Gelfand–Zeitlin algebra $\mathcal{GZ}_n$, which is the $\mathbb{k}$ -algebra generated by the centers $Z(\mathbb{k}S_m)$ of the group algebras $\mathbb{k}S_m$ for $m \leq n$, and it plays an analogous role to that of a Cartan subalgebra of a semisimple Lie algebra.

It is well known that conjugacy classes in S_n are also parametrized by $\mathcal{P}_n$. The fact that $\text{Irr}_{\mathbb{k}} S_n$ and conjugacy classes in S_n have the same parametrization is more than a coincidence. Let G be a finite group, and let $\mathbb{k}$ be an algebraically closed field such that $\text{char } \mathbb{k}$ does not divide $|G|$. Then the characters χ_V for $V \in \text{Irr}_{\mathbb{k}} G$ and the

characteristic functions of the conjugacy classes in G form two bases for the vector space of $\mathbb{k}$ -valued conjugation invariant functions on G . In particular, $\#\text{Irr}_{\mathbb{k}} G$ equals the number of conjugacy classes in G . Maschke's theorem [L18, §3.4.1] states that under the above assumption on $\text{char } \mathbb{k}$, the group algebra $\mathbb{k}G$ is semisimple and decomposes as a direct sum of matrix algebras,

$$\mathbb{k}G \simeq \bigoplus_{V \in \text{Irr}_{\mathbb{k}} G} V \otimes V^{\vee} \simeq \bigoplus_{V \in \text{Irr}_{\mathbb{k}} G} \text{Mat}_{d_V}(\mathbb{k}),$$

where $d_V = \dim_{\mathbb{k}} V$ and $V^{\vee}$ denotes the contragredient representation of V . The first isomorphism is in fact an isomorphism as $\mathbb{k}G$ -bimodules.

Many classical purely group-theoretic theorems were proven using representation theory as a tool; see the historical account in [C99]. Assume here that $\mathbb{k} = \mathbb{C}$ for simplicity. The Frobenius divisibility theorem states that $\dim_{\mathbb{k}} V$ divides $|G|$ for every $V \in \text{Irr}_{\mathbb{k}} G$, which was sharpened by N. Itô as that $\dim_{\mathbb{k}} V$ divides $|G/A|$ for any normal abelian subgroup A of G . The proof of the Frobenius divisibility theorem can be applied to establish the Burnside theorem that every group of order $p^a q^b$, where p and q are primes, is solvable. Another example is the Brauer–Fowler theorem, which states that given n , there are only finitely many finite nonabelian simple groups containing an involution with centralizer of order n . In proving this result, one employs an important notion called the Frobenius–Schur indicators for $V \in \text{Irr}_{\mathbb{k}} G$, which is defined to be

$$\nu_n(V) = \frac{1}{|G|} \sum_{g \in G} \chi_V(g^n).$$

In particular, it is known that $\nu_2(V) = 0, 1$ or -1 for any $V \in \text{Irr}_{\mathbb{k}} G$, indicating the three possibilities that V is non-self-dual, symmetric or alternating. More classical examples for the applications of representation theory can be found in [L18, §3.6].

We have seen that representations of G over $\mathbb{k}$ are equivalent to representations of the group algebra $\mathbb{k}G$. This leads to consideration for the representation theory of general associative unital algebras A over a field $\mathbb{k}$, which are not necessarily semisimple. For instance $\mathbb{k}G$ is not semisimple when $\text{char } \mathbb{k}$ is a prime p dividing $|G|$. In this case if $\mathbb{k} = \bar{\mathbb{k}}$ is algebraically closed, then $\#\text{Irr}_{\mathbb{k}} G$ equals the number of p -regular conjugacy classes in G , i.e., those that consist of elements whose common order is not divisible by p . However it is not sufficient to consider $\text{Irr}_{\mathbb{k}} G$ only, but one also needs to study indecomposable representations of G . Serre's classic monograph [S77, Part III] contains an excellent introduction to the modular representation theory of finite groups.

Some fundamental results can be stated for representations of a $\mathbb{k}$ -algebra A . By the Krull–Schmidt theorem, any finite-dimensional representation of A is a direct sum of indecomposable representations, and such a decomposition is unique up to permutations and isomorphisms of the summands. The Jordan–Hölder theorem states that in the composition series of a representation of finite length, the irreducible composition factors are unique up to permutations and isomorphisms.

The set $\text{Irr}_{\mathbb{k}} A$ is naturally associated to the set $\text{Prim } A$ of left primitive ideals of A , namely the kernels of $A \rightarrow \text{End}_{\mathbb{k}} V$, $V \in \text{Irr}_{\mathbb{k}} A$. Then

$$\text{MaxSpec } A \subset \text{Prim } A \subset \text{Spec } A.$$

The Jacobson radical of A is then $\text{rad } A = \bigcap_{\mathfrak{p} \in \text{Prim } A} \mathfrak{p}$. If A is a finitely generated commutative algebra over $\mathbb{k} = \bar{\mathbb{k}}$, then the first inclusion is an equality by the Hilbert *Nullstellensatz*.

If A is finite dimensional, then the Wedderburn structure theorem [L18, §1.4.4] gives that

$$A/\text{rad } A \simeq \bigoplus_{V \in \text{Irr}_{\mathbb{k}} A} \text{Mat}_{d_V}(D_V^{\text{op}}),$$

where D_V is the Schur division algebra $\text{End}_A(V)$, and $d_V = \dim_{D_V} V$. This can be viewed as a general version of Maschke's theorem. In particular, a finite-dimensional $\mathbb{k}$ -algebra A is semisimple if and only if $\text{rad } A = 0$.

Another rich source of representation theory that also falls into the framework of $\mathbb{k}$ -algebra representations arises from Lie algebras and Lie groups, named after Sophus Lie (1842–1859). Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over a field $\mathbb{k}$, with Lie bracket $[\cdot, \cdot]$. The binary operation $[\cdot, \cdot]$ is generally nonassociative. However the representations of $\mathfrak{g}$ are equivalent to the representations of the universal enveloping algebra $U(\mathfrak{g})$, which is the quotient of the tensor algebra $T(\mathfrak{g})$ by the two-sided ideal I generated by $x \otimes y - y \otimes x - [x, y]$, $x, y \in \mathfrak{g}$.

It is therefore natural to consider the $\mathbb{k}$ -algebra $A = U(\mathfrak{g})$ and $\text{Irr}_{\mathbb{k}} A$. The weak *Nullstellensatz* asserts that the Schur division algebra $D_V = \text{End}_A(V)$ is algebraic over $\mathbb{k}$ for any $V \in \text{Irr}_{\mathbb{k}} A$. By a result of Dixmier, if $\mathfrak{g}$ is nilpotent, then $\text{MaxSpec } A = \text{Prim } A$, which recovers the previously mentioned result when $\mathfrak{g}$ is abelian. On the other hand, mathematicians have developed sophisticated theory for the structures and representations of semisimple or reductive Lie algebras $\mathfrak{g}$, for which [H72] has been a standard reference for many generations.

The basic example is $\mathfrak{g} = \mathfrak{sl}_2$ over $\mathbb{k} = \mathbb{C}$, with standard generators e, f, h , subject to relations

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h.$$

The symmetric powers $V_m = \text{Sym}^m(\mathbb{k}^2)$ of the standard representation $\mathbb{k}^2$, $m \geq 0$, exhaust all the isomorphism classes of finite-dimensional irreducible representations of $\mathfrak{sl}_2$, and any finite-dimensional representation of $\mathfrak{sl}_2$ is completely reducible. The operator e (resp., f) acts on V_m as a raising (resp., lowering) operator that shifts the weights by 2, and the formal character of V_m is given by

$$\text{ch } V_m = t^m + t^{m-2} + \cdots + t^{-m} = \frac{t^{m+1} - t^{-m-1}}{t - t^{-1}},$$

which is the simplest case of the Weyl character formula.

The infinite-dimensional members in $\text{Irr}_{\mathbb{k}} A$, where $A = U(\mathfrak{sl}_2)$, are also known and can be described using primitive ideals of A [L18, Theorem 5.48]. Let $c = h^2 + 2ef + 2fe \in A$ be the Casimir element so that the center $\mathcal{Z}(A)$ of A equals $\mathbb{k}[c]$. The ideals $\mathfrak{p}_{\lambda} = (c - \lambda)A$, $\lambda \in \mathbb{k}$ are primitive and are the minimal nonzero primes of A . If λ is not of the form $m^2 + m$, $m \in \mathbb{Z}_+$, then $\mathfrak{p}_{\lambda}$ is also maximal; if $\lambda = m^2 + m$, then the kernel of V_m is the only proper ideal of A strictly containing $\mathfrak{p}_{\lambda}$. This description of $\text{Prim } A$ gives an exhaustion of $\text{Irr}_{\mathbb{k}} A$.

We continue to assume that $\mathbb{k} = \mathbb{C}$. In general, for $\mathfrak{g}$ semisimple, the set Λ_+ of dominant weights parametrizes finite-dimensional irreducible representations of $\mathfrak{g}$, e.g., $\Lambda_+ = \mathbb{Z}_+$ for $\mathfrak{g} = \mathfrak{sl}_2$. In brief, the construction proceeds as follows. Let $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$ be the standard Borel subalgebra that is used to define $\Lambda_+ \subset \mathfrak{h}^*$, where $\mathfrak{h}$ is the Cartan subalgebra and $\mathfrak{n}$ is the nilpotent radical of $\mathfrak{b}$. A weight $\mu \in \mathfrak{h}^* \simeq (\mathfrak{b}/[\mathfrak{b}, \mathfrak{b}])^*$ naturally gives a one-dimensional representation of $\mathfrak{b}$, denoted by $\mathbb{k}\mu$. The Verma

module M_μ , $\mu \in \Lambda_+$ is then defined as the parabolic induction $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{k}\mu$. It contains a unique maximal proper submodule M'_μ , and the quotient $V_\mu := M_\mu/M'_\mu$ is the finite-dimensional irreducible representation of $\mathfrak{g}$ corresponding to μ .

For $\mathfrak{g} = \mathfrak{sl}_d$ of type A_{d-1} with root system $\Phi = \{\varepsilon_i - \varepsilon_j : 1 \leq i \neq j \leq d\}$, the set Λ_+ consists of the dominant weights

$$\mu = l_1\mu_1 + \cdots + l_{d-1}\mu_{d-1},$$

where $\mu_i = \varepsilon_i - \frac{1}{d} \sum_{j=1}^d \varepsilon_j$ and $l_1 \geq \cdots \geq l_{d-1} \geq 0$. That is, Λ_+ is in bijection with partitions of lengths at most $d-1$.

As the Okounkov–Vershik approach suggests, representations of symmetric groups and semisimple Lie algebras share some similarities. Note also that S_n is the Weyl group of the root system of $\mathfrak{sl}_n$. An interesting explicit connection between representations of S_n and $\mathfrak{sl}_d$ is known as the Schur–Weyl duality. The starting point is the the Schur double centralizer theorem [L18, §4.7.2]. Let $V = \mathbb{k}^d$ and $G = \mathrm{GL}(V)$. Then the natural actions of G and S_n on $V^{\otimes n}$ commute with each other, and moreover the induced maps

$$\mathbb{k}G \twoheadrightarrow \mathrm{End}_{S_n}(V^{\otimes n}), \quad \mathbb{k}S_n \twoheadrightarrow \mathrm{End}_G(V^{\otimes n})$$

are both surjective. For a partition λ of n , denote by V^λ the corresponding irreducible representation of S_n . The Schur functor gives the finite-dimensional representation of G ,

$$\mathbb{S}^\lambda V := \mathrm{Hom}_{S_n}(V^\lambda, V^{\otimes n}).$$

The Schur–Weyl duality theorem [L18, Theorem 4.32, Proposition 8.39] states

- $\mathbb{S}^\lambda(V) \neq 0$ if and only if $\ell(\lambda) \leq d$;
- for $\lambda = (\lambda_1, \dots, \lambda_d)$ so that $\ell(\lambda) \leq d$, the space $\mathbb{S}^\lambda(V)$ is irreducible as an $\mathfrak{sl}_d$ -module and is isomorphic to $V_{\lambda'}$, where

$$\lambda' = \sum_{i=1}^d \lambda_i \mu_i = \sum_{i=1}^{d-1} (\lambda_i - \lambda_d) \mu_i \in \Lambda_+.$$

This phenomena of duality has various generalizations and applications. The invariant theory and representation theory of classical groups make a fruitful subject; see for example Weyl’s classic [We97] and the recent monograph [GW09].

Both the group algebra $\mathbb{k}G$ for a finite group G and the universal enveloping algebra $U(\mathfrak{g})$ for a Lie algebra $\mathfrak{g}$ fit into the category of Hopf algebras, named after Heinz Hopf (1894–1971). A $\mathbb{k}$ -algebra H is a Hopf algebra if it has additional structure given by a comultiplication homomorphism $\Delta : H \rightarrow H \otimes_{\mathbb{k}} H$, a co-unit homomorphism $\varepsilon : H \rightarrow \mathbb{k}$, and an antipode $S \in \mathrm{End}_{\mathbb{k}} H$ satisfying certain natural compatibility axioms. For the group case $H = \mathbb{k}G$, one takes $\Delta(g) = g \otimes g$, $\varepsilon(g) = 1$, and $S(g) = g^{-1}$ for $g \in G$; for the Lie algebra case $H = U(\mathfrak{g})$, one takes $\Delta(x) = x \otimes 1 + 1 \otimes x$, $\varepsilon(x) = 0$, and $S(x) = -x$ for $x \in \mathfrak{g}$. To mention another important family of examples, the category of commutative Hopf algebras over $\mathbb{k}$ is anti-equivalent to the category of affine group schemes over $\mathbb{k}$ [W79]. The theory of Hopf algebras thereby provides a unified framework for many separate constructions.

Important theorems have been extended from finite-group algebras to finite-dimensional Hopf algebras. Let H be a finite-dimensional Hopf algebra over $\mathbb{k}$. Then the antipode S is bijective (in fact of finite order), and the space $\int_H^l$ of H -invariants in the left regular representation is one dimensional. Then the Hopf

version of Maschke's theorem [L18, §12.3.1] gives that H is semisimple if and only if $\varepsilon(\int_H^l) \neq 0$. This generalizes the group case where $\int_{\mathbb{k}G}^l = \mathbb{k}\Lambda$, $\Lambda = \sum_{g \in G} g$ and $\varepsilon(\Lambda) = |G|$.

To explain further results, recall the commonly used Sweedler notation $\Delta(h) = \sum_i h_{i,1} \otimes h_{i,2} =: h_{(1)} \otimes h_{(2)} \in H \otimes_{\mathbb{k}} H$, and generally $\Delta_n(h) = h_{(1)} \otimes \cdots \otimes h_{(n)} \in H^{\otimes n}$, where $\Delta_2 = \Delta$ and $\Delta_{n+1} = (\Delta_n \otimes \text{Id}) \circ \Delta$. The Sweedler power is then $h^{[n]} := h_{(1)} h_{(2)} \cdots h_{(n)} \in H$.

For simplicity, assume that $\mathbb{k} = \mathbb{C}$ and that H is semisimple. Let $\Lambda \in \int_H^l$ be such that $0 \neq \varepsilon(\Lambda) \in \mathbb{Z}$. The Frobenius divisibility for Hopf algebras [L18, Theorem 12.18] gives that $\dim_{\mathbb{k}} V$ divides $\varepsilon(\Lambda)$ for every $V \in \text{Irr}_{\mathbb{k}} H$ if and only if the Casimir element $c_{\Lambda} := S(\Lambda_{(1)}) \otimes \Lambda_{(2)}$ is integral over $\mathbb{Z}$.

By Maschke's theorem, there is a unique $\Lambda \in \int_H^l$ such that $\varepsilon(\Lambda) = 1$. The higher Frobenius-Schur indicator for $V \in \text{Irr}_{\mathbb{k}} H$ is then defined by

$$\nu_n(V) = \chi_V(\Lambda^{[n]}).$$

It is also known that the second indicator $\nu_2(V) \in \{0, 1, -1\}$ indicates the three possible quadratic types of V ; see [L18, Theorem 12.26]. This generalizes the classical result of Frobenius and Schur for the group case.

The monograph *A tour of representation theory* by Martin Lorenz gives an excellent introduction to the various rudiments of representation theory. Influenced by the wide-ranging style of [EGH+11], the book adopts a panoramic viewpoint and offers a detailed and clear exposition to three distinct flavors of representation theory, namely representations of groups, Lie algebras, and Hopf algebras. It explains the similarities and connections, and it brings together different aspects of the theory. Certain important topics are looped around, such as the Schur-Weyl duality and Frobenius-Schur indicators.

The text logically subdivides into four parts with appendices. Part I presents fundamental materials for the representation theory of associative algebras, and it also introduces two topics, namely projective modules and Frobenius algebras. Part II mainly deals with nonmodular finite-dimensional representations of finite groups with applications. It includes a detailed exposition of the Okounkov-Vershik method for the representation theory of symmetric groups in characteristic 0. Part III introduces Lie algebras and enveloping algebras, root systems, and representation theory of semisimple Lie algebras. Part IV treats the general theory of Hopf algebras, and discusses affine algebraic groups and finite-dimensional Hopf algebras. The appendices provide various background materials, including category theory and some commutative algebra. The book is written in an accessible style, supplemented by many companion exercises.

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DONGWEN LIU

ZHEJIANG UNIVERSITY

HANGZHOU, PEOPLES REPUBLIC OF CHINA

Email address: malieu@zju.edu.cn